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ON A GENERALIZED FUNCTION-TO-SEQUENCE TRANSFORM

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By attaching a sequence $\{\alpha_n\}_{n\in\mathbb{N}_0}$ to the binomial transform, a new operator $\mathcal{D}_\alpha$ is obtained. We use the same sequence to define a new transform $\mathcal{T}_\alpha$ mapping derivatives to the powers of $\mathcal{D}_\alpha$, and integrals to $\mathcal{D}_\alpha^{-1}$. The inverse transform $\mathcal{B}_\alpha$ of $\mathcal{T}_\alpha$ is introduced and its properties are studied. For $\alpha_n = (-1)^n$, $\mathcal{B}_\alpha$ reduces to the Borel transform. Applying $\mathcal{T}_\alpha$ to Bessel's differential operator $\frac{d}{dx}x\frac{d}{dx}$, we obtain Bessel's discrete operator $D_\alpha n\mathcal{N}_\alpha$. Its eigenvectors correspond to eigenfunctions of Bessel's differential operator.

1. INTRODUCTION AND PRELIMINARIES

One can pose a question regarding the character of the reality. Is it discrete or continuous? The answer is simple: it is dual. There are numerous examples of duality in the real world such as light. Louis de Broglie showed its particle and wave nature.

Obvious example of duality is the **Poisson distribution** in the probability theory. It is **discrete**, expressing the probability of a given number of events occurring in a fixed interval of time or space if these events occur with a known constant rate and independently of the time since the last event.

The exponential distribution is the **continuous** probability distribution that describes the time between events in a Poisson point process, that is, a process in which events occur **continuously** and independently at a constant average rate.

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Very often, if interpreted as discrete, problems concerned with continuity can be solved more easily, and the other way around. Complicated integrals of continuous functions are reduced in numerical analysis to finite sums, and to be solved, some difference equations have to be transformed into differential ones. In this sense, we develop a specific transformation mapping the continuous into the discrete and vice versa.

Let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Sequences $\{s_n\}_{n\in\mathbb{N}_0}$ and $\{t_n\}_{n\in\mathbb{N}_0}$ are mapped to each other by applying the binomial transform and its inverse defined respectively by (see [**4**])

$$t_n = \sum_{k=0}^{n} \binom{n}{k} s_k, \qquad s_n = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} t_k,$$

$$T = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 1 & 1 & 0 & \dots & 0 \\ 1 & 2 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 1 & \binom{n}{1} & \binom{n}{2} & \dots & \binom{n}{n} \end{pmatrix}. \tag{1}$$

The binomial transform can be regarded as an infinite-dimensional linear operator, with its regular lower triangular matrix T.

Further on, for simplicity's sake, to denote a sequence, we write curly brackets and omit $n \in \mathbb{N}_0$, i.e. instead of $\{t_n\}_{n\in\mathbb{N}_0}$, we write only $\{t_n\}$. First of all, we are referring to Newton's forward difference formula

$$t_n = \sum_{k=0}^{n} \binom{n}{k} \Delta^k t_0, \tag{2}$$

where Δ is the forward difference operator, i.e. $\Delta t_n = t_{n+1} - t_n$, which is a special case of the binomial transform (1) for $s_k = \Delta^k t_0$. Taking into account linearity of Δ, using the representation (2), there follows

$$\Delta t_n = \sum_{k=0}^{n} \binom{n}{k} \Delta^{k+1} t_0, \tag{3}$$

and by means of the method of mathematical induction in $p \in \mathbb{N}$, one can easily prove

$$\Delta^p t_n = \sum_{k=0}^{n} \binom{n}{k} \Delta^{k+p} t_0. \tag{4}$$

On the other hand, by virtue of (2) and the inverse binomial transform (1), we have

$$\Delta^n t_0 = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} t_k. \tag{5}$$

2. A NEW OPERATOR

If we multiply terms $\binom{n}{k}\Delta^{k+1}t_0$ in (3) by c_k, where $\{c_n\}$ is an arbitrary sequence of real numbers, we obtain a more general linear difference operator.

In the sequel we shall stick to the specific structure of $c_k = -\frac{\alpha_{k+1}}{\alpha_k}$ used in [**2**]. So, we deal with a linear generalized difference operator $\mathcal{D}_\alpha$, associated with a sequence of real numbers $\{\alpha_n\}$, $\alpha_0 = 1$ and $\alpha_n \neq 0$, $n \in \mathbb{N}$,

$$\mathcal{D}_\alpha t_n = \sum_{k=0}^{n} \Big(-\frac{\alpha_{k+1}}{\alpha_k} \Big) \binom{n}{k} \Delta^{k+1} t_0. \tag{6}$$

Thus, for $\alpha_k = (-1)^k$, (6) becomes (3), implying that the generalized difference operator $\mathcal{D}_\alpha$ reduces to the forward difference operator Δ.

A formula similar to (4) can be derived for $\mathcal{D}_\alpha$, namely, there holds

Lemma 1. *For the linear operator $\mathcal{D}_\alpha$ and a sequence $\{t_n\}$, there holds*

$$\mathcal{D}_\alpha^m t_n = (-1)^m \sum_{k=0}^{n} \frac{\alpha_{k+m}}{\alpha_k} \binom{n}{k} \Delta^{k+m} t_0. \tag{7}$$

where $m \in \mathbb{N}$.

Proof. For $m = 1$, we have the formula (6). Now we assume (7) holds for an arbitrary m. However, making use of (5), where we replace n with $k+m$, the equality (7) becomes

$$\mathcal{D}_\alpha^m t_n = (-1)^m \sum_{k=0}^{n} \frac{\alpha_{k+m}}{\alpha_k} \binom{n}{k} \sum_{j=0}^{k+m} (-1)^{k+m-j} \binom{k+m}{j} t_j$$

Applying the linear operator $\mathcal{D}_\alpha$, and using (6), we obtain

$$\mathcal{D}_\alpha^{m+1} t_n = (-1)^{m+1} \sum_{k=0}^{n} \frac{\alpha_{k+m}}{\alpha_k} \binom{n}{k} \sum_{j=0}^{k+m} (-1)^{k+m-j} \binom{k+m}{j} \sum_{i=0}^{j} \frac{\alpha_{i+1}}{\alpha_i} \binom{j}{i} \Delta^{i+1} t_0. \tag{8}$$

After interchanging the last two sums in (8), we find

$$\sum_{j=0}^{k+m} (-1)^{k+m-j} \binom{k+m}{j} \sum_{i=0}^{j} \frac{\alpha_{i+1}}{\alpha_i} \binom{j}{i} \Delta^{i+1} t_0 = \sum_{i=0}^{k+m} \frac{\alpha_{i+1}}{\alpha_i} \binom{k+m}{i} \Delta^{i+1} t_0 \sum_{j=0}^{k+m-i} (-1)^{k+m-i-j} \binom{k+m-i}{j}.$$

For $0 \leqslant i < k+m$, the second sum on the right-hand side of the equality is zero, and for $i = k+m$, it equals 1, so that we obtain

$$\sum_{j=0}^{k+m} (-1)^{k+m-j} \binom{k+m}{j} \sum_{i=0}^{j} \binom{j}{i} \frac{\alpha_{i+1}}{\alpha_i} \Delta^{i+1} t_0 = \frac{\alpha_{k+m+1}}{\alpha_{k+m}} \Delta^{k+m+1} t_0.$$

Now (8) takes the form of

$$\mathcal{D}_\alpha^{m+1} t_n = (-1)^{m+1} \sum_{k=0}^{n} \frac{\alpha_{k+m+1}}{\alpha_k} \binom{n}{k} \Delta^{k+m+1} t_0,$$

which assures us that (7) holds for $m+1$, and by the principle of mathematical induction, we conclude that (7) is true, which proves the lemma. □

For a sequence $\{t_n\}$ and any $p \in \mathbb{N}_0$, there holds a more general case of (2)

$$t_{n+p} = \sum_{k=0}^{p} \binom{p}{k} \Delta^k t_n = t_n + \binom{p}{1} \Delta t_n + \binom{p}{2} \Delta^2 t_n + \cdots + \Delta^p t_n. \tag{9}$$

Setting $n = 0$ in (7), we have

$$\mathcal{D}_\alpha^m t_0 = (-1)^m \frac{\alpha_m}{\alpha_0} \Delta^m t_0 = (-1)^m \alpha_m \Delta^m t_0 \quad \Rightarrow \quad \Delta^m t_0 = \frac{(-1)^m}{\alpha_m} \mathcal{D}_\alpha^m t_0.$$

Using the last equality, for $n = 0$, the equality (9) becomes

$$t_p = \sum_{k=0}^{p} \frac{(-1)^k}{\alpha_k} \binom{p}{k} \mathcal{D}_\alpha^k t_0 = t_0 - \frac{1}{\alpha_1} \binom{p}{1} \mathcal{D}_\alpha t_0 + \frac{1}{\alpha_2} \binom{p}{2} \mathcal{D}_\alpha^2 t_0 + \cdots + \frac{(-1)^p}{\alpha_p} \mathcal{D}_\alpha^p t_0.$$

Now there arises a question, whether one can obtain a formula similar to (9), but in terms of $\mathcal{D}_\alpha^k t_n$, $k = 1, 2, \ldots, p$.

Lemma 2. *The equality*

$$\begin{aligned} t_{n+p} &= \sum_{k=0}^{p} \frac{(-1)^k}{\alpha_k} \binom{p}{k} \mathcal{D}_\alpha^k t_n \\ &= t_n - \frac{1}{\alpha_1} \binom{p}{1} \mathcal{D}_\alpha t_n + \frac{1}{\alpha_2} \binom{p}{2} \mathcal{D}_\alpha^2 t_n + \cdots + \frac{(-1)^p}{\alpha_p} \mathcal{D}_\alpha^p t_n. \end{aligned} \tag{10}$$

is valid, if and only if the sequence α_k *satisfy the condition* $\alpha_{k+m} = \alpha_k \alpha_m$.

Proof. Replacing $(-1)^k \mathcal{D}_\alpha^k t_n$ in (10) with the sum on the right-hand side of (7), and making use of the imposed condition, we find

$$\begin{aligned} t_{n+p} &= \sum_{k=0}^{p} \frac{1}{\alpha_k} \binom{p}{k} \sum_{j=0}^{n} \frac{\alpha_{j+k}}{\alpha_j} \binom{n}{j} \Delta^{j+k} t_0 = \sum_{k=0}^{p} \binom{p}{k} \sum_{j=0}^{n} \binom{n}{j} \Delta^{j+k} t_0 \\ &= \sum_{k=0}^{p} \binom{p}{k} \Delta^k t_n = t_{n+p}, \end{aligned}$$

which proves (10).

The other way round, if (10) holds true, then using (9) and again (7), we have

$$\begin{aligned} t_{n+p} &= \sum_{k=0}^{p} \binom{p}{k} \Delta^k t_n = \sum_{k=0}^{p} \frac{1}{\alpha_k} \binom{p}{k} \sum_{j=0}^{n} \frac{\alpha_{j+k}}{\alpha_j} \binom{n}{j} \Delta^{j+k} t_0 \\ &= \sum_{k=0}^{p} \binom{p}{k} \sum_{j=0}^{n} \frac{\alpha_{j+k}}{\alpha_k \alpha_j} \binom{n}{j} \Delta^{j+k} t_0, \end{aligned}$$

whence there follows

$$\Delta^k t_n = \sum_{j=0}^{n} \frac{\alpha_{j+k}}{\alpha_k \alpha_j} \binom{n}{j} \Delta^{j+k} t_0.$$

Because of (4), we conclude $\dfrac{\alpha_{j+k}}{\alpha_k \alpha_j} = 1$. □

It is obvious that from the condition $\alpha_{k+m} = \alpha_k \alpha_m$, there follows $\alpha_k = \alpha_1^k$, since $\alpha_0 = 1$.

3. THE FUNCTION-TO-SEQUENCE TRANSFORM AND ITS INVERSE

Taking into account that $\mathcal{D}_\alpha$ reduces to Δ for $\alpha_k = (-1)^k$, and relying on (1), we introduce the **generalized binomial transform**

$$(11) \qquad t_n = \sum_{k=0}^{n} \frac{(-1)^k}{\alpha_k} \binom{n}{k} s_k, \quad T = \begin{pmatrix} \frac{1}{\alpha_0} & 0 & 0 & \dots & 0 \\ \frac{1}{\alpha_0} & -\frac{1}{\alpha_1} & 0 & \dots & 0 \\ \frac{1}{\alpha_0} & -\frac{2}{\alpha_1} & \frac{1}{\alpha_2} & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ \frac{1}{\alpha_0} & -\frac{\binom{n}{1}}{\alpha_1} & \frac{\binom{n}{2}}{\alpha_2} & \dots & \frac{(-1)^n \binom{n}{n}}{\alpha_n} \end{pmatrix}.$$

The binomial transform is a sequence transformation, however, after replacing the sequence $\{s_n\}$ with the sequence $\{f^{(n)}(0)\}$ of a infinitely differentiable function $f(x)$ in (11), we obtain a more general form of a sequence transformation, a specific linear transform mapping a set of functions into a set of sequences. We denote it by $\mathcal{T}_\alpha$.

Let S_c be a set of real functions f having continued derivatives of all orders at $x = 0$ for which there exists constant $M > 0$, such that $|f^{(k)}(0)| \leqslant M$ for every $k \in \mathbb{N}_0$, and S_q be a set of one-parametric sequences $\{t_n^{(m)}\}$, with $m \in \mathbb{N}_0$ as a parameter, for which $|\mathcal{D}_\alpha^k t_0| \leqslant M$ for every $k \in \mathbb{N}_0$.

Definition 1. *The transform $\mathcal{T}_{\alpha,m}$, with respect to the sequence $\{\alpha_n\}$ and a non-negative integer m, determined by the equalities*

$$(12) \qquad \mathcal{T}_{\alpha,m} f(x) = \{t_n^{(m)}\}, \quad t_n^{(m)} = \sum_{k=m}^{n} \frac{(-1)^{k-m}}{\alpha_{k-m}} \binom{n}{k} \frac{d^k}{dx^k} \big(x^m f(x)\big)_{x=0},$$

maps a function $f \in S_c$ to a sequence $\{t_n^{(m)}\} \in S_q$.

In the case $m=0$, the transform $\mathcal{T}_{\alpha,0}$ and sequence $\{t_n^{(0)}\}$ are for the sake of simplicity designated respectively by $\mathcal{T}_\alpha$ and $\{t_n\}$, so (12) takes the form

$$\mathcal{T}_\alpha f(x) = \{t_n\}, \quad t_n = \sum_{k=0}^{n} \frac{(-1)^k}{\alpha_k} \binom{n}{k} f^{(k)}(0), \tag{13}$$

and its matrix is (11).

Theorem 1. *If $\mathcal{T}_\alpha f(x) = \{t_n\}$, knowing that $n^{(m)} = n(n-1)\cdots(n-m+1)$, for the sequence $\{t_n^{(m)}\}$ from* (12) *there holds*

$$t_n^{(m)} = n^{(m)} \sum_{k=0}^{n-m} \frac{(-1)^k}{\alpha_k} \binom{n-m}{k} f^{(k)}(0) = n^{(m)} t_{n-m}. \tag{14}$$

Proof. In view of $m \leqslant k \leqslant n$, after setting $x=0$ in the derivatives

$$\frac{d^k}{dx^k}\big(x^m f(x)\big) = \sum_{j=0}^{k} \binom{k}{j} (x^m)^{(k-j)} f^{(j)}(x), \qquad k = m, m+1, \dots, n$$

there remain only the terms $\binom{k}{k-m} m^{(m)} f^{(k-m)}(0) = \binom{k}{k-m} m! f^{(k-m)}(0)$. Thus, (12) becomes

$$\begin{aligned} t_n^{(m)} &= \sum_{k=m}^{n} \frac{(-1)^{k-m}}{\alpha_{k-m}} \binom{n}{k} \binom{k}{k-m} m! f^{(k-m)}(0) \\ &= \sum_{k=0}^{n-m} \frac{(-1)^k}{\alpha_k} \binom{n}{k+m} \binom{k+m}{k} m! f^{(k)}(0). \end{aligned}$$

However, in view of $0 \leqslant k \leqslant n-m$, we have

$$\binom{n}{k+m}\binom{k+m}{k} m! = \frac{n!}{(n-k-m)!k!} = n(n-1)\cdots(n-m+1)\frac{(n-m)!}{k!(n-m-k)!},$$

so that we finally obtain

$$t_n^{(m)} = n^{(m)} \sum_{k=0}^{n-m} \frac{(-1)^k}{\alpha_k} \binom{n-m}{k} f^{(k)}(0) = n^{(m)} t_{n-m},$$

which proves (14). □

In order to study properties of $\mathcal{T}_\alpha$, we make use of $\mathcal{D}_\alpha$.

Theorem 2. *Let $\mathcal{T}_\alpha f(x) = \{t_n\}$. Then for $p \in \mathbb{N}$ there holds*

$$1^\circ\ \mathcal{T}_\alpha f^{(p)}(x) = \{\mathcal{D}_\alpha^p t_n\}, \qquad 2^\circ\ \mathcal{T}_\alpha \int_0^x f(t)dt = \{\mathcal{D}_\alpha^{-1} t_n\},$$

where $\mathcal{D}_\alpha^{-1}$ is the inverse operator of $\mathcal{D}_\alpha$.

Proof. We prove the statement 1°. Applying the finite differences method for sequences to $\{t_n\}$, and using (13), we have

$$t_n = \sum_{k=0}^{n} \binom{n}{k} \Delta^k t_0 = \sum_{k=0}^{n} \binom{n}{k} \frac{(-1)^k}{\alpha_k} f^{(k)}(0).$$

we conclude

$$\Delta^k t_0 = \frac{(-1)^k}{\alpha_k} f^{(k)}(0) \quad \Leftrightarrow \quad (-1)^k \alpha_k \Delta^k t_0 = f^{(k)}(0), \quad k \in \mathbb{N}_0. \tag{15}$$

That means $|(-1)^k \alpha_k \Delta^k t_0| = |f^{(k)}(0)|$ for every $k \in \mathbb{N}_0$. Since $f \in S_c$, on the basis of Definition 1 there follows $|\mathcal{D}_\alpha^k t_0| < aP^k$, and that means $\mathcal{T}_\alpha f \in S_q$. However, for any $p \in \mathbb{N}$, $f^{(p)} \in S_c$ as well, so in view of (12) we find

$$\mathcal{T}_\alpha f^{(p)}(x) = \{r_n\}, \quad r_n = \sum_{k=0}^{n} \binom{n}{k} \frac{(-1)^k}{\alpha_k} f^{(k+p)}(0). \tag{16}$$

From (15), for any $p \in \mathbb{N}$ there holds $\Delta^{k+p} t_0 = \frac{(-1)^{k+p}}{\alpha_{k+p}} f^{(k+p)}(0)$. Replacing this in (7), we have

$$\mathcal{D}_\alpha^p t_n = (-1)^p \sum_{k=0}^{n} \frac{\alpha_{k+p}}{\alpha_k} \binom{n}{k} \Delta^{k+p} t_0 = \sum_{k=0}^{n} \binom{n}{k} \frac{(-1)^k}{\alpha_k} f^{(k+p)}(0) = r_n. \tag{17}$$

As for the statement 2°, let $\mathcal{T}_\alpha g(x) = \{s_n\}$, where $g(x) = \int_0^x f(t)dt \in S_c$. Using (13), we conclude $s_0 = g(0) = 0$. Setting $p = 1$ in the statement 1°, we find

$$\{t_n\} = \mathcal{T}_\alpha f(x) = \mathcal{T}_\alpha \frac{d}{dx} \int_0^x f(t)dt = \mathcal{T}_\alpha g'(x) = \{\mathcal{D}_\alpha s_n\}.$$

Using (6) and applying the representation of the sequence $\{t_n\}$ by forward differences, gives rise to

$$t_n = \sum_{k=0}^{n} \binom{n}{k} \Delta^k t_0 = \mathcal{D}_\alpha s_n = \sum_{k=0}^{n} c_k \binom{n}{k} \Delta^{k+1} s_0. \tag{18}$$

On the other hand, from (18) there follows $\Delta^k t_0 = c_k \Delta^{k+1} s_0$, $k \in \mathbb{N}_0$ or $\Delta^{k-1} t_0 = c_{k-1} \Delta^k s_0$, $k \in \mathbb{N}$. So, in view of (6), taking into account $s_0 = g(0) = 0$, the sequence $\{s_n\}$ is expressed in terms of forward differences of the sequence $\{t_n\}$

$$s_n = \sum_{k=0}^{n} \binom{n}{k} \Delta^k s_0 = s_0 + \sum_{k=1}^{n} \binom{n}{k} \Delta^k s_0 = \sum_{k=1}^{n} \frac{1}{c_{k-1}} \binom{n}{k} \Delta^{k-1} t_0. \tag{19}$$

Thus, the inverse linear operator $\mathcal{D}_\alpha^{-1}$ operator is defined. On the basis of (19), we form a system of linear equations. Expressing the forward differences $\Delta^{k-1} t_0$ $(k = 1, \ldots, n)$ in matrix form in terms of $t_0, \ldots, t_{n-1}$, we have

$$\begin{pmatrix} s_1 \\ s_2 \\ \cdot\cdot \\ s_n \end{pmatrix} = \begin{pmatrix} \frac{1}{c_0} & 0 & \dots & 0 \\ \frac{2}{c_0} & \frac{1}{c_1} & \dots & 0 \\ \dots & \dots & \dots & \dots \\ \frac{\binom{n}{1}}{c_0} & \frac{\binom{n}{2}}{c_1} & \dots & \frac{1}{c_{n-1}} \end{pmatrix} \begin{pmatrix} 1 & 0 & \dots & 0 \\ -1 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ (-1)^{n-1} & (-1)^{n-2}\binom{n-1}{1} & \dots & 1 \end{pmatrix} \begin{pmatrix} t_0 \\ t_1 \\ \dots \\ t_{n-1} \end{pmatrix},$$

where the first matrix is lower triangular with diagonal coefficients $\frac{1}{c_0}, \frac{1}{c_1}, \dots, \frac{1}{c_{n-1}}$. After multiplying these matrices, we arrive at the matrix representation of $\mathcal{D}_\alpha^{-1}$. Finally, we obtain

$$\text{(20)} \qquad \mathcal{T}_\alpha \int_0^x f(t)dt = \{\mathcal{D}_\alpha^{-1} t_n\}, \qquad \begin{aligned} \mathcal{D}_\alpha^{-1} t_n &= \sum_{k=1}^{n} \frac{1}{c_{k-1}} \binom{n}{k} \Delta^{k-1} t_0 \\ &= \sum_{k=1}^{n} \frac{\alpha_{k-1}}{\alpha_k} \binom{n}{k} \Delta^{k-1} t_0. \end{aligned}$$

Thus, by applying the transform $\mathcal{T}_\alpha$ to inverse operations differentiation and integration, sequences generated by the linear operators $\mathcal{D}_\alpha$ and its inverse $\mathcal{D}_\alpha^{-1}$ are obtained. □

Example 1. *Taking $n = 5$, for the matrix $\mathcal{D}_\alpha^{-1}$ we find*

$$\text{(21)} \quad \begin{pmatrix} \frac{1}{c_0} & 0 & 0 & 0 & 0 \\ \frac{2}{c_0} - \frac{1}{c_1} & \frac{1}{c_1} & 0 & 0 & 0 \\ \frac{3}{c_0} - \frac{3}{c_1} + \frac{1}{c_2} & \frac{3}{c_1} - \frac{2}{c_2} & \frac{1}{c_2} & 0 & 0 \\ \frac{4}{c_0} - \frac{6}{c_1} + \frac{4}{c_2} - \frac{1}{c_3} & \frac{6}{c_1} - \frac{8}{c_2} + \frac{3}{c_3} & \frac{4}{c_2} - \frac{3}{c_3} & \frac{1}{c_3} & 0 \\ \frac{5}{c_0} - \frac{10}{c_1} + \frac{10}{c_2} - \frac{5}{c_3} + \frac{1}{c_4} & \frac{10}{c_1} - \frac{20}{c_2} + \frac{15}{c_3} - \frac{4}{c_4} & \frac{10}{c_2} - \frac{15}{c_3} + \frac{6}{c_4} & \frac{5}{c_3} - \frac{4}{c_4} & \frac{1}{c_4} \end{pmatrix},$$

which is obviously regular, since its determinant is $\frac{1}{c_0} \cdots \frac{1}{c_4}$, so its inverse matrix is

$$\begin{pmatrix} c_0 & 0 & 0 & 0 & 0 \\ c_0 - 2c_1 & c_1 & 0 & 0 & 0 \\ c_0 - 4c_1 + 3c_2 & 2c_1 - 3c_2 & c_2 & 0 & 0 \\ c_0 - 6c_1 + 9c_2 - 4c_3 & 3c_1 - 9c_2 + 6c_3 & 3c_2 - 4c_3 & c_3 & 0 \\ c_0 - 8c_1 + 18c_2 - 16c_3 + 5c_4 & 4c_1 - 18c_2 + 24c_3 - 10c_4 & 6c_2 - 16c_3 + 10c_4 & 4c_3 - 5c_4 & c_4 \end{pmatrix},$$

and this is the matrix of the linear operator $\mathcal{D}_\alpha$ defined by the equations (18).

Theorem 3. *If $\mathcal{T}_\alpha f(x) = \{t_n\}$ and $m, p \in \mathbb{N}_0$, then*

$$\text{(22)} \qquad \mathcal{T}_\alpha x^m f^{(p)}(x) = \{s_n^{(m)}\}, \quad s_n^{(m)} = n^{(m)} \mathcal{D}_\alpha^p t_{n-m}.$$

Proof. Applying the statement 1° of Theorem 2, yields $\mathcal{T}_\alpha f^{(p)}(x) = \{\mathcal{D}_\alpha^p t_n\}$. If we denote $g(x) = f^{(p)}(x)$ and $s_n = \mathcal{D}_\alpha^p t_n$, then $\mathcal{T}_\alpha g(x) = \{s_n\}$. Making use of Theorem 1, there follows

$$\mathcal{T}_\alpha x^m f^{(p)}(x) = \mathcal{T}_\alpha x^m g(x) = \{s_n^{(m)}\}, \quad s_n^{(m)} = n^{(m)} s_{n-m} = n^{(m)} \mathcal{D}_\alpha^p t_{n-m}.$$

which proves the lemma. □

3.1 $\mathcal{T}_\alpha$-TRANSFORM OF BASIC ELEMENTARY FUNCTIONS

We are going to derive mappings of some basic elementary functions.

Lemma 3. *Let $m = 0$ be set in (12) and $\lambda, C \in \mathbb{R}$. Then $\mathcal{T}\{C\} = \{r_n\}$, where $r_n = 1$, $n \in \mathbb{N}_0$. Also, if $\mathcal{T}\{f(x)\} = \{t_n\}$, then $\mathcal{T}\{\lambda f(x)\} = \lambda\{t_n\}$.*

Proof. Replacing derivatives of the function $f(x) = C$ in

$$r_n = \sum_{k=0}^{n} \frac{(-1)^k}{\alpha_k} \binom{n}{k} f^{(k)}(0),$$

we see that all terms equal zero, except for the first, which is 1.

Since $(\lambda f(x))^{(k)} = \lambda f^{(k)}(x)$, we immediately obtain the second statement. □

Lemma 4. *For $m = 0$ and $f(x) = x^r$ in (12), let $\mathcal{T}_\alpha x^r = \{t_n\}$. Then*

$$t_n = \frac{(-1)^r}{\alpha_r} n^{(r)},$$

where $n^{(r)} = n(n-1)\cdots(n-r+1)$, $r \leqslant n$.

Proof. We consider the function $f(x) = x^r$, $r \in \mathbb{N}$. Here

$$t_n = \sum_{k=0}^{n} \frac{(-1)^k}{\alpha_k} \binom{n}{k} f^{(k)}(0) = \sum_{k=0}^{n} \frac{(-1)^k}{\alpha_k} \binom{n}{k} r^{(k)} x^{r-k} \mid_{x=0} .$$

All the terms of the sum are zero except for $k = r$, so we obtain

(23) $$t_n = \frac{(-1)^r r!}{\alpha_r} \binom{n}{r} = \frac{(-1)^r}{\alpha_r} \frac{n!r!}{r!(n-r)!} = \frac{(-1)^r}{\alpha_r} n^{(r)}, \quad r \leqslant n,$$

which proves the lemma. □

Lemma 5. *For $a \in \mathbb{R}$, the function $f(x) = (1+x)^a$ is mapped to the sequence $\{t_n\}$, where*

(24) $$t_n = \sum_{k=0}^{n} \frac{(-1)^k}{\alpha_k} \binom{n}{k} a^{(k)}, \quad a^{(k)} = a(a-1)\cdots(a-k+1).$$

Proof. We have $f^{(k)}(x) = a^{(k)}(1+x)^{a-k}$, with $f^{(k)}(0) = a^{(k)}$, so we easily obtain (24). □

Lemma 6. *For $a \in \mathbb{R}$, there holds $\mathcal{T}_\alpha e^{ax} = \{t_n\}$,* $\quad t_n = \sum_{k=0}^{n} \frac{(-1)^k}{\alpha_k} \binom{n}{k} a^k.$

Proof. Considering that for $f(x) = e^{ax}$, we have $f^{(k)}(x) = a^k e^{ax}$ and $f^{(k)}(0) = a^k$, there immediately follows the equality. □

Lemma 7. *The function $f(x) = \ln(1 + x)$ is mapped to the sequence $\{t_n\}$, where*

$$t_n = \sum_{k=1}^{n} \frac{(-1)^{k-1}(k-1)!}{\alpha_k} \binom{n}{k} = \sum_{k=1}^{n} \frac{(-1)^{k-1} n^{(k)}}{k\, \alpha_k}. \tag{25}$$

Proof. For the function $f(x) = \ln(1 + x)$, we find $f(0) = 0$ and $f^{(k)}(x) = (-1)^{k-1}\dfrac{(k-1)!}{(1+x)^k}$, $k = 1, 2, \dots$, and $f^{(k)}(0) = (-1)^{k-1}(k-1)!$, $k = 1, 2, \dots$. Replacing this in (12), we obtain (25). □

Lemma 8. *For $a \in \mathbb{R}$, the following equalities hold*

$$\mathcal{T}_\alpha \sin ax = \{t_n\}, \quad t_n = \sum_{k=0}^{[\frac{n-1}{2}]} \frac{(-1)^{k+1}}{\alpha_{2k+1}} \binom{n}{2k+1} a^{2k+1}, \tag{26}$$

$$\mathcal{T}_\alpha \cos ax = \{t_n\}, \quad t_n = \sum_{k=0}^{[\frac{n}{2}]+1} \frac{(-1)^{k}}{\alpha_{2k}} \binom{n}{2k} a^{2k}. \tag{27}$$

Proof. First, we consider $f(x) = \sin ax$. By virtue of

$$f^{(k)}(x) = a^k \sin\Big(\frac{k\pi}{2} + ax\Big), \quad f^{(k)}(0) = a^k \sin\Big(\frac{k\pi}{2}\Big), \quad k = 1, 2, \dots$$

we have $f^{(2k)}(0) = 0$, $f^{(2k+1)}(0) = a^{2k+1}\cos k\pi = (-1)^{k+1}a^{2k+1}$. So we deal only with odd numbers in (12), that is why (12) becomes (26). Similarly, considering $f(x) = \cos ax$, on account of

$$f^{(k)}(x) = a^k \cos\Big(\frac{k\pi}{2} + ax\Big), \quad f^{(k)}(0) = a^k \cos\Big(\frac{k\pi}{2}\Big), \quad k = 1, 2, \dots$$

we have $f^{(2k)}(0) = (-1)^k a^{2k}$, $f^{(2k+1)}(0) = 0$, and deal only with even numbers in (12), that is why (12) now becomes (27). □

Example 2. *Bernoulli polynomials can be expanded into the Fourier series* (see [**1**])

$$B_{2k}(x) = 2(-1)^{k+1}\frac{(2k)!}{(2\pi)^{2k}} \sum_{j=1}^{\infty} \frac{\cos 2j\pi x}{j^{2k}}, \quad B_{2k+1}(x) = (-1)^{k+1}\frac{2(2k+1)!}{(2\pi)^{2k+1}} \sum_{j=1}^{\infty} \frac{\sin 2j\pi x}{j^{2k+1}},$$

where $k = 1, 2, \dots$ and $0 \leqslant x \leqslant 1$, the second expansion holds also for $k = 0$ and $0 < x < 1$.

In the paper [**10**], *we derived the closed form expression for trigonometric series in terms of the Riemann zeta functions*

$$\sum_{j=1}^{\infty} \frac{\cos jx}{j^{2k}} = \frac{(-1)^k \pi x^{2k-1}}{2(2k-1)!} + \sum_{j=0}^{k} \frac{(-1)^j \zeta(2k-2j)}{(2j)!} x^{2j},$$

$$\sum_{j=1}^{\infty} \frac{\sin jx}{j^{2k+1}} = \frac{(-1)^k \pi x^{2k}}{2(2k)!} + \sum_{j=0}^{k} \frac{(-1)^j \zeta(2k-2j)}{(2j+1)!} x^{2j+1}.$$

Substituting $2\pi x$ for x, we have

$$\sum_{j=1}^{\infty}\frac{\cos 2j\pi x}{j^{2k}}=\frac{(-1)^k(2\pi)^{2k}x^{2k-1}}{4(2k-1)!}+\sum_{j=0}^{k}\frac{(-1)^j\zeta(2k-2j)}{(2j)!}(2\pi x)^{2j},$$
$$\sum_{j=1}^{\infty}\frac{\sin 2j\pi x}{j^{2k+1}}=\frac{(-1)^k\pi(2\pi)^{2k+1}x^{2k}}{4(2k)!}+\sum_{j=0}^{k}\frac{(-1)^j\zeta(2k-2j)}{(2j+1)!}(2\pi x)^{2j+1}.$$

and express Bernoulli polynomials in terms of the Riemann zeta functions

$$B_{2k}(x)=-kx^{2k-1}-2\sum_{j=0}^{k}(-1)^{k+j}(2k)^{(2j)}(2\pi)^{2j-2k}x^{2j}\zeta(2k-2j),$$
$$B_{2k+1}(x)=-\frac{(2k+1)x^{2k}}{2}-2\sum_{j=0}^{k}(-1)^{k+j}(2k+1)^{(2j+1)}(2\pi)^{2j-2k}x^{2j+1}\zeta(2k-2j).$$

Applying $\mathcal{T}_\alpha$ and relying on Lemma 3 *and* Lemma 4, *we obtain* ***generalized Bernoulli numbers*** *with respect to the sequence α*

$$B_{2k,n}=\frac{k\,n^{(2k-1)}}{\alpha_{2k-1}}-2\sum_{j=0}^{k}(-1)^{k+j}(2k)^{(2j)}(2\pi)^{2j-2k}\frac{n^{(2j)}}{\alpha_{2j}}\zeta(2k-2j),$$
$$B_{2k+1,n}=-\frac{(2k+1)\,n^{(2k)}}{2\alpha_{2k}}+2\sum_{j=0}^{k}(-1)^{k+j}(2k+1)^{(2j+1)}(2\pi)^{2j-2k}\frac{n^{(2j+1)}}{\alpha_{2j+1}}\zeta(2k-2j).$$

3.2 THE INVERSE TRANSFORM

Definition 2. *For any sequence $\{t_n\}\in S_q$ and the linear operator $\mathcal{D}_\alpha$ introduced by* (6), *the function $f(x)$ defined by*

$$(28)\qquad \mathcal{B}_\alpha\{t_n\}=f(x),\quad f(x)=\sum_{k=0}^{\infty}\mathcal{D}_\alpha^k t_0\frac{x^k}{k!}=\sum_{k=0}^{\infty}\alpha_k(-1)^k\Delta^k t_0\frac{x^k}{k!}.$$

is called the $\mathcal{B}_\alpha$-transform.

On the basis of Definition 1, $f(x)$ is expandable into Taylor's series, and comparing it with the expansion of $f(x)$ from Definition 2, we conclude that $f^{(k)}(0)=\mathcal{D}_\alpha^k t_0=(-1)^k\alpha_k\Delta^k t_0$ for every $k\in\mathbb{N}_0$, so relying again on Definition 1, for $\{t_n\}\in S_q$, there is a constant $M>0$, such that $|\mathcal{D}_\alpha^k t_0|\leqslant M$, which means $|f^{(k)}(0)|\leqslant M$, so $f\in S_c$, and we have $\mathcal{B}_\alpha\{t_n\}\in S_c$.

Theorem 4. *Transforms $\mathcal{T}_\alpha$ and $\mathcal{B}_\alpha$ are inverse to each other.*

Proof. Let $\mathcal{T}_\alpha f(x)=\{t_n\}$. That means (12) and (15) hold. Since $f\in S_c$, the function $f(x)$ can be expanded into Taylor's series, and the replacing of $f^{(k)}(0)$ with $(-1)^k\alpha_k\Delta^k t_0$ gives rise to (28), whereby we prove that $\mathcal{B}_\alpha\{t_n\}=f(x)$.

Let $\{t_n\} \in S_q$. Relying on Definition 2, and applying $\mathcal{B}_\alpha$-transform, we obtain the function $f(x)$ defined by (28), i.e. $\mathcal{B}_\alpha\{t_n\} = f(x)$. However, we have noted above that $f^{(k)}(0) = (-1)^k \alpha_k \Delta^k t_0$, that is $\frac{(-1)^k}{\alpha_k} f^{(k)}(0) = \Delta^k t_0$, whence, relying on (12), we find

$$\sum_{k=0}^{n} \binom{n}{k} \frac{(-1)^k}{\alpha_k} f^{(k)}(0) = \sum_{k=0}^{n} \binom{n}{k} \Delta^k t_0 = t_n,$$

which means $\{t_n\} = \mathcal{T}_\alpha f(x)$. Thereby we have proved the theorem. □

We note that the matrix of $\mathcal{B}_\alpha$ is the inverse of T in (11), i.e. infinity-dimensional matrix

$$T^{-1} = \begin{pmatrix} \alpha_0 & 0 & 0 & \dots & 0 \\ \alpha_1 & -\alpha_1 & 0 & \dots & 0 \\ \alpha_2 & -2\alpha_2 & \alpha_2 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ \alpha_n & -\binom{n}{1}\alpha_n & \binom{n}{2}\alpha_n & \dots & (-1)^n\binom{n}{n}\alpha_n \\ \dots & \dots & \dots & \dots & \dots \end{pmatrix}.$$

Expressing $\Delta^k t_0$, $k \in \mathbb{N}_0$, in terms of $t_0, t_1, t_2, \dots$, the equality (28) can be obtained in the matrix form

$$\mathcal{B}_\alpha\{t_n\} = \begin{pmatrix} 1 & \frac{x}{1!} & \frac{x^2}{2!} & \dots & \frac{x^n}{n!} & \dots \end{pmatrix} \begin{pmatrix} \alpha_0 & 0 & 0 & \dots & 0 \\ \alpha_1 & -\alpha_1 & 0 & \dots & 0 \\ \alpha_2 & -2\alpha_2 & \alpha_2 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ \alpha_n & -\binom{n}{1}\alpha_n & \binom{n}{2}\alpha_n & \dots & (-1)^n\binom{n}{n}\alpha_n \\ \dots & \dots & \dots & \dots & \dots \end{pmatrix} \begin{pmatrix} t_0 \\ t_1 \\ t_2 \\ \dots \\ t_n \\ \dots \end{pmatrix}.$$

Example 3. *It is well-known that the sequence $\{F_n\}$ of Fibonacci numbers is defined by the recurrence relation $F_{n+2} = F_{n+1} + F_n$, with $F_0 = 0, F_1 = 1$. Using its basic properties, one gets*

$$F_k = (-1)^{k+1} \Delta^k F_0 \quad \Rightarrow \quad (-1)^k \Delta^k F_0 = -F_k. \tag{29}$$

Applying the above matrix form where the sequence $\{t_n\}$ is replaced with $\{F_n\}$, and relying on (29), we find

$$\mathcal{B}_\alpha\{F_n\} = f(x) = \sum_{k=0}^{\infty} \alpha_k (-1)^k \Delta^k F_0 \frac{x^k}{k!} = -\sum_{k=0}^{\infty} \alpha_k F_k \frac{x^k}{k!}.$$

Placing, for instance, $\alpha_k = (-1)^k$, we obtain

$$\sum_{k=0}^{\infty} (-1)^{k+1} \frac{F_k}{k!} x^k = \frac{e^{-\frac{x}{2}\left(1+\sqrt{5}\right)} \left(e^{x\sqrt{5}} - 1\right)}{\sqrt{5}}.$$

3.3 TRANSFORMS $\mathcal{T}_\alpha$ AND $\mathcal{B}_\alpha$ AND CONVOLUTION OF SEQUENCES

At first, we introduce convolution of sequences associated with the operator $\mathcal{D}_\alpha$.

Definition 3. *Convolution of the sequences $\{t_n\}, \{s_n\} \in S_q$ with respect to the sequence α, is defined by*

$$r_n = t_n *_\alpha s_n = \sum_{k=0}^{n} \binom{n}{k} \sum_{j=0}^{k} \binom{k}{j} \frac{\alpha_j \alpha_{k-j}}{\alpha_k} \Delta^j t_0 \Delta^{k-j} s_0. \tag{30}$$

Here we give some properties concerned with the convolution. For the sequences $\{r_n\}$, $\{s_n\}$, $\{t_n\} \in S_q$ and $c \in \mathbb{R}$, there holds

1° $c *_\alpha t_n = ct_n$, 2° $t_n *_\alpha s_n = s_n *_\alpha t_n$, 3° $r_n *_\alpha (s_n + t_n) = r_n *_\alpha s_n + r_n *_\alpha t_n$.

Theorem 5. *For the sequences $\{t_n\}, \{s_n\} \in S_q$ and $\mathcal{B}_\alpha$-transform, the equality*

$$\mathcal{B}_\alpha\{t_n *_\alpha s_n\} = \mathcal{B}_\alpha\{t_n\}\mathcal{B}_\alpha\{s_n\}$$

holds if and only if the convolution $t_n *_\alpha s_n$ is defined by (30).

Proof. Assume that (30) holds. Let $\mathcal{T}_\alpha f(x) = \{t_n\}$ and $\mathcal{T}_\alpha g(x) = \{s_n\}$. Substituting $f^{(j)}(0)$ for $(-1)^j \alpha_j \Delta^j t_0$ and $g^{(k-j)}(0)$ for $(-1)^{k-j}\alpha_{k-j}\Delta^{k-j} s_0$ in (30), we obtain

$$\begin{aligned} t_n *_\alpha s_n &= \sum_{k=0}^{n} \binom{n}{k} \frac{(-1)^k}{\alpha_k} \sum_{j=0}^{k} \binom{k}{j} f^{(j)}(0) g^{(k-j)}(0) \\ &= \sum_{k=0}^{n} \binom{n}{k} \frac{(-1)^k}{\alpha_k} (f(x)\, g(x))^{(k)}\Big|_{x=0}, \end{aligned} \tag{31}$$

i.e. $\{t_n *_\alpha s_n\} = \mathcal{T}_\alpha f(x)\, g(x)$. In other words, in view of Theorem 4, $\mathcal{B}_\alpha\{t_n *_\alpha s_n\} = f(x)\, g(x)$, that is $\mathcal{B}_\alpha\{t_n *_\alpha s_n\} = \mathcal{B}_\alpha\{t_n\}\mathcal{B}\{s_n\}$, since $\mathcal{B}_\alpha\{t_n\} = f(x)$ and $\mathcal{B}_\alpha\{s_n\} = g(x)$.

Conversely, assume that $\mathcal{B}_\alpha\{t_n *_\alpha s_n\} = \mathcal{B}_\alpha\{t_n\}\mathcal{B}_\alpha\{s_n\}$ holds true. So we have

$$\begin{aligned} \mathcal{B}_\alpha\{t_n\}\mathcal{B}_\alpha\{s_n\} = f(x)\, g(x) &= \sum_{k=0}^{\infty} (-1)^k \frac{\alpha_k x^k}{k!} \Delta^k t_0 \sum_{k=0}^{\infty} (-1)^k \frac{\alpha_k x^k}{k!} \Delta^k s_0 \\ &= \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k \sum_{k=0}^{\infty} \frac{g^{(k)}(0)}{k!} x^k = \sum_{k=0}^{\infty} \sum_{j=0}^{k} \frac{f^{(j)}(0) g^{(k-j)}(0)}{j!(k-j)!} x^k \\ &= \sum_{k=0}^{\infty} \frac{x^k}{k!} \sum_{j=0}^{k} \binom{k}{j} f^{(j)}(0) g^{(k-j)}(0) \\ &= \sum_{k=0}^{\infty} (-1)^k \alpha_k \frac{x^k}{k!} \sum_{j=0}^{k} \binom{k}{j} \frac{\alpha_j \alpha_{k-j}}{\alpha_k} \Delta^j t_0 \Delta^{k-j} s_0 = \mathcal{B}_\alpha\{t_n *_\alpha s_n\}. \end{aligned}$$

Denote $r_n = t_n *_\alpha s_n$. According to (30), there must be

$$\Delta^k r_0 = \sum_{j=0}^{k} \binom{k}{j} \frac{\alpha_j \alpha_{k-j}}{\alpha_k} \Delta^j t_0 \Delta^{k-j} s_0.$$

By virtue of $r_n = \sum_{k=0}^{n} \binom{n}{k} \Delta^k r_0$, we have

$$r_n = \sum_{k=0}^{n} \binom{n}{k} \sum_{j=0}^{k} \binom{k}{j} \frac{\alpha_j \alpha_{k-j}}{\alpha_k} \Delta^j t_0 \Delta^{k-j} s_0,$$

and thereby arrive at (30). □

Let $\mathcal{T}_\alpha e^{ax} = \{s_n\}$, $\mathcal{T}_\alpha f(x) = \{t_n\}$ and $\alpha_k = (-a)^k$. Using the results from Theorem 5, we obtain

$$\begin{aligned} e^{ax} f(x) = \mathcal{B}_\alpha\{s_n\}\mathcal{B}_\alpha\{t_n\} = \mathcal{B}_\alpha\{s_n *_\alpha t_n\} &= \sum_{k=0}^{\infty}\sum_{j=0}^{k} \frac{(e^{ax})^{(k-j)}\big|_{x=0} f^{(j)}(0)}{j!(k-j)!} x^k \\ &= \sum_{k=0}^{\infty} \frac{x^k}{k!} \sum_{j=0}^{k} \binom{k}{j} a^{k-j} f^{(j)}(0) = \sum_{k=0}^{\infty} \frac{x^k}{k!} a^k \sum_{j=0}^{k} \frac{1}{a^j} \binom{k}{j} f^{(j)}(0) \\ &= \sum_{k=0}^{\infty} t_k \frac{(ax)^k}{k!}, \end{aligned}$$

which is the Borel transform of $e^{ax} f(x)$ (see [**5**]), whence we get

$$f(x) = e^{-ax} \sum_{k=0}^{\infty} t_k \frac{(ax)^k}{k!}.$$

Example 4. *Let $\{t_n\}$ be the Fibonacci sequence $\{F_n\}$. Denote $g(x) = \dfrac{x}{1-x-x^2}$. It is the generating function of F_n, i.e. $g(x) = \sum_{n=0}^{\infty} F_n x^n$. As a Borel transform of $g(x)$ we obtain*

$$e^{-x} \sum_{n=0}^{\infty} \frac{F_n}{n!} x^n = \frac{e^{-\frac{x}{2}\left(1+\sqrt{5}\right)} \left(e^{\sqrt{5}x} - 1\right)}{\sqrt{5}}.$$

which is the same function obtained in Example 3 *by $\mathcal{B}_\alpha$-transform of $\{F_n\}$.*

Making use of the transforms $\mathcal{T}_\alpha$ and $\mathcal{B}_\alpha$ enables us to map Bessel's differential operator $\frac{d}{dx} x \frac{d}{dx}$ (see [**3**]) to the discrete Bessel operator.

Theorem 6. *$\mathcal{T}_\alpha$-transform maps Bessel's operator $\frac{d}{dx} x \frac{d}{dx}$ to the operator $\mathcal{D}_\alpha n \mathcal{N}_\alpha$, where the operator $\mathcal{N}_\alpha$ is introduced by the relation $\mathcal{N}_\alpha t_n = \mathcal{D}_\alpha t_{n-1}$.*

Proof. Let $\mathcal{T}_\alpha f(x) = \{t_n\}$. Relying on the statement 1° of Theorem 2, we have

$$\mathcal{T}_\alpha f'(x) = \{\mathcal{D}_\alpha t_n\}.$$

According to Lemma 3, we have

$$\mathcal{T}_\alpha x f'(x) = \{n\mathcal{D}_\alpha t_{n-1}\}. \tag{32}$$

Denote $g(x) = xf'(x)$. Then, because of (32), $\mathcal{T}_\alpha g(x) = \{n\mathcal{D}_\alpha t_{n-1}\}$, and using again the statement 1° of Theorem 2, we get

$$\mathcal{T}_\alpha g'(x) = \{\mathcal{D}_\alpha n\mathcal{D}_\alpha t_{n-1}\}.$$

Thus

$$\mathcal{T}_\alpha \frac{d}{dx} x \frac{d}{dx} f(x) = \{\mathcal{D}_\alpha n\mathcal{D}_\alpha t_{n-1}\}. \tag{33}$$

Let us take $\alpha_k = (-1)^k$. In this case the operator $\mathcal{D}_\alpha$ becomes Δ, and (33) yields

$$\mathcal{T}_\alpha \frac{d}{dx} x \frac{d}{dx} f(x) = \{\Delta n\Delta t_{n-1}\} = \{\Delta n(t_n - t_{n-1})\} = \{\Delta n\nabla t_n\},$$

i.e. $\mathcal{T}_\alpha$-transform maps Bessel's operator $\frac{d}{dx}x\frac{d}{dx}$ to the operator $\Delta n\nabla$. Since $\Delta t_{n-1} = \nabla t_n$, it reasonable to have a new operator corresponding to ∇. Thus we define $\mathcal{D}_\alpha t_{n-1} = \mathcal{N}_\alpha t_n$, and now consider $\mathcal{D}_\alpha n\mathcal{N}_\alpha$ the generalized discrete Bessel's operator. □

Example 5. *Laguerre-type exponential function, L-exponential function for short, is defined by the series* (see [**6, 7, 8**])

$$e_1(x) = \sum_{k=0}^{\infty} \frac{x^k}{(k!)^2}.$$

Considering that Bessel's differential operator is linear, we find

$$\frac{d}{dx} x \frac{d}{dx} e_1(x) = e_1(x), \tag{34}$$

which means that L-exponential function is its eigenvector.

Let $\mathcal{T}_\alpha e_1(x) = \{s_n\}$. *By virtue of the uniform convergence of the left-hand series everywhere, we are allowed to exchange summation and differentiation, and find*

$$\begin{aligned} s_n &= \sum_{k=0}^{n} \frac{(-1)^k}{\alpha_k}\binom{n}{k} e_1^{(k)}(0) = \sum_{k=0}^{n} \frac{(-1)^k}{\alpha_k}\binom{n}{k} \frac{k^{(k)}}{(k!)^2} \\ &= \sum_{k=0}^{n} \frac{(-1)^k}{\alpha_k k!}\binom{n}{k} = \sum_{k=0}^{n} \frac{(-1)^k n^{(k)}}{(k!)^2 \alpha_k}. \end{aligned} \tag{35}$$

Applying $\mathcal{T}_\alpha$ *to* (34), *because of* (33), *we have*

$$\mathcal{T}_\alpha \frac{d}{dx} x \frac{d}{dx} e_1(x) = \{\mathcal{D}_\alpha n\mathcal{D}_\alpha s_{n-1}\} = \{\mathcal{D}_\alpha n\mathcal{N}_\alpha s_n\} = \mathcal{T}_\alpha e_1(x) = \{s_n\}.$$

In other words

$$\mathcal{D}_\alpha n \mathcal{N}_\alpha s_n = s_n. \tag{36}$$

So, s_n is the eigenvector of the generalized discrete Bessel operator $\mathcal{D}_\alpha n \mathcal{N}_\alpha$.

As special case, for $\alpha_k = (-1)^k$, applying $\mathcal{T}_\alpha$ to (34)*, because of* (33)*, we have*

$$\{\Delta n \Delta s_{n-1}\} = \{\Delta n \nabla s_n\} = \mathcal{T} \frac{d}{dx} x \frac{d}{dx} e_1(x) = \mathcal{T} e_1(x) = \{s_n\}.$$

In other words

$$\Delta n \nabla s_n = s_n. \tag{37}$$

So, s_n is the eigenvector of the discrete Bessel operator $\Delta n \nabla$.

REFERENCES

1. M. Abramowitz I. A. Stegun: *Handbook of Mathematical Functions, with Formulas, Graphs and Mathematical Tables*, New York, Dover, 1972.
2. W.A. Al-Salam, M.E. H. Ismail: *A familiy of operational calculi*, Mahtematica Japonica **22** (1978) 571-583.
3. H. Batamen, A. Erdélyi: *Higher transcendental function, vol. 2*, New York Toronto London, Mc Graw-Hill Company, 1953.
4. M. Bernstein, N. J. A. Sloane: *Some Canonical Sequences of Integers*, Lin. Alg. Appl. **226-228** (1995) 57-72.
5. E. Borel: *Mémoire sur les séries divergentes*, Annales scientifiques de l'École Normale Supérieure, Série 3 :**16** (1899) , p. 9-131.
6. G. Dattoli, A. Arena, P.E. Ricci: *Laguerrian eigenvalue problems and Wright functions*, Math. Comp. Model. **40** (7/8) (2004) 877-881.
7. G. Dattoli, M.X. He, P.E. Ricci: *Eigenfunctions of Laguerre-type Operators and Generalized Evolution Problems*, Wiley, Math. Comp. Model, **42** (2005) 1263-1268.
8. G. Dattoli, P.E. Ricci: *Laguerre-type exponentials and relevant L-circular and L-hyperbolic functions*, Georgian Math. J. **10** (7/8) (2003) 481-494.
9. M.Z. Spivey, L.L. Steil: *The k-Binomial transform and Hankel Transform*, J. Int. Seq. **9** (2006) Article 06.1.1
10. S.B. Tričković, M.V. Vidanović, M.S. Stanković: *On the Summation of Series in Terms of Bessel Functions* Zeitsch. Anal. Anw. Vol. 25 No. 3 (2006) 393-406.

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