

## GROUP INVERSE MATRIX OF THE NORMALIZED LAPLACIAN ON SUBDIVISION NETWORKS

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In this paper we consider a subdivision of a given network and we show how the group inverse matrix of the normalized laplacian of the subdivision network is related to the group inverse matrix of the normalized laplacian of the initial given network. Our approach establishes a relationship between solutions of related Poisson problems on both structures and takes advantage on the properties of the group inverse matrix. As a consequence we get formulae for effective resistances and the Kirchhoff Index of the subdivision network expressed in terms of its corresponding in the base network. Finally, we study two examples where the base network are the star and the wheel, respectively.

### 1. PRELIMINARIES

In recent years many papers are devoted to the study of different parameters of composite graphs and operations on graphs, for instance the subdivision graph operation. Parameters such that the effective resistances and Kirchhoff index of the subdivision graph have been considered in different works under several hypothesis such as regular graphs in [6, 11], bipartite graphs in [12], or operations between graphs that involve the subdivision concept as well, see [3] for instance.

In [14], the author extends the previous results to general graphs and computes the Kirchhoff index of subdivision graphs in terms of the Kirchhoff index, the multiplicative degree–Kirchhoff index, the additive degree–Kirchhoff index, the

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number of vertices, and the number of edges of a graph. Simultaneously, Sun *et al.* in [12], gave the formulae for the Kirchhoff index in terms of a  $\{1\}$ -inverse of the combinatorial Laplacian. In [5], the authors extend the subdivision concept to the case of networks, in such a way that electrical compatibility formulae are satisfied and we compute the effective resistances and the Kirchhoff index with respect to the combinatorial laplacian of the network, which allows us to obtain some generalization of known results.

All the cited works, except the last one, compute the parameters associated with the combinatorial Laplacian for graphs; that is, without taking into account conductances. In this work, we consider the so-called *normalized Laplacian* introduced in [9]. The increasing interest on the normalized Laplacian comes from its probabilistic interpretation that allows to know many measures for random walks, see for instance [7, 8, 10, 13]. In particular, in [13] the eigenvalues of iterated subdivision graphs were obtained.

In the present paper, we consider the subdivision of a network by interpreting it as an electric circuit, and hence each edge has got assigned a positive number that corresponds with the conductance of a wire connecting two nodes, its inverse is the resistance. Then, we decompose each edge into two new edges taking into account electrical compatibility of the circuit, specifically, the series sum rule for resistances. As a consequence, we would get that after the subdivision process, the effective resistance between any pair of old vertices should remain unchanged.

Main results are obtained in Section 2, where an expression for the Green function of the normalized laplacian for the subdivision network is given in terms of the Green function of the base network and some other known parameters. Then using the relation between effective resistances and Green functions we obtain the formula for these parameters and also the corresponding Kirchhoff index. In the last section we consider two examples that show the efficiency of the method.

In this paper  $\Gamma = (V, E, c)$  denotes a simple *network*; that is, a finite, with no loops, nor multiple edges, connected graph. Let  $n$  be the *order* of the network, that is, the number of vertices in  $V$ , and let  $m$  be the *size* of the network, thus the number of edges in  $E$ . We call *conductance* to the symmetric function  $c : V \times V \rightarrow [0, +\infty)$  satisfying  $c(x, y) > 0$  iff  $x \sim y$ , which means that  $\{x, y\} \in E$  is an edge in  $\Gamma$ . For every vertex in  $V$ , let  $k(x) = \sum_{y \in V} c(x, y)$  be the *degree* of vertex  $x$ ; then the *volume* of  $\Gamma$  is  $\text{vol}(\Gamma) = \sum_{x \in V} k(x)$ .

Let  $\mathcal{C}(V)$  be the set of real functions on  $V$ . The standard inner product on  $\mathcal{C}(V)$  is denoted by  $\langle \cdot, \cdot \rangle$  and hence if  $u, v \in \mathcal{C}(V)$  then  $\langle u, v \rangle = \sum_{x \in V} u(x)v(x)$ . For any  $x \in V$ ,  $\varepsilon_x \in \mathcal{C}(V)$  is the Dirac function or characteristic function of the set  $\{x\}$ , with the only non-null value that it takes is one at  $x$ . The *normalized laplacian* of  $\Gamma$ , introduced by Chung and Langlands in [9], is the linear operator  $\mathcal{L} : \mathcal{C}(V) \rightarrow \mathcal{C}(V)$

that assigns to each  $u \in \mathcal{C}(V)$ , the real function  $\mathcal{L}(u)$  defined at every  $x \in V$  by

$$\mathcal{L}(u)(x) = \frac{1}{\sqrt{k(x)}} \sum_{y \in V} c(x, y) \left( \frac{u(x)}{\sqrt{k(x)}} - \frac{u(y)}{\sqrt{k(y)}} \right).$$

Easily  $\ker(\mathcal{L}) = \text{span}(\sqrt{k})$  and given  $f \in \mathcal{C}(V)$ , the Poisson problem  $\mathcal{L}(u)(x) = f(x)$  is compatible iff  $\langle f, \sqrt{k} \rangle = 0$ . In this case, two different solutions differ up to a multiple of  $\sqrt{k}$ , so there exists a unique solution orthogonal to  $\sqrt{k}$  for every compatible linear system  $\mathcal{L}(u) = f$  (Fredholm’s alternative).

The *Green function of  $\Gamma$*  is the kernel  $G : V \times V \rightarrow \mathbb{R}$  such that for each  $y \in V$ ,  $G(\cdot, y)$  is the unique orthogonal to  $\sqrt{k}$  solution of the Poisson equation  $\mathcal{L}u = \varepsilon_y - \frac{\sqrt{k(y)}}{\text{vol}(\Gamma)}\sqrt{k}$ . Related to it, the *Green operator of  $\Gamma$*  is the linear operator  $\mathcal{G} : \mathcal{C}(V) \rightarrow \mathcal{C}(V)$  defined on every  $f \in \mathcal{C}(V)$  as

$$\mathcal{G}(f)(x) = \sum_{y \in V} G(x, y)f(y), \quad x \in V.$$

Once a labelling of  $V$  is considered, linear operators can be identified with matrices and functions can be identified with  $n$  dimensional vectors. Then, the matrix corresponding to  $\mathcal{L}$ , will be denoted by  $L$  and, the matrix corresponding to  $\mathcal{G}$ , which is the *group inverse matrix* of  $L$ , will be denoted by  $G$ .

Given  $x, y \in V$ , the *effective resistance between  $x$  and  $y$* , is the value

$$(1) \quad R(x, y) = \frac{G(x, x)}{k(x)} + \frac{G(y, y)}{k(y)} - 2 \frac{G(x, y)}{\sqrt{k(x)}\sqrt{k(y)}},$$

see [1, Proposition 4.3]. Moreover, the *Kirchhoff index* of  $\Gamma$  is defined in [2] as

$$k(\Gamma) = \text{vol}(\Gamma) \sum_{x \in V} G(x, x) = \frac{1}{2} \sum_{x, y \in V} R(x, y)k(x)k(y).$$

The *subdivision network*  $\Gamma^S = (V^S, E^S, c^S)$  of  $\Gamma$  is obtained from it by inserting a new vertex in every edge, so that each  $\{x, y\} \in E$  is replaced by two new edges, say  $\{x, v_{xy}\}$  and  $\{v_{xy}, y\}$  where  $v_{xy}$  is the new inserted vertex. We denote by  $V'$  the set of new generated vertices assuming that  $v_{xy} = v_{yx}$ . Thus,  $V^S = V \cup V'$  and the order of the subdivision network is  $n + m$  while its size is  $2m$ . As we are interested in keeping electrical compatibility in the new network, we define conductances as  $c^S(x, v_{xy}) = c^S(y, v_{xy}) = 2c(x, y)$ , so that series rule for resistors is fulfilled

$$\frac{1}{c(x, y)} = \frac{1}{c^S(x, v_{xy})} + \frac{1}{c^S(y, v_{xy})}.$$

To our knowledge, the most common is to assume that the value of conductances on the subdivision structure is one and hence, the electrical compatibility condition fails.

The degree function on  $\Gamma^S$ ,  $k^S \in \mathcal{C}(V^S)$ , satisfies  $k^S(x) = 2k(x)$  for any  $x \in V$ , and  $k^S(v_{xy}) = 4c(x, y)$  for those vertices in  $V'$ . Moreover, it holds that  $\text{vol}(\Gamma^S) = 4\text{vol}(\Gamma)$ .

## 2. THE GROUP INVERSE OF THE NORMALIZED LAPLACIAN FOR THE SUBDIVISION NETWORK

The first main result we present in this section sets the precise relation between the solution of a compatible Poisson problem for the normalized laplacian on a subdivision network  $\Gamma^S$  and the solution of a conveniently well posed Poisson problem for the normalized laplacian on the base network  $\Gamma$ .

With the aim of usefulness we consider two operators:

(a) Let  $h \in \mathcal{C}(V^S)$  its *contraction* to  $\mathcal{C}(V)$  is

$$\underline{h}(x) = h(x) + \frac{1}{\sqrt{2k(x)}} \sum_{y \sim x} \sqrt{c(x, y)} h(v_{xy}).$$

(b) For any  $u \in \mathcal{C}(V)$  its *extension* to  $\mathcal{C}(V^S)$ , and related to  $h \in \mathcal{C}(V^S)$  is

$$u^h(v_{xy}) = h(v_{xy}) + \frac{\sqrt{c(x, y)}}{\sqrt{2}} \left( \frac{u(x)}{\sqrt{k(x)}} + \frac{u(y)}{\sqrt{k(y)}} \right)$$

for  $v_{xy} \in V'$ , while  $u^h(x) = u(x)$  for those vertices in  $V$ .

The following result links the solution of a Poisson problem in the subdivision network with an appropriate Poisson problem on the base network. From now on  $\mathcal{L}_S$  denotes the normalized laplacian of the subdivision network  $\Gamma^S$ .

**Theorem 1.** *Given  $h \in \mathcal{C}(V^S)$  such that  $\langle h, \sqrt{k^S} \rangle_{V^S} = 0$ , then  $\langle \underline{h}, \sqrt{k} \rangle_V = 0$ . Moreover,  $\bar{u} \in \mathcal{C}(V^S)$  is a solution of the Poisson equation  $\mathcal{L}_S(\bar{u}) = h$  in  $V^S$  iff  $u = \bar{u}|_V$  is a solution of the Poisson equation  $\mathcal{L}(u) = 2\underline{h}$  in  $V$ . In this case, the identity  $\bar{u} = u^h$  holds.*

*Proof.* Firstly we note that  $\langle \underline{h}, \sqrt{k} \rangle_V = \frac{1}{\sqrt{2}} \langle h, \sqrt{k^S} \rangle_{V^S}$  as

$$\begin{aligned} \sum_{x \in V} \underline{h}(x) \sqrt{k(x)} &= \sum_{x \in V} h(x) \sqrt{\frac{k^S(x)}{2}} + \frac{1}{\sqrt{2}} \sum_{x \in V} \frac{1}{\sqrt{k(x)}} \sum_{y \sim x} \sqrt{c(x, y)} h(v_{xy}) \sqrt{k(x)} \\ &= \frac{1}{\sqrt{2}} \sum_{x \in V} h(x) \sqrt{k^S(x)} + \frac{1}{\sqrt{2}} \sum_{v_{xy} \in V'} h(v_{xy}) \sqrt{k^S(v_{xy})}. \end{aligned}$$

So the first statement holds. Then,

$$\begin{aligned} \mathcal{L}_s(\bar{u})(v_{xy}) &= \bar{u}(v_{xy}) - \frac{c^s(v_{xy}, x)}{\sqrt{k^s(v_{xy})}\sqrt{k^s(x)}}\bar{u}(x) - \frac{c^s(v_{xy}, y)}{\sqrt{k^s(v_{xy})}\sqrt{k^s(y)}}\bar{u}(y) \\ &= \bar{u}(v_{xy}) - \frac{\sqrt{c(x, y)}}{\sqrt{2}} \frac{\bar{u}(x)}{\sqrt{k(x)}} - \frac{\sqrt{c(x, y)}}{\sqrt{2}} \frac{\bar{u}(y)}{\sqrt{k(y)}}. \end{aligned}$$

So we obtain that,

$$\bar{u}(v_{xy}) = \mathcal{L}_s(\bar{u})(v_{xy}) + \frac{\sqrt{c(x, y)}}{\sqrt{2}} \left( \frac{\bar{u}(x)}{\sqrt{k(x)}} + \frac{\bar{u}(y)}{\sqrt{k(y)}} \right).$$

Also, for the former vertex in the given network,

$$\mathcal{L}_s(\bar{u})(x) = \frac{1}{\sqrt{k^s(x)}} \sum_{y \sim x} c^s(x, v_{xy}) \left[ \frac{\bar{u}(x)}{\sqrt{k^s(x)}} - \frac{\bar{u}(v_{xy})}{\sqrt{k^s(v_{xy})}} \right].$$

Substituting the precedent expression for  $\bar{u}(v_{xy})$  we obtain

$$\begin{aligned} \mathcal{L}_s(\bar{u})(x) &= \frac{1}{2\sqrt{k(x)}} \sum_{y \sim x} c(x, y) \left[ \frac{2\bar{u}(x)}{\sqrt{k(x)}} - \frac{\sqrt{2}\mathcal{L}_s(\bar{u})(v_{xy})}{\sqrt{c(x, y)}} - \frac{\bar{u}(x)}{\sqrt{k(x)}} - \frac{\bar{u}(y)}{\sqrt{k(y)}} \right] \\ &= \frac{1}{2\sqrt{k(x)}} \sum_{y \sim x} c(x, y) \left( \frac{\bar{u}(x)}{\sqrt{k(x)}} - \frac{\bar{u}(y)}{\sqrt{k(y)}} \right) - \frac{1}{\sqrt{2k(x)}} \sum_{y \sim x} \sqrt{c(x, y)} \mathcal{L}_s(\bar{u})(v_{xy}). \end{aligned}$$

Finally, we get, if  $u = \bar{u}|_V$

$$\mathcal{L}_s(\bar{u})(x) = \frac{1}{2} \mathcal{L}(u)(x) - \frac{1}{\sqrt{2k(x)}} \sum_{y \sim x} \sqrt{c(x, y)} \mathcal{L}_s(\bar{u})(v_{xy}). \quad \square$$

**Corollary 1.** *Given  $h \in \mathcal{C}(V^S)$ , such that  $\langle h, \sqrt{k^S} \rangle_{V^S} = 0$ , let  $\underline{h} \in \mathcal{C}(V)$  be its contraction to  $V$ ;  $u \in \mathcal{C}(V)$  be the unique solution of  $\mathcal{L}(u) = 2\underline{h}$  that satisfies  $\langle u, \sqrt{k} \rangle_V = 0$ ; and the constant*

$$\lambda = -\frac{1}{2\text{vol}(\Gamma)} \sum_{r \sim s} h(v_{rs}) \sqrt{c(r, s)}.$$

*Then,  $u^\perp = u^h + \lambda \sqrt{k^S} \in \mathcal{C}(V^S)$  is the unique solution of the Poisson problem  $\mathcal{L}_s(u) = h$  that satisfies  $\langle u^\perp, \sqrt{k^S} \rangle_{V^S} = 0$ . Specifically,*

$$\begin{aligned} u^\perp(x) &= u(x) - \frac{\sqrt{k(x)}}{\sqrt{2\text{vol}(\Gamma)}} \sum_{r \sim s} h(v_{rs}) \sqrt{c(r, s)}, \\ u^\perp(v_{xy}) &= h(v_{xy}) + \frac{\sqrt{c(x, y)}}{\sqrt{2}} \left( \frac{u(x)}{\sqrt{k(x)}} + \frac{u(y)}{\sqrt{k(y)}} \right) - \frac{\sqrt{c(x, y)}}{\text{vol}(\Gamma)} \sum_{r \sim s} h(v_{rs}) \sqrt{c(r, s)}, \end{aligned}$$

for any  $x \in V$  and  $v_{xy} \in V'$ .

*Proof.* As two solutions differ on a multiple of  $\sqrt{k^S}$ , we have that  $u^\perp = u^h + \gamma\sqrt{k^S}$ ,  $\gamma \in \mathbb{R}$ . Then,

$$\begin{aligned} 0 &= \langle u^\perp, \sqrt{k^S} \rangle_{V^S} = \langle u^h, \sqrt{k^S} \rangle_{V^S} + \gamma \sum_{x \in V^S} k^S(x) \\ &= \sqrt{2} \sum_{x \in V} u(x) \sqrt{k(x)} + \sum_{v_{xy} \in V'} u^h(v_{xy}) \sqrt{k^S(v_{xy})} + \gamma \text{vol}(\Gamma^S) \\ &= 2 \sum_{v_{xy} \in V'} u^h(v_{xy}) \sqrt{c(x, y)} + 4\gamma \text{vol}(\Gamma), \end{aligned}$$

because  $\langle u, \sqrt{k} \rangle_V = 0$ , and hence

$$\begin{aligned} \lambda &= -\frac{1}{2\text{vol}(\Gamma)} \sum_{r \sim s} u^h(v_{rs}) \sqrt{c(r, s)} \\ &= -\frac{1}{2\text{vol}(\Gamma)} \sum_{r \sim s} \left[ h(v_{rs}) \sqrt{c(r, s)} + \frac{c(r, s)}{\sqrt{2}} \left( \frac{u(r)}{\sqrt{k(r)}} + \frac{u(s)}{\sqrt{k(s)}} \right) \right] \\ &= -\frac{1}{2\text{vol}(\Gamma)} \sum_{r \sim s} h(v_{rs}) \sqrt{c(r, s)} - \frac{1}{2\sqrt{2}\text{vol}(\Gamma)} \sum_{r \in V} \frac{u(r)}{\sqrt{k(r)}} \sum_{s \sim r} c(r, s) \\ &= -\frac{1}{2\text{vol}(\Gamma)} \sum_{r \sim s} h(v_{rs}) \sqrt{c(r, s)} - \frac{1}{2\sqrt{2}\text{vol}(\Gamma)} \sum_{r \in V} u(r) \sqrt{k(r)} \\ &= -\frac{1}{2\text{vol}(\Gamma)} \sum_{r \sim s} h(v_{rs}) \sqrt{c(r, s)}. \end{aligned}$$

□

Taking into account the relation between Poisson problems for the normalized laplacian on  $\Gamma^S$  and  $\Gamma$ , we obtain the expression of the Green function for the normalized laplacian of the subdivision network  $G_S$ , in terms of the Green function of the base network  $G$ .

**Theorem 2.** *Let  $\Gamma^S$  be the subdivision network of  $\Gamma$ , then for any  $x, z \in V$  and  $v_{xy}, v_{zt} \in V'$ , the Green function of  $\Gamma^S$  is given by*

$$\begin{aligned} G_S(x, z) &= 2G(x, z) + \frac{\sqrt{k(x)}\sqrt{k(z)}}{4\text{vol}(\Gamma)}, \\ G_S(v_{xy}, z) &= \sqrt{2}\sqrt{c(x, y)} \left( \frac{G(x, z)}{\sqrt{k(x)}} + \frac{G(y, z)}{\sqrt{k(y)}} - \frac{\sqrt{k(z)}}{4\text{vol}(\Gamma)} \right), \\ G_S(v_{xy}, v_{zt}) &= \sqrt{c(x, y)c(z, t)} \left( \frac{G(x, z)}{\sqrt{k(x)k(z)}} + \frac{G(x, t)}{\sqrt{k(x)k(t)}} + \frac{G(y, z)}{\sqrt{k(y)k(z)}} + \frac{G(y, t)}{\sqrt{k(y)k(t)}} \right) \\ &\quad - \frac{3\sqrt{c(x, y)c(z, t)}}{2\text{vol}(\Gamma)} + \varepsilon_{v_{zt}}(v_{xy}). \end{aligned}$$

*Proof.* For the first case, suppose  $z \in V$ , and let  $h_z = \varepsilon_z - \frac{\sqrt{k^S(z)}}{4\text{vol}(\Gamma)}\sqrt{k^S}$ . After Theorem 1, for every  $x \in V$  the data function to be used for the Poisson problem on  $\Gamma$  must be

$$\begin{aligned} \underline{h}_z(x) &= \varepsilon_z(x) - \frac{\sqrt{k^S(z)}\sqrt{k^S(x)}}{4\text{vol}(\Gamma)} - \frac{1}{\sqrt{2}} \frac{\sqrt{k^S(z)}}{4\text{vol}(\Gamma)} \sum_{y \sim x} \frac{\sqrt{c(x,y)}}{\sqrt{k(x)}} \sqrt{k^S(v_{xy})} \\ &= \varepsilon_z(x) - \frac{\sqrt{k^S(x)}\sqrt{k^S(z)}}{2\text{vol}(\Gamma)} = \varepsilon_z(x) - \frac{\sqrt{k(x)}\sqrt{k(z)}}{\text{vol}(\Gamma)}. \end{aligned}$$

The unique solution to the Poisson problem  $\mathcal{L}(u_z) = 2h_z$ , orthogonal to  $\sqrt{k}$ , using the Green function for  $\Gamma$ , is  $u_z(x) = 2G(x, z)$ , and from Corollary 1

$$\begin{aligned} G_s(x, z) &= 2G(x, z) + \frac{\sqrt{k(x)}\sqrt{k(z)}}{4\text{vol}(\Gamma)^2} \sum_{r \sim s} \sqrt{k^S(v_{rs})}\sqrt{c(r, s)} \\ &= 2G(x, z) + \frac{\sqrt{k(x)}\sqrt{k(z)}}{2\text{vol}(\Gamma)^2} \sum_{r \sim s} c(r, s) = 2G(x, z) + \frac{\sqrt{k(x)}\sqrt{k(z)}}{4\text{vol}(\Gamma)}. \end{aligned}$$

On the other hand, for every  $v_{xy} \in V'$ ,

$$\begin{aligned} G_s(v_{xy}, z) &= \sqrt{2}\sqrt{c(x, y)} \left( \frac{G(x, z)}{\sqrt{k(x)}} + \frac{G(y, z)}{\sqrt{k(y)}} \right) \\ &\quad - \frac{\sqrt{k(z)}\sqrt{c(x, y)}}{\sqrt{2}\text{vol}(\Gamma)} + \frac{\sqrt{c(x, y)}\sqrt{k(z)}}{\sqrt{2}\text{vol}(\Gamma)^2} \sum_{r \sim s} c(r, s) \\ &= \sqrt{2}\sqrt{c(x, y)} \left( \frac{G(x, z)}{\sqrt{k(x)}} + \frac{G(y, z)}{\sqrt{k(y)}} - \frac{\sqrt{k(z)}}{4\text{vol}(\Gamma)} \right). \end{aligned}$$

Finally, we complete the proof by considering the case where the pole is a new generated vertex by the subdivision procedure. So suppose now  $v_{zt} \in V'$ , and let

$h_{v_{zt}} = \varepsilon_{v_{zt}} - \frac{\sqrt{c(z, t)}}{2\text{vol}(\Gamma)}\sqrt{k^S}$ . Then, for every  $x \in V$

$$\begin{aligned} \underline{h}_{v_{zt}}(x) &= -\frac{\sqrt{2}\sqrt{c(z, t)}}{2\text{vol}(\Gamma)}\sqrt{k(x)} + \frac{1}{\sqrt{2}} \sum_{y \sim x} \frac{\sqrt{c(x, y)}}{\sqrt{k(x)}} - \left( \varepsilon_{v_{zt}}(v_{xy}) - \frac{\sqrt{c(z, t)}}{\text{vol}(\Gamma)}\sqrt{c(x, y)} \right) \\ &= -\frac{2\sqrt{c(z, t)}}{\sqrt{2}\text{vol}(\Gamma)}\sqrt{k(x)} + \frac{1}{\sqrt{2}} \left[ \frac{\sqrt{c(z, t)}}{\sqrt{k(z)}}\varepsilon_z(x) + \frac{\sqrt{c(z, t)}}{\sqrt{k(t)}}\varepsilon_t(x) \right] \\ &= \frac{1}{\sqrt{2}} \frac{\sqrt{c(z, t)}}{\sqrt{k(z)}} \left( \varepsilon_z(x) - \frac{\sqrt{k(z)}}{\text{vol}(\Gamma)}\sqrt{k(x)} \right) \\ &\quad + \frac{1}{\sqrt{2}} \frac{\sqrt{c(z, t)}}{\sqrt{k(t)}} \left( \varepsilon_t(x) - \frac{\sqrt{k(t)}}{\text{vol}(\Gamma)}\sqrt{k(x)} \right). \end{aligned}$$

Hence, the Poisson problem to solve is  $\mathcal{L}(u_{v_{zt}}) = 2h_{v_{zt}}$  and, using the Green function for  $\Gamma$ , we obtain

$$u_{v_{zt}}(x) = \sqrt{2}\sqrt{c(z,t)} \left( \frac{G(x,z)}{\sqrt{k(z)}} + \frac{G(x,t)}{\sqrt{k(t)}} \right).$$

Then, from Corollary 1, we get that

$$\begin{aligned} G_S(v_{xy}, v_{zt}) &= \varepsilon_{v_{zt}}(v_{xy}) - \frac{\sqrt{c(x,y)}\sqrt{c(z,t)}}{\text{vol}(\Gamma)} \\ &+ \sqrt{c(x,y)c(z,t)} \left( \frac{G(x,z)}{\sqrt{k(x)k(z)}} + \frac{G(x,t)}{\sqrt{k(x)k(t)}} + \frac{G(y,z)}{\sqrt{k(y)k(z)}} + \frac{G(y,t)}{\sqrt{k(y)k(t)}} \right) \\ &- \frac{\sqrt{c(x,y)c(z,t)}}{2\text{vol}(\Gamma)}. \quad \square \end{aligned}$$

From Eq. (1), we easily calculate the values of the effective resistances for the subdivision network. Moreover, we also give the expression of its Kirchhoff index.

**Theorem 3.** *Let  $\Gamma^S$  be the subdivision network of  $\Gamma$ , then for any  $x, z \in V$  and  $v_{xy}, v_{zt} \in V'$ , the effective resistances between vertices of  $\Gamma^S$  are given by*

$$\begin{aligned} R_S(x, z) &= R(x, z), \\ R_S(v_{xy}, z) &= \frac{1}{4} \frac{1}{c(x,y)} + \frac{1}{2} R(x, z) + \frac{1}{2} R(y, z) - \frac{1}{4} R(x, y), \\ R_S(v_{xy}, v_{zt}) &= \frac{1}{4} \left( \frac{1}{c(x,y)} + \frac{1}{c(z,t)} \right) \\ &+ \frac{1}{4} (R(x, z) + R(x, t) + R(y, z) + R(y, t) - R(x, y) - R(z, t)), \end{aligned}$$

for any  $v_{xy} \neq v_{zt}$ .

Moreover, the Kirchhoff index of the subdivision network is

$$k(\Gamma^S) = 16k(\Gamma) + 2\text{vol}(\Gamma)(2m - 2n + 1).$$

*Proof.* The expressions for the effective resistance follow directly from Eq. (1) and



Theorem 2. On the other hand, from the definition of Kirchhoff index we get

$$\begin{aligned}
 k(\Gamma^S) &= 8k(\Gamma) + 2\text{vol}(\Gamma)(2m - 1) \\
 &+ 4\text{vol}(\Gamma) \sum_{x \sim y} c(x, y) \left( \frac{G(x, x)}{k(x)} + 2 \frac{G(x, y)}{\sqrt{k(x)k(y)}} + \frac{G(y, y)}{k(y)} \right) \\
 &= 8k(\Gamma) + 2\text{vol}(\Gamma)(2m - 1) \\
 &+ 4\text{vol}(\Gamma) \sum_{x \in V} \frac{G(x, x)}{k(x)} \sum_{y \sim x} c(x, y) + 4\text{vol}(\Gamma) \sum_{x \in V} \frac{1}{\sqrt{k(x)}} \sum_{y \sim x} c(x, y) \frac{G(x, y)}{\sqrt{k(y)}} \\
 &= 12k(\Gamma) + 2\text{vol}(\Gamma)(2m - 1) + 4\text{vol}(\Gamma) \sum_{x \in V} \frac{G(x, x)}{k(x)} \sum_{y \sim x} c(x, y) \\
 &+ 4\text{vol}(\Gamma) \sum_{x \in V} \frac{1}{\sqrt{k(x)}} \sum_{y \sim x} c(x, y) \left( \frac{G(x, y)}{\sqrt{k(y)}} - \frac{G(x, x)}{\sqrt{k(x)}} \right) \\
 &= 16k(\Gamma) + 2\text{vol}(\Gamma)(2m - 1) - 4\text{vol}(\Gamma) \sum_{x \in V} \left( 1 - \frac{\sqrt{k(x)}}{\text{vol}(\Gamma)} \sqrt{k(x)} \right) \\
 &= 16k(\Gamma) + 2\text{vol}(\Gamma)(2m - 2n + 1).
 \end{aligned}$$

□

### 3. SOME EXAMPLES

In this final section we add results corresponding to two examples of subdivision networks obtained from the  $n$ -Star and the  $n$ -Wheel, respectively.

Firstly we consider the  $n$ -Star network, see Fig. 1 (left), that has  $n + 1$  vertices,  $\{x_0, x_1, \dots, x_n\}$ , and constant conductance  $a > 0$ , i.e.,  $c(x_0, x_i) = a$ , for  $i = 1, \dots, n$ , and zero otherwise. Thus, the degree function is  $k(x_0) = na$  and  $k(x_i) = a$  for  $i = 1, \dots, n$ , while  $\text{vol}(\Gamma) = 2na$ . Hence, the subdivision of the  $n$ -Star has  $n$  new inserted vertices, those white in Fig. 1 (right), that we denote as  $v_{x_0x_i}$ . Accordingly to the definition of the conductances, the degree function of the subdivision  $n$ -Star network is  $k^S(x_0) = 2na, k^S(x_i) = 2a$ , and  $k^S(v_{x_0x_i}) = 4a$ ,  $i = 1, \dots, n$ ,

Normalized laplacian matrices of the former network and its subdivision are, respectively

$$L = \begin{bmatrix} 1 & -\frac{1}{\sqrt{n}}\mathbf{1}^T \\ -\frac{1}{\sqrt{n}}\mathbf{1} & I \end{bmatrix} \quad \text{and} \quad L_S = \begin{bmatrix} 1 & 0^T & -\frac{1}{\sqrt{2n}}\mathbf{1}^T \\ 0 & I & -\frac{1}{\sqrt{2}}I \\ -\frac{1}{\sqrt{2n}}\mathbf{1} & -\frac{1}{\sqrt{2}}I & I \end{bmatrix}$$

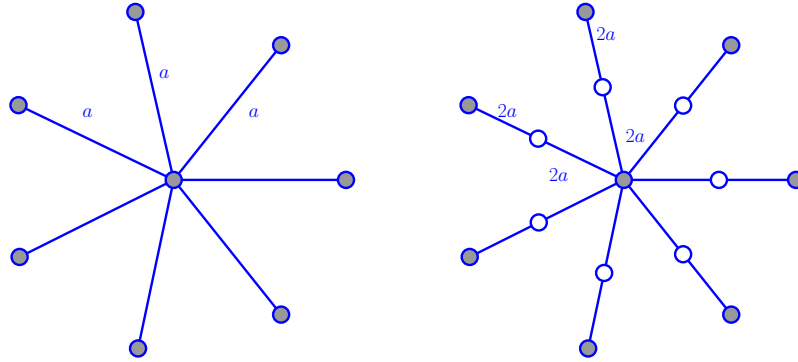


Figure 1: The Star network and its subdivision network

being  $\mathbf{0}$  and  $\mathbf{1}$   $n$  entries vectors (all zeros, all ones) and  $\mathbf{I}$  the  $n \times n$  identity matrix. It can be proved, see [4] for instance, that the group inverse matrix for such an  $n$ -Star network is

$$\mathbf{G} = \begin{bmatrix} \frac{1}{4} & -\frac{1}{4\sqrt{n}}\mathbf{1}^T \\ -\frac{1}{4\sqrt{n}}\mathbf{1} & \mathbf{A} \end{bmatrix}$$

being  $\mathbf{A}$  an  $n \times n$  matrix whose values are  $1 - \frac{3}{4n}$  on the diagonal, and  $-\frac{3}{4n}$  otherwise.

For the  $n$ -Star network, effective resistances are

$$R(x_0, x_i) = \frac{1}{a}, \quad R(x_i, x_j) = \frac{2}{a},$$

while its corresponding Kirchhoff index is  $k(\Gamma) = n(2n - 1)a$ .

Hence, using Theorem 2, we calculate

$$\mathbf{G}_s = \begin{bmatrix} \frac{5}{8} & -\frac{3}{8\sqrt{n}}\mathbf{1}^T & -\frac{\sqrt{2}}{8\sqrt{n}}\mathbf{1}^T \\ -\frac{3}{8\sqrt{n}}\mathbf{1} & \mathbf{A}_1 & \mathbf{A}_2 \\ \frac{\sqrt{2}}{8\sqrt{n}}\mathbf{1} & \mathbf{A}_2^T & \mathbf{A}_3 \end{bmatrix}$$

where matrices  $\mathbf{A}_1, \mathbf{A}_2$  and  $\mathbf{A}_3$  have all the same “shape”; that is, a constant value on the diagonal and a different one off the diagonal, so they can be expressed in terms of the identity matrix and  $\mathbf{J}$  the all ones matrix as

$$\mathbf{A}_1 = 2\mathbf{I} - \frac{11}{8n}\mathbf{J}, \quad \mathbf{A}_2 = \sqrt{2}\mathbf{I} - \frac{9\sqrt{2}}{8n}\mathbf{J} \quad \text{and} \quad \mathbf{A}_3 = 2\mathbf{I} - \frac{7}{4n}\mathbf{J}.$$

After using Theorem 3, effective resistances for the subdivision network of  $n$ -Star network are to be

$$\begin{aligned}
 R_S(x_0, x_i) &= R_S(v_{x_0x_i}, x_0x_j) = \frac{1}{a}, \\
 R_S(x_0, v_{x_0x_i}) &= R_S(x_i, v_{x_0x_i}) = \frac{1}{2a}, \\
 R_S(x_i, x_j) &= \frac{2}{a} \text{ and } R_S(x_i, v_{x_0x_j}) = \frac{3}{2a},
 \end{aligned}$$

for  $i, j = 1, \dots, n, i \neq j$ . Please note that as the subdivided star is a tree, the values of the effective resistances do agree with those obtained by direct application of simple electrical properties.

And also, the Kirchhoff index for the subdivision network of  $n$ -Star network is,

$$k(\Gamma^S) = 4na(8n - 5).$$

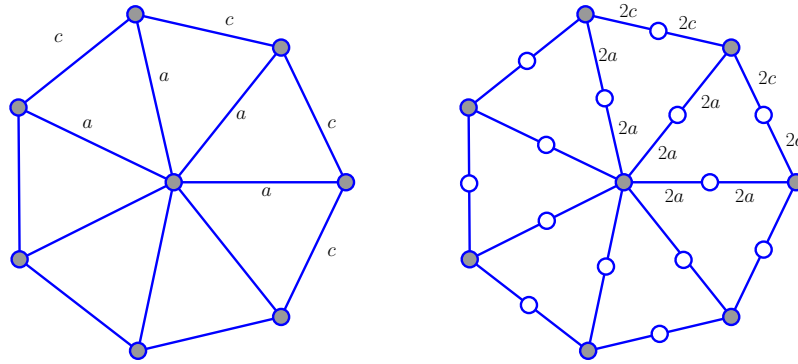


Figure 2: The Wheel network and its subdivision network

Finally, in our second example, we consider the  $n$ -Wheel network, see Fig. 2 (left),  $W_n$ , that has  $n + 1$  vertices labelled  $\{x_0, x_1, \dots, x_n\}$ . The only non null conductances are  $c(x_0, x_i) = a > 0$  and  $c(x_i, x_{i+1}) = c > 0$ , for  $i = 1, \dots, n$ , assuming  $x_{n+1} = x_1$ . Thus, the degree function is defined as  $k(x_0) = na$  and  $k(x_i) = a + 2c$  for  $i = 1, \dots, n$ . In addition,  $\text{vol}(W_n) = 2n(a + c)$ .

As the normalized laplacian operator on a network can be seen as a particular Schrödinger operator, we use the results in [4] again to obtain the Green function

of the normalized laplacian for the  $n$ -Wheel network. And it is

$$\begin{aligned} G(x_0, x_0) &= \frac{(a+2c)^2}{(2(a+c))^2} \\ G(x_0, x_i) &= -\frac{(a+2c)\sqrt{na(a+2c)}}{n(2(a+c))^2}, \quad i = 1, \dots, n \\ G(x_i, x_j) &= -\frac{(a+2c)^2}{2na(a+c)} \left( \frac{a}{2(a+c)} + 1 \right) + p \frac{U_{n-1-|i-j|}(p) + U_{|i-j|-1}(p)}{T_n(p) - 1} \end{aligned}$$

$i, j = 1, \dots, n$ , where  $p = 1 + \frac{a}{2c}$  and  $T_k(p)$  and  $U_k(p)$  are the first and second kind Chebyshev polynomials respectively.

Then effective resistances are

$$\begin{aligned} R(x_0, x_i) &= \frac{n(a+c)}{c} \frac{U_{n-1}(p)}{(T_n(p) - 1)} \\ R(x_i, x_j) &= 2 \frac{n(a+c)}{c} \frac{(U_{n-1}(p) + U_{n-1-|i-j|}(p) + U_{|i-j|-1}(p))}{(T_n(p) - 1)}, \end{aligned}$$

$i, j = 1, \dots, n, i \neq j$  and the Kirchoff index is

$$\begin{aligned} k(W_n) &= 2n(a+c) \left( -\frac{(a+2c)^2}{2a(a+c)} + \frac{n(a+2c)}{2c} \frac{U_{n-1}(p)}{T_n(p) - 1} \right) \\ &= 2n(a+c) \left( \frac{p^2}{(2p-1)(1-p)} + \sum_{j=0}^{n-1} \frac{p}{p - \cos\left(\frac{2\pi j}{n}\right)} \right), \end{aligned}$$

taking into account that  $\frac{nU_{n-1}(p)}{T_n(p) - 1} = \sum_{j=0}^{n-1} \frac{1}{p - \cos\left(\frac{2\pi j}{n}\right)}$ .

Let us now consider the subdivision network of the  $n$ -Wheel network. We denote the new white vertices in Fig.2 (right), by  $v_{x_0x_i}$  and  $v_{x_ix_{i+1}}$ ,  $i = 1, \dots, n$  provided  $x_{n+1} = x_1$ , as before. According to the notation, the degree of the vertices in the subdivision network of the  $n$ -Wheel network are  $k^S(x_0) = 2na$ ,  $k^S(x_i) = 2(a+2c)$ ,  $k^S(v_{x_0x_i}) = 4a$  and  $k^S(v_{x_ix_{i+1}}) = 4c$ .

From Theorem 2, we obtain the expression of the values of the Green function for the subdivision of the Wheel network case by case.

Initially when only former vertices are concerned

$$\begin{aligned}
 G_S(x_0, x_0) &= \frac{5a^2 + 17ac + 16c^2}{8(a+c)^2}, \\
 G_S(x_0, x_i) &= -\frac{\sqrt{na(a+2c)}}{8n(a+c)^2}(3a+7c), \quad i = 1, \dots, n; \\
 G_S(x_i, x_j) &= -\frac{(a+2c)^2}{8an(a+c)^2}(11a^2 + 31ac + 16c^2), \\
 &\quad + \frac{2p(U_{n-1-|i-j|}(p) + U_{|i-j|-1}(p))}{(T_n(p) - 1)}, \quad i, j = 1, \dots, n.
 \end{aligned}$$

Secondly when both, former and later, kinds of vertices are involved

$$\begin{aligned}
 G_S(x_0, v_{x_0x_i}) &= \frac{\sqrt{2}}{8\sqrt{n}} \frac{8c^2 + 3ac - a^2}{(a+c)^2}, \quad i = 1, \dots, n; \\
 G_S(x_0, v_{x_ix_{i+1}}) &= \frac{\sqrt{2nac}}{8n(a+c)^2}(-5a - 9c), \quad i = 1, \dots, n \text{ and assuming } x_{n+1} = x_1; \\
 G_S(x_j, v_{x_0x_i}) &= -\frac{\sqrt{2a(a+2c)}}{8na(a+c)^2}(9a^2 + 21ac + 8c^2) \\
 &\quad + \frac{p\sqrt{2a}}{\sqrt{a+2c}} \frac{(U_{n-1-|i-j|}(p) + U_{|i-j|-1}(p))}{(T_n(p) - 1)}, \quad i, j = 1, \dots, n; \\
 G_S(x_j, v_{x_ix_{i+1}}) &= \frac{\sqrt{2(a+2c)c}}{8an(a+c)^2}(-11a^2 - 33ac - 16c^2), \quad i, j = 1, \dots, n \text{ with } x_{n+1} = x_1.
 \end{aligned}$$

And finally when only new vertices are taken into account

$$\begin{aligned}
 G_S(v_{x_0x_i}, v_{x_0x_j}) &= -\frac{a(7a+11c)}{4n(a+c)^2} + \frac{a}{2c} \frac{(U_{n-1-|i-j|}(p) + U_{|i-j|-1}(p))}{(T_n(p) - 1)} \\
 &\quad + \varepsilon_{v_{x_0x_i}}(v_{x_0x_j}), \quad i, j = 1, \dots, n; \\
 G_S(v_{x_0x_i}, v_{x_jx_{j+1}}) &= -\sqrt{ac} \frac{(11a^2 + 23ac + 8c^2)}{4an(a+c)^2} \\
 &\quad + \frac{\sqrt{ac}}{2c} \frac{(U_{n-1-|i-j|}(p) + U_{n-1-|i-j-1|}(p))}{T_n(p) - 1} \\
 &\quad + \frac{\sqrt{ac}}{2c} \frac{(U_{|i-j|-1}(p) + U_{|i-j-1|-1}(p))}{T_n(p) - 1}, \\
 G_S(v_{x_ix_{i+1}}, v_{x_jx_{j+1}}) &= -\frac{c(15a^2 + 35ac + 16c^2)}{4an(a+c)^2} \\
 &\quad + (1+p) \frac{U_{n-1-|i-j|}(p) + U_{|i-j|-1}(p)}{T_n(p) - 1} + \varepsilon_{x_i}(x_j).
 \end{aligned}$$

with  $i, j = 1, \dots, n$  in every case and, as usual with  $x_{n+1} = x_1$  when required.

Applying Theorem 3 we can obtain the effective resistances for the subdivision network of the  $n$ -Wheel. In what follows we compute just some of them as examples.

$$\begin{aligned}
 R_S(x_i, x_j) &= R(x_i, x_j), \quad i, j = 0, \dots, n \quad i \neq j; \\
 R_S(x_0, v_{x_0 x_i}) &= \frac{1}{4a} + \frac{n(a+c)}{c} \frac{U_{n-1}(p)}{T_n(p)-1}, \quad i = 1, \dots, n; \\
 R_S(x_j, v_{x_0 x_i}) &= \frac{1}{4a} + \frac{5n(a+c)}{4c} \frac{U_{n-1}(p)}{T_n(p)-1} \\
 &\quad + \frac{n(a+c)}{c} \frac{U_{n-1-|i-j|}(p) + U_{|i-j|-1}(p)}{T_n(p)-1}, \quad i, j = 1, \dots, n; \\
 R_S(v_{x_i x_{i+1}}, x_j) &= \frac{1}{2c} + \frac{n(a+c)}{c} \frac{U_{n-1}(p)}{T_n(p)-1} - \frac{n(a+c)}{c} \frac{(U_{n-2}(p)+1)}{T_n(p)-1} \\
 &\quad + \frac{n(a+c)}{c} \frac{(U_{n-1-|i-j|}(p) + U_{|i-j|-1}(p))}{T_n(p)-1} \\
 &\quad + \frac{n(a+c)}{c} \frac{(U_{n-1}(p) + U_{n-1-|i+1-j|}(p) + U_{|i+1-j|-1}(p))}{T_n(p)-1}.
 \end{aligned}$$

with  $i, j = 1, \dots, n$  and assuming  $x_{n+1} = x_1$  once more.

In addition, the Kirchhoff index for the subdivision network of the  $n$ -Wheel is

$$k(\Gamma^S) = n + 4np \frac{U_{n-1}(p)}{(T_n(p)-1)} - \frac{5a^2 + 17ac + 16c^2}{2a(a+c)}.$$

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