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CONVOLUTIONS OF THE GENERALIZED JACOBSTHAL AND GENERALIZED LUCAS NUMBERS

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In this paper we consider the generalized Jacobsthal $J_{n,m}$ and the generalized Jacobsthal–Lucas numbers $j_{n,m}$. Also, we introduce new sequences of numbers $\mathcal{A}_{n,m}$, $\mathcal{B}_{n,m}$, $\mathcal{C}_{n,m}$ and $\mathcal{D}_{n,m}$. Namely, these new sequences are convolutions of the sequences $J_{n,m}$ and $j_{n,m}$. Further, we find the generating functions and some recurrence relations for these sequences of numbers.

1. INTRODUCTION

At first, we remind to [**5**], where the generalized Fibonacci polynomials $f_n^m(\lambda)$ and the generalized Lucas polynomials $l_n^m(\lambda)$ were introduced and studied. Also, in [**5**], the incomplete polynomials of the Fibonacci type $f_{r,n}(\lambda)$ and the incomplete polynomials of the Lucas type $l_{r,n}(\lambda)$ were defined and studied. Furthermore, in [**5**], the corresponding incomplete polynomials $F_{r,n}^m(\lambda)$ and $L_{r,n}^m(\lambda)$ were investigated.

The paper [**6**] is very interesting. In [**6**] the author obtains a closed–form expression for the sum of the elements lying on the nth diagonal of a Fibonacci triangle. Thereby the author utilizes the ordinary generating functions of two subsequences of the sequence of diagonal sums.

Next, in [**10**], the new family of the q–Fibonacci polynomials – $F_n^k(x, s, q)$, as well as the new family of the q–Lucas polynomials – $F_n^k(x, s, q)$ were introduced. Further, the authors [**10**] give several properties and generating functions of each of these families.

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Two interesting classes of polynomials were considered in papers [**1**] and [**2**]: the generalized Jacobsthal $J_{n,m}(x)$ and the generalized Jacobsthal–Lucas polynomials $j_{n,m}(x)$. These polynomials are given by the following recurrence relations ([**1, 2**]):

$$J_{n,m}(x) = J_{n-1,m}(x) + 2xJ_{n-m,m}(x), \;\; n \geq m, \;\; m \geq 2, \tag{1}$$

with $J_{0,m}(x) = 0, \;\; J_{n,m}(x) = 1, \;\; n = 1, \ldots, m-1;$

$$j_{n,m}(x) = j_{n-1,m}(x) + 2xj_{n-m,m}(x), \;\; n \geq m, \;\; m \geq 2,$$

with $j_{0,m}(x) = 2, \;\; j_{n,m}(x) = 1, \;\; n = 1, \ldots, m-1.$

The corresponding generating functions for these polynomials are given by

$$G_m^J(x,t) = \frac{1}{1-t-2xt^m} = \sum_{n=1}^{\infty} J_{n,m}(x)\, t^{n-1}, \tag{2}$$

$$g_m^j(x,t) = \frac{1+4xt^{m-1}}{1-t-2xt^m} = \sum_{n=1}^{\infty} j_{n,m}(x)\, t^{n-1}. \tag{3}$$

Explicit formulas for above mentioned polynomials are given by the following formulas:

$$J_{n,m}(x) = \sum_{k=0}^{[(n-1)/m]} \binom{n-1-(m-1)k}{k} (2x)^k, \tag{4}$$

$$j_{n,m}(x) = \sum_{k=0}^{[n/m]} \frac{n-(m-2)k}{n-(m-1)k} \binom{n-(m-1)k}{k} (2x)^k. \tag{5}$$

Remark 1. *The generalized Jacobsthal polynomials $J_{n,m}(x)$ are the special case of the Humbert polynomials, i.e.,*

$$J_{n,m}(x) = P_n(m, x/m, -2, -1, 1),$$

where (see [**3, 8**]*)*

$$\sum_{n=0}^{\infty} P_n(m,x,y,p,C)\, t^n = (C - mxt + yt^m)^p \,.$$

For $x = 1$, from (4) and (5), we obtain the generalized Jacobsthal numbers (see [**4**]*)*

$$J_{n,m} = J_{n,m}(1) = \sum_{k=0}^{[(n-1)/m]} \binom{n-1-(m-1)k}{k} 2^k$$

and the generalized Jacobsthal – Lucas numbers

$$j_{n,m} = j_{n,m}(1) = \sum_{k=0}^{[n/m]} \frac{n-(m-2)k}{n-(m-1)k} \binom{n-(m-1)k}{k} 2^k.$$

Having in view the relation (2), we get the corresponding generating function of the numbers $J_{n,m}$:

$$G_m^J(1,t) = \frac{1}{1-t-2t^m}.$$

Similarly, having in view the relation (3), we obtain the corresponding generating function of the numbers $j_{n,m}$:

$$g_m^j(1,t) = \frac{1+4t^{m-1}}{1-t-2t^m}.$$

Remark 2. *The particular cases of the numbers $J_{n,m}$ and $j_{n,m}$ are so–called Jacobsthal numbers $J_n = J_{n,2}$ and the Jacobsthal–Lucas numbers $j_n = j_{n,2}$, which were investigated earlier by Horadam* [**7**]. *(See also a systematic investigation by Raina and Srivastava* [**9**]*, dealing with an interesting class of numbers associated with the familiar Lucas numbers).*

Motivated by [**1, 2, 5, 6, 9, 10**], in this paper we introduce and we consider some new sequences of numbers which are convolutions of the generalized Jacobsthal $J_{n,m}$ and the generalized Jacobsthal–Lucas numbers $j_{n,m}$.

At first, we find some initial members of the sequences $J_{n,m}$ and $j_{n,m}$.

Table 1:

n	$J_{n,2}$	$j_{n,2}$	$J_{n,3}$	$j_{n,3}$	$J_{n,4}$	$j_{n,4}$
0	0	2	0	2	0	2
1	1	1	1	1	1	1
2	1	5	1	1	1	1
3	3	7	1	5	1	1
4	5	17	3	7	1	5
5	11	31	5	9	3	7
6	21	65	7	19	5	9
7	43	127	13	33	7	11
8	85	257	23	51	9	21
9	171	511	37	89	15	35
10	341	1025	63	155	25	53
11	683	2047	109	257	39	75
12	1365	4097	183	435	57	117

Table 2:

n	$J_{n,m}$	$j_{n,m}$
0	0	2
1	1	1
2	1	1
3	1	1
...	...	...
$m-1$	1	1
m	1	5
$m+1$	3	7
$m+2$	5	9
...	...	...
$2m$	$2m+1$	$2m+13$
$2m+1$	$2m+7$	$2m+27$
$2m+2$	$2m+17$	$2m+45$

We can easily prove the following relations

$$\begin{aligned} j_{n,m} &= J_{n,m} + 4J_{n+1-m,m} \\ &= J_{n+1,m} + 2J_{n+1-m,m}. \end{aligned} \tag{6}$$

2. SOME PROPERTIES OF SEQUENCES OF NUMBERS $J_{N,M}$ AND $J_{N,M}$

In this section we are going to prove the following statement.

Theorem 1. *For $n \geq m \geq 2$ and $s \geq m-2$, the generalized Jacobsthal numbers $J_{n,m}$ satisfy the following quadratic relation*

$$J_{n+s,m} = J_{1+s,m} J_{n,m} + 2\sum_{\nu=2}^{m} J_{\nu+s-m,m} J_{n+1-\nu,m}. \tag{7}$$

Proof. We start from

$$\begin{aligned} (1-t-2t^m)\sum_{n=1}^{\infty} J_{n+s,m}\, t^n &= \sum_{n=1}^{\infty} J_{n+s,m}\, t^n - \sum_{n=1}^{\infty} J_{n+s,m}\, t^{n+1} - 2\sum_{n=1}^{\infty} J_{n+s,m}\, t^{n+m} \\ &= \sum_{n=1}^{\infty} J_{n+s,m}\, t^n - \sum_{n=2}^{\infty} J_{n-1+s,m}\, t^n - 2\sum_{n=m+1}^{\infty} J_{n-m+s,m}\, t^n \\ &= \sum_{\nu=1}^{\infty} A_\nu\, t^\nu, \end{aligned}$$

where

$$A_\nu = \begin{cases} J_{1+s,m}, & \nu = 1, \\ J_{\nu+s,m} - J_{\nu-1+s,m}, & \nu = 2,\dots,m, \\ J_{\nu+s,m} - J_{\nu-1+s,m} - 2J_{\nu-m+s,m}, & \nu \geq m+1, \end{cases}$$

i.e.,

$$A_\nu = \begin{cases} J_{1+s,m}, & \nu = 1, \\ 2J_{\nu+s-m,m}, & \nu = 2,\dots,m, \\ 0, & \nu \geq m+1, \end{cases} \tag{8}$$

because of the recurrence relation (1). Then, using the generating function (2) for $x = 1$, we have

$$\sum_{n=1}^{\infty} J_{n+s,m}\, t^n = \frac{1}{1-t-2t^m}\left(\sum_{\nu=1}^{\infty} A_\nu\, t^\nu\right) = \left(\sum_{k=1}^{\infty} J_{k,m}\, t^{k-1}\right)\left(\sum_{\nu=1}^{\infty} A_\nu\, t^\nu\right),$$

i.e.,

$$\sum_{n=1}^{\infty} J_{n+s,m}\, t^n = \sum_{n=1}^{\infty}\left(\sum_{\nu=1}^{n} J_{n+1-\nu,m} A_\nu\right) t^n,$$

from which we obtain

$$J_{n+s,m} = \sum_{\nu=1}^{n} J_{n+1-\nu,m} A_\nu.$$

According to (8), we finally get

$$J_{n+s,m} = J_{1+s,m} J_{n,m} + 2\sum_{\nu=2}^{m} J_{\nu+s-m,m} J_{n+1-\nu,m}.$$

□

Corollary 1. *The Jacobsthal numbers $J_{n,m}$ for $m = 2$ and $m = 3$ satisfy the relations*

$$J_{n+s,2} = J_{1+s,2} J_{n,2} + 2J_{s,2} J_{n-1,2}$$

and

$$J_{n+s,3} = J_{1+s,3} J_{n,3} + 2J_{s-1,3} J_{n-1,3} + 2J_{s,3} J_{n-2,3},$$

respectively.

Example 1. *From Table 1 and by (7), we find:*

$$J_{5+3,3} = J_{4,3} J_{5,3} + 2J_{2,3} J_{4,3} + 2J_{3,3} J_{3,3} = 23.$$
$$J_{6+3,3} = J_{4,3} J_{6,3} + 2J_{2,3} J_{5,3} + 2J_{3,3} J_{4,3} = 37.$$
$$J_{5+3,4} = J_{4,4} J_{5,4} + 2J_{1,4} J_{4,4} + 2J_{2,4} J_{3,4} + 2J_{3,4} J_{2,4} = 9.$$

Theorem 2. *Jacobsthal–Lucas numbers $j_{n,m}$ satisfy the following relation*

$$j_{n+s,m} = J_{1+s,m} \cdot j_{n,m} + 2J_{s+2-m,m} \cdot j_{n-1,m} + \cdots + 2J_{s,m} \cdot j_{n+1-m,m}. \tag{9}$$

Proof. Using relations (7) and (6), we arrive to (9). □

Example 2. *From Table 1, using (9), we get*

$$j_{4+3,3} = J_{4,3} \cdot j_{4,3} + 2J_{2,3} \cdot j_{3,3} + 2J_{3,3} \cdot j_{2,3} = 33.$$

3. NEW SEQUENCES OF NUMBERS

In this section we introduce and we consider new sequences of numbers: $\mathcal{A}_{n,m}$, $\mathcal{B}_{n,m}$, $\mathcal{C}_{n,m}$ and $\mathcal{D}_{n,m}$. For these sequences we find generating functions and some interesting properties.

Firstly we define the sequence $\mathcal{A}_{n,m}$.

Definition 1. *For $m \geq 2$, the sequence of numbers $\mathcal{A}_{n,m}$ is given by*

$$\mathcal{A}_{n,m} = \sum_{i=1}^{[(n+m-1)/m]} J_{i,m} J_{n-m(i-1),m}, \tag{10}$$

where $n = mp + l$, $l = 0, 1, \ldots, m-1$.

Now we are going to prove the following

Theorem 3. *For the sequence of numbers $\mathcal{A}_{n,m}$ the following formulas are correct*

$$\mathcal{A}_{n,m} = \begin{cases} \mathcal{A}_{n-1,m} + 2\mathcal{A}_{n-m,m}, & l \neq 1, \\ \mathcal{A}_{n-1,m} + 2\mathcal{A}_{n-m,m} + J_{p+1,m}, & l = 1. \end{cases} \tag{11}$$

Proof. Using the formula (10), we get:

1° let $l = 0$, i.e., $n = mp$, then

$$\begin{aligned} \mathcal{A}_{n,m} &= J_{1,m}J_{n,m} + J_{2,m}J_{n-m,m} + \cdots + J_{p,m}J_{n-m(p-1),m} \\ \mathcal{A}_{n-1,m} &= J_{1,m}J_{n-1,m} + J_{2,m}J_{n-1-m,m} + \cdots + J_{p,m}J_{n-1-m(p-1),m}, \\ \mathcal{A}_{n-m,m} &= J_{1,m}J_{n-m,m} + J_{2,m}J_{n-2m,m} + \cdots + J_{p-1,m}J_{n-m-m(p-2),m}. \end{aligned}$$

So we find that

$$\begin{aligned} \mathcal{A}_{n-1,m} + 2\mathcal{A}_{n-m,m} &= J_{1,m}J_{n,m} + J_{2,m}J_{n-m,m} \\ &\quad + \cdots + J_{p-1,m}J_{2m,m} + J_{p,m}J_{m-1,m} = \mathcal{A}_{n,m}, \end{aligned}$$

where $J_{m-1,m} = J_{m,m} = 1$.

2° For $l = 1$, we get:

$$\begin{aligned} \mathcal{A}_{n,m} &= J_{1,m}J_{n,m} + J_{2,m}J_{n-m,m} + \cdots + J_{p,m}J_{n-m(p-1),m} + J_{p+1,m}J_{n-mp,m}, \\ \mathcal{A}_{n-1,m} &= J_{1,m}J_{n-1,m} + J_{2,m}J_{n-1-m,m} + \cdots + J_{p,m}J_{n-1-m(p-1),m}, \\ \mathcal{A}_{n-m,m} &= J_{1,m}J_{n-m,m} + J_{2,m}J_{n-2m,m} + \cdots + J_{p,m}J_{n-m-m(p-1),m}, \end{aligned}$$

hence

$$\mathcal{A}_{n,m} = \mathcal{A}_{n-1,m} + 2\mathcal{A}_{n-m,m} + J_{p+1,m}.$$

3° If $l \geq 2$, then

$$\begin{aligned} \mathcal{A}_{n,m} &= J_{1,m}J_{n,m} + J_{2,m}J_{n-m,m} + \cdots + J_{p+1,m}J_{n-mp,m}, \\ \mathcal{A}_{n-1,m} &= J_{1,m}J_{n-1,m} + J_{2,m}J_{n-1-m,m} + \cdots + J_{p+1,m}J_{n-1-mp,m}, \\ \mathcal{A}_{n-m,m} &= J_{1,m}J_{n-m,m} + J_{2,m}J_{n-2m,m} + \cdots + J_{p,m}J_{n-m-m(p-1),m}, \end{aligned}$$

hence

$$\mathcal{A}_{n,m} = \mathcal{A}_{n-1,m} + 2\mathcal{A}_{n-m,m}.$$

This proves the formulas (11). □

Example 3. *Using Table 1 and by (10), we get:*

$$\begin{aligned} \mathcal{A}_{6,3} &= J_{1,3}J_{6,3} + J_{2,3}J_{3,3} = 8, \\ \mathcal{A}_{5,3} &= J_{1,3}J_{5,3} + J_{2,3}J_{2,3}J_{2,3} = 6, \\ \mathcal{A}_{3,3} &= J_{1,3}J_{3,3} = 1, \\ \mathcal{A}_{6,3} &= \mathcal{A}_{5,3} + 2\mathcal{A}_{3,3}. \end{aligned}$$

Example 4. *Again, from Table 1 and by (10), we get*

$$\begin{aligned} &\mathcal{A}_{7,3} = 17, \;\; \mathcal{A}_{6,3} = 8, \;\; \mathcal{A}_{4,3} = J_{1,3}J_{4,3} + J_{2,3}J_{1,3} = 4, \\ &\mathcal{A}_{7,3} = \mathcal{A}_{6,3} + 2\mathcal{A}_{4,3} + 1. \end{aligned}$$

In the next step we find the generating function of the series of numbers $\mathcal{A}_{n,m}$.

Theorem 4. *The generating function of $\mathcal{A}_{n,m}$ is given by*

$$G_m^{\mathcal{A}}(t) = \frac{t}{(1-t-2t^m)(1-t^m-2t^{m^2})} = \sum_{n=1}^{\infty} \mathcal{A}_{n,m}\, t^n. \tag{12}$$

Proof.

$$\begin{aligned}
G_m^{\mathcal{A}}(t) &= \sum_{n=1}^{\infty} \mathcal{A}_{n,m}\, t^n \\
&= \mathcal{A}_{1,m} t + \mathcal{A}_{2,m} t^2 + \mathcal{A}_{3,m} t^3 + \cdots + \mathcal{A}_{m,m} t^m + \mathcal{A}_{m+1,m}\, t^{m+1} + \ldots \\
&= (J_{1,m}J_{1,m})t + (J_{1,m}J_{2,m})t^2 + (J_{1,m}J_{3,m})t^3 + \cdots + (J_{1,m}J_{m,m})t^m \\
&\quad + (J_{1,m}J_{m+1,m} + J_{2,m}J_{1,m})\, t^{m+1} + \ldots \\
&= (J_{1,m} + J_{2,m}t + J_{3,m}t^2 + \cdots + J_{m,m}t^{m-1} + J_{m+1,m}t^m \ldots)\cdot \\
&\quad\ (J_{1,m}t + J_{2,m}t^{m+1} + J_{3,m}t^{2m+1} + \ldots) \\
&= \frac{1}{1-t-2t^m} \cdot \frac{t}{1-t^m-2t^{m^2}} = \frac{t}{(1-t-2t^m)(1-t^m-2t^{m^2})}.
\end{aligned}$$

□

We can see that

$$G_m^{\mathcal{A}}(t) = t \cdot G_m^J(1,t) \cdot G_m^J(1,t^m).$$

Corollary 2. *For $m = 2$ in (12), we get*

$$G_2^{\mathcal{A}}(t) = \frac{t}{(1-t-2t^2)(1-t^2-2t^4)},$$

which presents the generating function of the sequence $\mathcal{A}_{n,2}$.

Definition 2. *A convolution of the numbers $J_{n,m}$ and $j_{n,m}$ is given by*

$$\mathcal{B}_{n,m} = \sum_{i=1}^{[(n+m-1)/m]} J_{i,m} j_{n-m(i-1),m}, \tag{13}$$

where $n = mp + l,\ \ l = 0, 1, \ldots, m-1$.

Theorem 5. *For the series of numbers $\mathcal{B}_{n,m}$ the following relations are correct*

$$\mathcal{B}_{n,m} = \begin{cases} \mathcal{B}_{n-1,m} + 2\mathcal{B}_{n-m,m} + 4J_{p,m},\ n = mp, \\ \mathcal{B}_{n-1,m} + 2\mathcal{B}_{n-m,m} + J_{p+1,m},\ n = mp+1, \\ \mathcal{B}_{n-1,m} + 2\mathcal{B}_{n-m,m},\ n = mp + l,\ l \geq 2. \end{cases} \tag{14}$$

Proof. The proof of the formula (14) is similar to the proof of the formula (11). □

Example 5. *From Table 1 and by (13), we get:*

$$\begin{aligned}
&\mathcal{B}_{6,3} = J_{1,3}j_{6,3} + J_{2,3}j_{3,3} = 24,\\
&\mathcal{B}_{5,3} = J_{1,3}j_{5,3} + J_{2,3}j_{2,3} = 10,\\
&\mathcal{B}_{3,3} = J_{1,3}j_{3,3} = 5,\\
&\mathcal{B}_{6,3} = \mathcal{B}_{5,3} + 2\mathcal{B}_{3,3} + 4 \cdot J_{2,3}.\\
&\mathcal{B}_{7,3} = 41,\\
&\mathcal{B}_{4,3} = 8,\\
&\mathcal{B}_{7,3} = \mathcal{B}_{6,3} + 2\mathcal{B}_{4,3} + J_{3,3}.\\
&\mathcal{B}_{8,3} = 61 = \mathcal{B}_{7,3} + 2\mathcal{B}_{5,3}.
\end{aligned}$$

Theorem 6. *The generating function of the sequence $\mathcal{B}_{n,m}$ is given by*

$$G_m^{\mathcal{B}}(t) = \sum_{n=1}^{\infty} \mathcal{B}_{n,m}\, t^n = \frac{t(1+4t^{m-1})}{(1-t-2t^m)(1-t^m-2t^{m^2})}.$$

Proof. Since

$$\begin{aligned}
\sum_{n=1}^{\infty}\mathcal{B}_{n,m}\, t^n &= \mathcal{B}_{1,m}t + \mathcal{B}_{2,m}t^2 + \cdots + \mathcal{B}_{m,m}t^m + \mathcal{B}_{m+1,m}t^{m+1} + \dots\\
&= (J_{1,m}j_{1,m})t + (J_{1,}j_{2,m})t^2 + \cdots + (J_{1,m}j_{m,m})t^m\\
&\quad + (J_{1,m}j_{m+1,m} + J_{2,m}j_{1,m})t^{m+1} + \dots\\
&= \big(j_{1,m} + j_{2,m}t + j_{3,m}t^2 + \cdots + j_{m,m}t^{m-1} + j_{m+1,m}t^m + \dots\big)\\
&\quad \cdot \big(J_{1,m}t + J_{2,m}t^{m+1} + J_{3,m}t^{2m+1} + J_{4,m}t^{3m+1} + \dots\big),
\end{aligned}$$

hence

$$\begin{aligned}
G_m^{\mathcal{B}}(t) &= \frac{1+4t^{m-1}}{1-t-2t^m} \cdot \frac{t}{1-t^m-2t^{m^2}}\\
&= \frac{t(1+4t^{m-1})}{(1-t-2t^m)(1-t^m-2t^{m^2})}.
\end{aligned}$$

□

We can notice that

$$G_m^{\mathcal{B}}(t) = t \cdot G_m^J(1,t^m) \cdot g_m^j(1,t).$$

Finally, we introduce two more sets of numbers, $\mathcal{C}_{n,m}$ and $\mathcal{D}_{n,m}$. For these sets of numbers we find recurrent relations and functions of generatrices. We also give a number of examples to illustrate the properties which are found. For these sequences we give some properties without proofs.

On this occasion, we give results without proofs, because they are similar to those related to sequences $\mathcal{A}_{n,m}$ and $\mathcal{B}_{n,m}$.

Definition 3. *A convolution of $j_{n,m}$ and $J_{n,m}$ is given by*

$$\mathcal{C}_{n,m} = \sum_{i=1}^{[(n+m-1)/m]} j_{i,m} J_{n-m(i-1),m}, \tag{15}$$

where $n = mp + l,\ \ l = 0, 1, \ldots, m-1$.

Theorem 7. *For the sequence of numbers $\mathcal{C}_{n,m}$ the following relations hold*

$$\mathcal{C}_{n,m} = \begin{cases} \mathcal{C}_{n-1,m} + 2\mathcal{C}_{n-m,m}, & l \neq 1, \\ \mathcal{C}_{n-1,m} + 2\mathcal{C}_{n-m,m} + j_{p+1,m}, & l = 1. \end{cases}$$

Example 6. *From (15) and using Table 1, we find:*

$$\begin{aligned} &\mathcal{C}_{6,4} = j_{1,4}J_{6,4} + j_{2,4}J_{2,4} = 8, \\ &\mathcal{C}_{5,4} = j_{1,4}J_{5,4} + j_{2,4}J_{1,4} = 4, \\ &\mathcal{C}_{2,4} = j_{1,4}J_{2,4} = 1, \ \textit{then} \\ &\mathcal{C}_{6,4} = \mathcal{C}_{5,4} + 2\mathcal{C}_{2,4} = 6. \end{aligned}$$

The generating function of the sequence $\mathcal{C}_{n,m}$ is given by

$$G_m^{\mathcal{C}}(t) = \sum_{n=1}^{\infty} \mathcal{C}_{n,m} t^n = \frac{t\left(1 + 4t^{m(m-1)}\right)}{(1 - t - 2t^m)\left(1 - t^m - 2t^{m^2}\right)}. \tag{16}$$

Now we can see that

$$G_m^{\mathcal{C}}(t) = t \cdot G_m^J(1,t) \cdot g_m^j(1,t^m).$$

Corollary 3. *For $m = 2$, the relation (16) becomes*

$$G_2^{\mathcal{C}}(t) = \frac{t\left(1 + 4t^2\right)}{(1 - t - 2t^2)\,(1 - t^2 - 2t^4)}$$

Definition 4. *The sequence of numbers $\mathcal{D}_{n,m}$ is defined by*

$$\mathcal{D}_{n,m} = \sum_{n=1}^{(n+m-1)/m]} j_{i,m} j_{n-m(i-1),m}, \tag{17}$$

where $n = mp + l,\ \ l = 0, 1, \ldots, m-1$.

We see that $\mathcal{D}_{n,m}$ is a convolution of Jacobsthal–Lucas numbers $j_{n,m}$.

Theorem 8. *The sequence of numbers $\mathcal{D}_{n,m}$ satisfies the following relations*

$$\mathcal{D}_{n,m} = \begin{cases} \mathcal{D}_{n-1,m} + 2\mathcal{D}_{n-m,m} + 4j_{p,m}, & l = 0, \\ \mathcal{D}_{n-1,m} + 2\mathcal{D}_{n-m,m} + j_{p+1,m}, & l = 1, \\ \mathcal{D}_{n-1,m} + 2\mathcal{D}_{n-m,m}, & l \geq 2. \end{cases}$$

Example 7. *Using (17) and from Table 1, we find:*

$$\begin{aligned}\mathcal{D}_{6,3} &= j_{1,3}j_{6,3} + j_{2,3}j_{3,3} = 24,\\ \mathcal{D}_{5,3} &= j_{1,3}j_{5,3} + j_{2,3}j_{2,3} = 10,\\ \mathcal{D}_{3,3} &= j_{1,3}j_{3,3} = 5, \text{ then}\\ \mathcal{D}_{6,3} &= \mathcal{D}_{5,3} + 2\mathcal{D}_{3,3} + 4 \cdot j_{2,3}.\end{aligned}$$

Finally, we give the following statement.

Theorem 9. *The generating function of the sequence* $\mathcal{D}_{n,m}$ *is*

$$G_m^{\mathcal{D}}(t) = \sum_{n=1}^{\infty} \mathcal{D}_{n,m} t^n = \frac{t\left(1+4t^{m-1}\right)\left(1+4t^{m(m-1)}\right)}{\left(1-t-2t^m\right)\left(1-t^m-2t^{m^2}\right)}. \tag{18}$$

We can easily realize that the next relation holds:

$$G_m^{\mathcal{D}}(t) = t \cdot g_m^j(1,t) \cdot g_m^j(1,t^m).$$

Corollary 4. *For* $m = 2$*, the relation (18) becomes*

$$G_2^{\mathcal{D}}(t) = \frac{t(1+4t)\left(1+4t^2\right)}{\left(1-t-2t^2\right)\left(1-t^2-2t^4\right)}.$$

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