Applicable Analysis and Discrete Mathematics
available online at http://pefmath.etf.rs

Appl. Anal. Discrete Math. **14** (2020), 094–105.
https://doi.org/10.2298/AADM190116012G

AN AFFIRMATIVE ANSWER TO TWO QUESTIONS CONCERNING SPECIAL CASE OF SIMSEK NUMBERS AND OPEN PROBLEMS

Mouloud Goubi

The purpose of this work is to give a positive answer to two questions asked by professor Yilmaz Simsek in a recent paper [**6**] concerning special numbers $B(n,k)$ for computing negative order Euler numbers.

1. INTRODUCTION

Let k a positive integer and $\lambda \in \mathbb{C}$. The Simsek numbers $y_1(n,k,\lambda)$ are defined by the following generating function (see [**6**])

$$F_{y_1}(t,k;\lambda) = \frac{1}{k!}\left(\lambda e^t + 1\right)^k = \sum_{n=0}^{\infty} y_1(n,k,\lambda)\frac{t^n}{n!}. \tag{1}$$

Explicitly from [**6**, Theorem 1] we get

$$y_1(n,k,\lambda) = \frac{1}{k!}\sum_{j=0}^{k}\binom{k}{j} j^n \lambda^j. \tag{2}$$

and then $y_1(n,k,\lambda)$ is a polynomial on variable λ of degree k. In the case $\lambda = 1$, let the numbers

$$B(n,k) = k! y_1(n,k,1) \tag{3}$$

2010 Mathematics Subject Classification. 11B83, 12D10, 11B68, 11S40, 11S80, 26C05, 26C10.
Keywords and Phrases. Euler numbers; Simsek numbers; generating functions.

and from the identity (2) we deduce that

$$B(n,k)=\sum_{j=0}^{k}\binom{k}{j}j^{n}. \tag{4}$$

Golombek proved in [**3**] that

$$B(n,k)=\frac{d^{n}}{dt^{n}}\left(e^{t}+1\right)^{k}|_{t=0} \tag{5}$$

and the same thinks for Simsek numbers

$$y_{1}\left(n,k,\lambda\right)=\frac{1}{k!}\frac{d^{n}}{dt^{n}}\left(\lambda e^{t}+1\right)^{k}|_{t=0} \tag{6}$$

Interesting computation formulae and connections relations to other combinatorial numbers such as the Stirling numbers and special numbers including the Apostol-type number are developed in the papers [**7**] and [**8**].

Only in the work [**6**] Y. Simsek conjectured that

$$B(n,k)=\left(k^{n}+x_{1}k^{n-1}+\cdots+x_{n-2}k^{2}+x_{n-1}k\right)2^{k-n}$$

where $x_1,\cdots,x_{n-1}$ are integers and n is a positive integer. Consequently, he arrives at the following open questions:
1)- How can we compute the coefficients $x_1,\cdots,x_{n-1}$.
2)- We assume that for $|x|<r$

$$\sum_{k=1}^{\infty}B(n,k)x^{k}=f_{n}\left(x\right)$$

Is it possible to find $f_n\left(x\right)$?

In this paper we extend the problem to Simsek numbers in order to prove a more general Simsek conjecture and construct the appropriate generating functions. Consequently we give a positive answer to the last questions with proof different of that given by Xu in [**10**].

We end this work by the connection between numbers $B(n,k;\lambda)=k!y_1\left(n,k,\lambda\right)$ and the first kind Apostol-Euler numbers $E_n^{(k)}(\lambda)$ of order k (cf. [**2**],[**5**]) defined by means of the following generating function:

$$\left(\frac{2}{\lambda e^{t}+1}\right)^{k}=\sum_{n\geq 0}E_{n}^{(k)}(\lambda)\frac{t^{n}}{n!},\ |t|<\pi. \tag{7}$$

Substituting $\lambda=k=1$ into (7) we obtain the first kind Euler numbers $E_n=E_n^{(1)}(1)$, which are defined by means of the following generating function (see [**9**] and references therein):

$$\frac{2}{e^t+1} = \sum_{n\geq 0} E_n \frac{t^n}{n!}, \ |t| < \pi. \tag{8}$$

Finally we prove the following interesting identity concerning Euler numbers.

$$2E_n + 1 = -\sum_{j=1}^{n-1} \binom{n}{j} E_{n-j}$$

2. EXPLICIT FORMULA OF SIMSEK NUMBERS

From the definition of $F_{y_1}(t, k; \lambda)$, we have in one hand

$$F'_{y_1}(t, k; \lambda) = \frac{\lambda e^t}{(k-1)!} \left(\lambda e^t + 1\right)^{k-1} = \lambda e^t F_{y_1}(t, k-1; \lambda).$$

Using Leibnitz formula (see [**4**]) for computing derivative at any order of product of two functions we conclude that

$$\frac{\partial^n}{\partial t^n} F_{y_1}(t, k; \lambda) = \lambda e^t \sum_{i=0}^{n-1} \binom{n-1}{i} \frac{\partial^i}{\partial t^i} F_{y_1}(t, k-1; \lambda). \tag{9}$$

This formula conducts to following new identity about Simsek numbers

$$y_1(n, k, \lambda) = \lambda \sum_{i=0}^{n-1} \binom{n-1}{i} y_1(i, k-1, \lambda). \tag{10}$$

In another hand we have

$$F'_{y_1}(t, k; \lambda) = kF_{y_1}(t, k; \lambda) - F_{y_1}(t, k-1; \lambda). \tag{11}$$

Then the successive derivatives are given by the recursion formula:

$$\frac{\partial^n}{\partial t^n} F_{y_1}(t, k; \lambda) = k \frac{\partial^{n-1}}{\partial t^{n-1}} F_{y_1}(t, k; \lambda) - \frac{\partial^{n-1}}{\partial t^{n-1}} F_{y_1}(t, k-1; \lambda). \tag{12}$$

Substitute $t = 0$ in the identity (12) we deduce the Theorem 6 in [**6**]

$$y_1(n, k, \lambda) = ky_1(n-1, k, \lambda) - y_1(n-1, k-1, \lambda). \tag{13}$$

Combining identities (10) and (13) we obtain

$$y_1(n, k, \lambda) = \lambda \sum_{i=0}^{n-1} \binom{n-1}{i} \left[(k-1)y_1(i-1, k-1, \lambda) - y_1(i-1, k-2, \lambda)\right].$$

Regarding the identity (13), one can conclude that $y_1(n, k, \lambda)$ is a polynomial on the variable k of degree n in the ring $\mathbb{C}[k]$. The following theorem provide the confirmation.

Theorem 2.1. *For any positive integers k, n and complex number λ we have*

$$y_1(n,k,\lambda) = \sum_{j=1}^{n} a_{n,k}(\lambda, j)k^j \tag{14}$$

where the coefficients $a_{n,k}(\lambda, j)$ are defined by

$$a_{n,k}(\lambda,1) = 0 \text{ if } n > k,\ a_{n,k}(\lambda,1) = (-1)^n \frac{(\lambda+1)^{k-n}}{(k-n)!} \text{ if } n \le k$$

and

$$a_{n,k}(\lambda, n) = \frac{(1+\lambda)^k}{k!}.$$

For others the recursive formulae

$$a_{n,k}(\lambda, j) = a_{n-1,k}(\lambda, j-1) - a_{n-1,k-1}(\lambda, j),\ 2 \le j \le n-1. \tag{15}$$

Proof. The proof is by recursion; first we have

$$y_1(0,k,\lambda) = \frac{(\lambda+1)^k}{k!},\ y_1(1,k,\lambda) = \frac{\lambda(\lambda+1)^{k-1}}{(k-1)!}.$$

And we suppose for every positive integer k that

$$y_1(n,k,\lambda) = \sum_{j=1}^{n} a_{n,k}(\lambda, j)k^j$$

Using the identity (13) we obtain

$$y_1(n+1,k,\lambda) = k\sum_{j=1}^{n} a_{n,k}(\lambda, j)k^j - \sum_{j=1}^{n} a_{n,k-1}(\lambda, j)k^j.$$

Furthermore

$$y_1(n+1,k,\lambda) = \sum_{j=2}^{n+1} a_{n,k}(\lambda, j-1)k^j - \sum_{j=1}^{n} a_{n,k-1}(\lambda, j)k^j$$

and

$$y_1(n+1,k,\lambda) = -a_{n,k-1}(\lambda,1) + \sum_{j=2}^{n}[a_{n,k}(\lambda, j-1) - a_{n,k-1}(\lambda, j)]k^j + a_{n,k}(\lambda, n)k^{n+1}.$$

We deduce that

$$y_1(n+1,k,\lambda) = \sum_{j=1}^{n+1} a_{n+1,k}(\lambda, j)k^j$$

where

$$a_{n+1,k}(\lambda,1) = -a_{n,k-1}(\lambda,1),\ a_{n+1,k}(\lambda,n+1) = a_{n,k}(\lambda,n)$$

and

$$a_{n+1,k}(\lambda,j) = a_{n,k}(\lambda,j-1) - a_{n,k-1}(\lambda,j) \text{ for } 2 \leq j \leq n.$$

But we remark that

$$a_{n+1,k}(\lambda,n+1) = a_{n,k}(\lambda,n) = \cdots = a_{0,k}(\lambda,0) = \frac{(1+\lambda)^k}{k!}$$

and $a_{n,k}(\lambda,1) = -a_{n-1,k-1}(\lambda,1)$. Thereafter for $n > k$,

$$a_{n,k}(\lambda,1) = -a_{n-1,k-1}(\lambda,1) = (-1)^k a_{n-k,0}(\lambda,1) = 0$$

and for $n \leq k$;

$$a_{n,k}(\lambda,1) = -a_{n-1,k-1}(\lambda,1) = (-1)^n a_{0,k-n}(\lambda,1) = (-1)^n \frac{(\lambda+1)^{k-n}}{(k-n)!}$$

Furthermore the Theorem (2.1) follows. □

Remark 2.1. *If $\lambda \in \mathbb{N}$, it follows from the identity (2) that $k!y_1(n,k,\lambda) \in \mathbb{N}$ and we can easily prove that $k!a_{n,k}(\lambda,j) \in \mathbb{N}$.*

For $\lambda = 1$, let us denoting $a_{n+1,k}(j) = a_{n+1,k}(\lambda,j)$. As a direct consequence of Theorem 2.1 we get the following corollary and the proof is left as an exercise.

Corollary 2.1. *For any positive integers k, n we have*

$$B(n,k) = k! \sum_{j=1}^{n} a_{n,k}(j) k^j \tag{16}$$

where the coefficients $a_{n,k}(j)$ are defined by

$$a_{n,k}(1) = 0 \text{ if } n > k,\ a_{n,k}(1) = (-1)^n \frac{2^{k-n}}{(k-n)!} \text{ if } n \leq k \text{ and } a_{n,k}(n) = \frac{2^k}{k!}.$$

For others; the recursive formulae

$$a_{n,k}(j) = a_{n-1,k}(j-1) - a_{n-1,k-1}(j),\ 2 \leq j \leq n-1. \tag{17}$$

In the following theorem we prove a new formula for the coefficients $a_{n,k}(j)$ and explicit formula for Simsek numbers.

Theorem 2.2.

$$y_1(n,k,\lambda) = (\lambda+1)^{k-n} \sum_{j=1}^{n} x_j(\lambda,n,k) k^j \tag{18}$$

where the coefficients $x_j(\lambda, n, k)$ *are defined by*

$$x_1(\lambda, n, k) = 0 \textit{ if } n > k,\ x_1(\lambda, n, k) = \frac{(-1)^n}{(k-n)!} \textit{ if } n \geq k \textit{ and } x_n(\lambda, n, k) = \frac{(1+\lambda)^n}{k!}.$$

For others; the recursive formula

$$x_j(\lambda, n, k) = (\lambda+1)\, x_{j-1}(\lambda, n-1, k) - x_j(\lambda, n-1, k-1),\ 2 \leq j \leq n-1.$$

Proof. The proof is by recursion, we have $a_{n,k}(\lambda, 1)$ is 0 or $(1+\lambda)^{k-n}\frac{(-1)^n}{(k-n)!}$ and

$$a_{n,k}(\lambda, n) = (1+\lambda)^{k-n}\frac{(1+\lambda)^n}{k!}$$

. Suppose for any positive integer k that

$$a_{n,k}(\lambda, j) = (\lambda+1)^{k-n}\, x_j(\lambda, n, k)$$

then

$$x_1(\lambda, n+1, k) = 0 \text{ if } n+1 > k \text{ and } x_1(\lambda, n+1, k) = \frac{(-1)^{n+1}}{(k-n-1)!} \text{ if } n+1 \geq k,$$

$$x_{n+1}(\lambda, n+1, k) = \frac{(1+\lambda)^{n+1}}{k!}$$

and for $2 \leq j \leq n$

$$\begin{aligned} a_{n+1,k}(\lambda, j) &= a_{n,k}(\lambda, j-1) - a_{n,k-1}(\lambda, j) \\ &= (\lambda+1)^{k-n}\, x_{j-1}(\lambda, n, k) - (\lambda+1)^{k-1-n}\, x_j(\lambda, n, k-1) \\ &= (\lambda+1)^{k-n-1}\left[(\lambda+1)\, x_{j-1}(\lambda, n, k) - x_j(\lambda, n, k-1)\right]. \end{aligned}$$

And then

$$a_{n+1,k}(\lambda, j) = (\lambda+1)^{k-n-1}\, x_j(\lambda, n+1, k)$$

with

$$x_j(\lambda, n+1, k) = (\lambda+1)\, x_{j-1}(\lambda, n, k) - x_j(\lambda, n, k-1), 2 \leq j \leq n.$$

□

Let us denoting $x_j(n, k) = x_j(1, n, k)$. The answer to Simsek conjecture can be found in the following corollary.

Corollary 2.2.

$$B(n, k) = 2^{k-n} k! \sum_{j=1}^{n} x_j(n, k) k^j \tag{19}$$

where the coefficients $x_j(n,k)$ *are defined by*

$$x_1(n,k) = 0 \text{ if } n > k, x_1(n,k) = \frac{(-1)^n}{(k-n)!} \text{ if } n \geq k \quad \text{and } x_n(n,k) = \frac{2^n}{k!}.$$

For others; the recursive formula

$$x_j(n,k) = 2x_{j-1}(n-1,k) - x_j(n-1,k-1), \; 1 \leq j \leq n-1$$

Proof. The proof is left as an exercise for the reader. □

Taking $x_j = k!x_j(n,k)$ we deduce that

$$B\,(n,k) = 2^{k-n} \sum_{j=0}^{n} x_j k^j$$

which provide that the Simsek conjecture is true. One remarks that $x_1 = 0$ and $x_0 = (-1)^n (k)_n$, where $(k)_n = k(k-1)\cdots(k-n+1)$. For more details about these numbers we refer to [**1**]. Others are computed from the recursion formula in Corollary 2.2.

3. GENERATING FUNCTIONS OF SIMSEK NUMBERS GENERATING FUNCTION

In the literature, only we are asked to compute generating functions for numbers or polynomials [**2**]. In this work we introduce the notion of generating functions for functions.

Definition 3.1. *Let for every positive integer* j *the function* $f_j(t)$ *defined on* $\mathbb{R}$. *We say that the family* f_j *admits a generating function in the interval* $I \subset \mathbb{R}$ *if and only if their exists a function* $F(t,x)$ *such that*

$$F(t,x) = \sum_{j \geq 0} f_j(t) x^j. \tag{20}$$

Example 3.1. *Let* $f(t)$ *a function on* $\mathbb{R}$ *such that* $|f(t)| \geq \theta > 0$ *for* $t \in I \subset \mathbb{R}$. *Then the generating function of the family* $f^j(t)$ *is* $\frac{1}{1-xf(t)}$. *More precisely we get*

$$\frac{1}{1-xf(t)} = \sum_{j \geq 0} f^j(t) x^j, \; |x| < \frac{1}{\theta} \text{ and } t \in I. \tag{21}$$

In those conditions

$$\frac{\partial^n}{\partial t^n} \frac{1}{1-xf(t)} = \sum_{j \geq 0} \left(\frac{d^n}{dt^n} f^j(t) \right) x^j, \; |x| < \frac{1}{\theta}. \tag{22}$$

and then

$$\frac{\partial^n}{\partial t^n}\frac{1}{1-xf(t)}|_{t=0}=\sum_{j\geq 0}\left(\frac{d^n}{dt^n}f^j(t)|_{t=0}\right)x^j,\ |x|<\frac{1}{\theta}. \tag{23}$$

is the generating function of the numbers $\frac{d^n}{dt^n}f^j(t)|_{t=0}$.

Now let the numbers $B(n,k;\lambda)=k!y_1(n,k,\lambda)$, then their generating function is given by the following theorem

Theorem 3.3. *Let $\lambda\neq -1$ and $t\geq 0$, the generating function of $B(n,k;\lambda)$ is*

$$f_n(\lambda,x)=\frac{\partial^n}{\partial t^n}\frac{1}{1-(\lambda e^t+1)\,x}|_{t=0}$$

i.e

$$f_n(\lambda,x)=\sum_{k\geq 0}B(n,k;\lambda)x^k,\ |x|<\frac{1}{|\lambda+1|}. \tag{24}$$

Proof. First we compute the generating function of the functions $\left(\lambda e^t+1\right)^k$;

$$\sum_{k\geq 0}\left(\lambda e^t+1\right)^k x^k=\frac{1}{1-(\lambda e^t+1)\,x}$$

if and only if $|\,(\lambda e^t+1)\,x|<1$. For $t\geq 0$ and $\lambda+1\neq 0$ which means that the condition of the convergence is $|x|<\frac{1}{|\lambda+1|}$.
We remark for $|x|\leq\rho<\frac{1}{|\lambda+1|}$ that the above series in normalcy convergent and we get

$$\frac{\partial^n}{\partial t^n}\sum_{k\geq 0}\left(\lambda e^t+1\right)^k x^k=\sum_{k\geq 0}x^k\frac{d^n}{t^n}\left(\lambda e^t+1\right)^k$$

then

$$\frac{\partial^n}{\partial t^n}\frac{1}{1-(\lambda e^t+1)\,x}=\sum_{k\geq 0}x^k\frac{d^n}{dt^n}\left(\lambda e^t+1\right)^k$$

Furthermore

$$\frac{\partial^n}{\partial t^n}\frac{1}{1-(\lambda e^t+1)\,x}|_{t=0}=\sum_{k\geq 0}\frac{d^n}{dt^n}\left(\lambda e^t+1\right)^k|_{t=0}x^k$$

but $B(n,k;\lambda)=\frac{d^n}{t^n}\left(\lambda e^t+1\right)^k|_{t=0}$ then the result (24) follows. □

Let $f_n(x)=f_n(1,x)$, the answer to second question of Y. Simsek is immediate

Corollary 3.3. *For $n\geq 1$ we have $B(n,0)=0$ and*

$$f_n(x)=\frac{\partial^n}{\partial t^n}\frac{1}{1-(e^t+1)\,x}|_{t=0}=\sum_{k\geq 1}B(n,k)x^k,\ |x|<\frac{1}{2}. \tag{25}$$

Proof. The proof is left as an exercise. □

The first successive generating functions are

$$f_0(x) = \frac{1}{1-2x}$$

,

$$f_1(x) = \frac{\partial}{\partial t}\frac{1}{1-(e^t+1)\,x}|_{t=0} = \frac{xe^t}{(1-(e^t+1)\,x)^2}|_{t=0} = \frac{x}{(1-2x)^2}$$

and

$$f_2(x) = \frac{\partial^2}{\partial t^2}\frac{1}{1-(e^t+1)\,x}|_{t=0} = \frac{\partial}{\partial t}\frac{xe^t}{(1-(e^t+1)\,x)^2}|_{t=0} = \frac{xe^t\,(1-(e^t+1)\,x)+2x^2e^{2t}}{(1-(e^t+1)\,x)^3}|_{t=0} = \frac{x}{(1-2x)^3}.$$

From the two generating functions and for a fixed integer n we remark that the generating function of $B(n,k)$ is of the form

$$f_n(x) = \frac{P_n(x)}{(1-2x)^{n+1}}$$

where $P_n(x)$ is a polynomial in $\mathbb{Z}[x]$ of degree at most n. This formula leads us to ask the following open question: How can we compute the polynomials $P_n(x)$.

Here we give a partial answer to this problem by explaining the method for computing successive derivatives of $f_0(x)$. Taking $f_0(x,t) = \frac{1}{1-(e^t+1)x}$ and $g(x,t) = 1-x-xe^t$. By using some techniques illustrated in the work [**4**] we deduce for $n \geq 1$ that

$$\sum_{j=0}^{n}\binom{n}{j}\frac{\partial^j}{\partial t^j}f_0(x,t)\frac{\partial^{n-j}}{\partial t^{n-j}}g(x,t) = 0$$

and then

$$\left(1-x-xe^t\right)\frac{\partial^n}{\partial t^n}f_0(x,t) = -xe^t\sum_{j=0}^{n-1}\binom{n}{j}\frac{\partial^j}{\partial t^j}f_0(x,t)$$

Finally

$$f_n(x) = -xf_0(x)\sum_{j=0}^{n-1}\binom{n}{j}f_{n-j}(x).$$

4. CONNECTION TO FIRST KIND APOSTOL-EULER NUMBERS

The following theorem explain how the numbers $B(n,k,\lambda)$ are connected to first kind Apostol-Euler numbers $E_n^{(k)}(\lambda)$ and their importance to obtain some statements about Euler numbers.

Theorem 4.4.

$$B(n,k,\lambda) = -\left(\frac{\lambda+1}{2}\right)^k \sum_{j=0}^{n-1} \binom{n}{j} B(j,k,\lambda) E_{n-j}^{(k)}(\lambda). \tag{26}$$

Furthermore for $n \geq 1$

$$2E_n + 1 = -\sum_{j=1}^{n-1} \binom{n}{j} E_{n-j}. \tag{27}$$

Proof. In one hand we have

$$\left(\lambda e^t + 1\right)^k \left(\frac{2}{\lambda e^t + 1}\right)^k = 2^k$$

and in another hand

$$\left(\lambda e^t + 1\right)^k \left(\frac{2}{\lambda e^t + 1}\right)^k = \left(\sum_{n\geq 0} B(n,k;\lambda) \frac{t^n}{n!}\right) \left(\sum_{j\geq 0} E_j^{(k)}(\lambda) \frac{t^j}{j!}\right).$$

From the well-known Cauchy product of two generating function [**4**] we deduce that

$$2^k = \sum_{n\geq 0} \left(\sum_{j=0}^{n} \binom{n}{j} B(j,k,\lambda) E_{n-j}^{(k)}(\lambda)\right) \frac{t^n}{n!}$$

hence $B(0,k,\lambda)E_0^{(k)}(\lambda) = 2^k$ and

$$\sum_{j=0}^{n} \binom{n}{j} B(j,k,\lambda) E_{n-j}^{(k)}(\lambda) = 0$$

which means that

$$B(n,k,\lambda) = -\frac{1}{E_0^{(k)}(\lambda)} \sum_{j=0}^{n-1} \binom{n}{j} B(j,k,\lambda) E_{n-j}^{(k)}(\lambda)$$

and

$$B(n,k,\lambda) = -\left(\frac{\lambda+1}{2}\right)^k \sum_{j=0}^{n-1} \binom{n}{j} B(j,k,\lambda) E_{n-j}^{(k)}(\lambda).$$

For $\lambda = 1$ we conclude that

$$B(n,k) = -\sum_{j=0}^{n-1} \binom{n}{j} B(j,k) E_{n-j}^{(k)}(1)$$

furthermore if $k = 1$ we obtain

$$B(n,1) = -\sum_{j=0}^{n-1} \binom{n}{j} B(j,1) E_{n-j}. \tag{28}$$

Since

$$e^t + 1 = \sum_{n \geq 0} B(n,1) \frac{t^n}{n!}$$

and comparing with the identity

$$e^t + 1 = 2 + \sum_{n \geq 1} \frac{t^n}{n!}$$

we deduce that $B(0,1) = 2$ and $B(n,1) = 1$ for $n \geq 1$. Returning back to the identity (28) we obtain for $n \geq 1$

$$2E_n + 1 = -\sum_{j=1}^{n-1} \binom{n}{j} E_{n-j}.$$

□

REFERENCES

1. M. Bona:*Introduction to Enumerative Combinatorics.* The McGraw-Hill Companies, Inc. New York, 2007.
2. G. B. Djordjevic, G. V. Milovanovic: *Special classes of polynomials.* University of Nis. Faculty of Technology Leskovac, 2014.
3. R. Golombek:: *Aufgabe 1088.* El. Math.,**49** (1994), 126–127.
4. M. Goubi: *Successive Derivatives of Fibonacci Type Polynomials of Higher Order in Two Variables.* Filomat, 32:14 (2018), 5149–5159
5. D. S. Kim, T. Kim: *Daehee numbers and polynomials.* Appl. Math. Sci., **7** (2013), 5969–5976.
6. Y. Simsek: *New families of special numbers for computing negative order Euler numbers and related numbers and polynomials.* Appl. Anal. Discrete Math., **12** (2018), 001–035.
7. Y. Simsek: *Computation methods for combinatorial sums and Euler type numbers related to new families of numbers.* Math. Meth. Appl. Sci., **40** (2017), 2347–2361.
8. Simsek, Y.: *Generating functions for finite sums involving higher powers of binomial coefficients: Analysis of hypergeometric functions including new families of polynomials and numbers.* J. Math. Anal. Appl., **477** (2019) 1328-1352
9. C. F. Wei, F. Qi: *Several closed expressions for the Euler Numbers.* J. Inequal. Appl., **219** (2015), 1–8.

10. A. Xu: *On an open problem of Simsek concerning the computation of a family of special numbers.* Appl. Anal. Discrete Math., **13** (1) (2019) 61–72.

Mouloud Goubi
Department of Mathematics
Faculty of Sciences
UMMTO University Algeria
E-mail:*mouloud.goubi@ummto.dz*

(Received 16.01.2019)
(Revised 13.02.2020)