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SOME NEW PROPERTIES OF THE BARNES G-FUNCTION AND RELATED RESULTS

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In this paper, we present several potentially useful properties of the Barnes G-function. The properties considered here include, for example, its integral representation, complete monotonicity, and continued-fraction approximation. We also derive continued-fraction approximations of the Glaisher-Kinkelin constant and the Choi-Srivastava constants.

1. INTRODUCTION AND MOTIVATION

The double Gamma function Γ_2 and the multiple Gamma functions Γ_n $(n \in \mathbb{N} \setminus \{1,2\})$ were introduced and investigated by Barnes in a series of papers (see, for example, [**2, 3, 4, 5**]), $\mathbb{N}$ being the set of positive integers. Barnes applied these functions in the theories of elliptic functions and theta functions. Nonetheless, except possibly for the citations of Γ_2 in the exercises by Whittaker and Watson [**46**, p. 264] and also by Gradshteyn and Ryzhik [**27**, p. 661, Entry 6.441 (4); p. 937, Entry 8.333], these functions were revived only in about the middle of the 1980s in the study of determinants of the Laplacians on the n-dimensional unit sphere S^n (see, e. g., [**18, 31, 34, 37, 44, 45**]). The theory of the double Gamma function Γ_2 has indeed found interesting applicationsin many other recent investigations (see, for details, the monographs by Srivastava and Choi [**41, 42**]).

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Barnes [**2**] defined the double Gamma function Γ_2 (or the Barnes G-function) $\Gamma_2 = 1/G$ satisfying each of the following properties:

(i) $G(z+1) = \Gamma(z)G(z)$ for all complex z;

(ii) $G(1) = 1$;

(iii) In the limit when $n \to \infty$,

$$\ln G(z+n+2) = \frac{n+1+z}{2}\ln(2\pi) + \left(\frac{n^2}{2} + n + \frac{5}{12} + \frac{z^2}{2} + (n+1)z\right)\ln n - \frac{3n^2}{4} - n - nz - \ln A + \frac{1}{12} + O\left(\frac{1}{n}\right) \qquad (n \to \infty), \tag{1}$$

where Γ is the (Euler's) Gamma function and A is the Glaisher-Kinkelin constant defined by

$$\ln A = \lim_{n\to\infty}\left\{\sum_{k=1}^{n} k\ln k - \left(\frac{n^2}{2} + \frac{n}{2} + \frac{1}{12}\right)\ln n + \frac{n^2}{4}\right\}, \tag{2}$$

the numerical value of A being $1.28242713\cdots$.

The Glaisher-Kinkelin constant A can be expressed as follows (see [**25**]):

$$A = \lim_{n\to\infty} n^{-(n^2/2)-(n/2)-(1/12)}\; e^{n^2/4} \prod_{k=1}^{n} k^k, \tag{3}$$

$$\frac{e^{\frac{1}{12}}}{A} = \lim_{n\to\infty}\frac{G(n+1)}{n^{(n^2/2)-(1/12)}\;(2\pi)^{n/2}\;e^{-3n^2/4}} \tag{4}$$

and (see [**19**, p. 129, Eq. (3.22)])

$$A = e^{\frac{1}{12}-\zeta'(-1)} = (2\pi)^{1/12}\left(e^{(\gamma\pi^2/6)-\zeta'(2)}\right)^{1/(2\pi^2)}, \tag{5}$$

where $\zeta'(z)$ is the derivative of the Riemann zeta function $\zeta(z)$ (see [**22**]; see also [**39**] and [**40**]), γ is the Euler-Mascheroni constant defined by

$$\gamma = \lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{1}{k} - \ln n\right) = 0.577215664901532860606512\cdots. \tag{6}$$

The Glaisher-Kinkelin constant A has drawn attention in many works (see, for example, [**10, 13, 19, 21, 22**]; see also [**2**]). Finch, in his book [**26**, pp.135–138], devoted a section introducing the Glaisher-Kinkelin constant A. This constant A indeed plays an important rôle in the study of the Barnes G-function (see, for details, [**42**, Section 1.4]).

The following integral representation for the Barnes G-function was established by Ferreira and López [**25**, Theorem 1]:

For $|\mathrm{Arg}(z)| < \pi$, *it is asserted that*

$$\ln G(z+1) = \frac{1}{4}\,z^2 + z\ln\Gamma(z+1) - \left(\frac{1}{2}\,z^2 + \frac{1}{2}\,z + \frac{1}{12}\right)\ln z - \ln A + \sum_{k=1}^{N-1} \frac{B_{2k+2}}{2k(2k+1)(2k+2)\,z^{2k}} + R_N(z) \quad (N \in \mathbb{N} := \{1,2,3,\cdots\}), \tag{7}$$

where B_n *are the Bernoulli numbers defined by*

$$\frac{t}{e^t - 1} = \sum_{n=0}^{\infty} B_n\,\frac{t^n}{n!} \qquad (|t| < 2\pi). \tag{8}$$

and the remainder term $R_N(z)$ *in* (7) *is given, for* $\Re(z) > 0$, *by*

$$R_N(z) = \int_0^{\infty} \left(\frac{t}{e^t - 1} - \sum_{k=0}^{2N} \frac{B_k}{k!}\,t^k\right)\,\frac{e^{-zt}}{t^3}\,\mathrm{d}t. \tag{9}$$

Estimates for $|R_N(z)|$ were also found by Ferreira and López [**25**], showing that the expansion is indeed an asymptotic expansion of $\ln G(z+1)$ in sectors of the complex plane cut along the negative real axis. Pedersen [**35**, Theorem 1.1] proved that, for any $N \in \mathbb{N}$, the function $x \mapsto (-1)^N R_N(x)$ is completely monotonic on the open interval $(0,\infty)$. Other asymptotic relations (avoiding the $\ln\Gamma$ term) was obtained by Ruijsenaars [**38**] and investigated by Pedersen [**36**], Koumandos [**29**] and Koumandos and Pedersen [**30**].

Some upper and lower bounds for the double Gamma function were derived in terms of the Gamma, Psi and Polygamma functions (see [**6, 7, 8, 14**]). Chen [**9**] and Mortici [**32**] established several inequalities and asymptotic expansions involving $\ln A$ in (2). Chen and Lin [**13**] and Chen [**10**] presented a class of asymptotic expansions related to the Glaisher-Kinkelin constant A and the Barnes G-function.

In the limit when $x \to \infty$, the following Stirling formula for the Barnes G-function is known (see [**41**, p. 26]):

$$\ln G(x+1) = \frac{x}{2}\ln(2\pi) - \ln A + \frac{1}{12} - \frac{3x^2}{4} + \left(\frac{x^2}{2} - \frac{1}{12}\right)\ln x + O\left(\frac{1}{x}\right) \quad (x \to \infty). \tag{10}$$

Quite recently, Chen [**11**] made use of the Stirling formula (10) in order to develop a complete asymptotic expansion given by

$$\begin{aligned}(11)\quad \ln G(x+1) &\sim \frac{x}{2}\ln(2\pi) - \ln A + \frac{1}{12} - \frac{3x^2}{4} + \left(\frac{x^2}{2} - \frac{1}{12}\right)\ln x + \sum_{k=1}^{\infty}\frac{q_k}{x^k} \\ &= \frac{x}{2}\ln(2\pi) - \ln A + \frac{1}{12} - \frac{3x^2}{4} + \left(\frac{x^2}{2} - \frac{1}{12}\right)\ln x - \frac{1}{240x^2} \\ &\quad + \frac{1}{1008x^4} - \frac{1}{1440x^6} + \frac{1}{1056x^8} - \frac{691}{327600x^{10}} + \frac{1}{144x^{12}} \\ &\quad - \frac{3617}{114240x^{14}} + \frac{43867}{229824x^{16}} - \frac{174611}{118800x^{18}} + \cdots \qquad (x \to \infty)\end{aligned}$$

with the coefficients q_k given by the following recurrence relation:

$$\begin{aligned}(12)\quad q_1 = 0 \ \text{ and } \ q_k = \frac{(-1)^{k+1}}{k}\Bigg[\sum_{j=1}^{k-1} q_j(-1)^j\binom{k}{k-j+1} + \frac{(-1)^k B_{k+2}}{(k+1)(k+2)} \\ + \frac{(k+6)(k-1)}{12(k+3)(k+2)(k+1)}\Bigg] \qquad (k \in \mathbb{N}\setminus\{1\}),\end{aligned}$$

where B_n denotes the Bernoulli numbers generated by (8).

It is well known that (see [**1**, p. 257, Eq. (6.1.40)])

$$(13)\quad \ln\Gamma(x+1) \sim x\ln x - x + \ln\sqrt{2\pi x} + \sum_{k=1}^{\infty}\frac{B_{2k}}{2k(2k-1)x^{2k-1}} \qquad (x \to \infty).$$

We then find from (7) and (13) that

$$(14)\quad \ln G(x+1) \sim \frac{x}{2}\ln(2\pi) - \ln A + \frac{1}{12} - \frac{3x^2}{4} + \left(\frac{x^2}{2} - \frac{1}{12}\right)\ln x + \sum_{k=1}^{\infty}\frac{q_{2k}}{x^{2k}} \qquad (x \to \infty),$$

where the coefficients q_{2k} are given explicitly by

$$(15)\quad q_{2k} = \frac{B_{2k+2}}{2k(2k+2)} \qquad (k \in \mathbb{N})$$

in terms of the Bernoulli numbers B_n defined by (8). This would obviously show that

$$(16)\quad q_{2k-1} = 0 \qquad (\forall\ k \in \mathbb{N}).$$

Finally, by applying the formulas (12) and (15), we find the following (presumably new) recursive relation of the Bernoulli numbers B_n defined by (8):

$$(17)\quad B_{k+2} = (-1)^{k+1}\left[\sum_{j=1}^{k-1}\frac{(-1)^j B_{j+2}}{j(j+2)}\,(k+1)\binom{k}{k-j+1} + \frac{(k+6)(k-1)}{12(k+3)(k+2)}\right]$$

$$(k \in \mathbb{N} \setminus \{1\}),$$

which can be written as follows:

$$B_k = (-1)^{k+1} \left[\sum_{j=1}^{k-3} \frac{(-1)^j B_{j+2}}{j(j+2)} \, (k-1) \binom{k-2}{k-j-1} + \frac{(k+4)(k-3)}{12k(k+1)} \right] \tag{18}$$

$$(k \in \mathbb{N} \setminus \{1, 2, 3\}).$$

It is well known that

$$B_0 = 1, \quad B_1 = -\frac{1}{2}, \quad B_2 = \frac{1}{6} \quad \text{and} \quad B_{2n+1} = 0 \qquad (n \in \mathbb{N}).$$

Our main object in this paper is to present several potentially useful properties of the Barnes G-function. We first recall a set of lemmas in Section 2, each of which will be useful in our investigation. The properties of the Barnes G-function, which are considered in Section 3, include its integral representation, complete monotonicity, and continued-fraction approximation. Then, in Section 4, we derive continued-fraction approximations of the Glaisher-Kinkelin constant and the Choi-Srivastava constants. Our last section (Section 5) is devoted to our concluding remarks and observations.

2. A SET OF LEMMAS

The following lemmas will be useful in our present investigation.

Lemma 1. (see [**15**]) *Let* $a_1 \neq 0$ *and suppose that*

$$A(x) \sim \sum_{j=1}^{\infty} \frac{a_j}{x^j} \qquad (x \to \infty) \tag{19}$$

and

$$A^*(x) \sim \sum_{j=1}^{\infty} \frac{a_j}{x^{2j}} \qquad (x \to \infty) \tag{20}$$

are two given asymptotic expansions. Then the asymptotic expansion (19) *can be transformed into the continued-fraction approximation of the following form*:

$$A(x) \approx \cfrac{a_1}{x + b_0 + \cfrac{b_1}{x + c_0 + \cfrac{c_1}{x + d_0 + \ddots}}} \qquad (x \to \infty). \tag{21}$$

Moreover, the asymptotic expansion (20) *can be transformed into the following continued-fraction approximation*:

$$A^*(x) \approx \cfrac{a_1}{x^2 + b_0 + \cfrac{b_1}{x^2 + c_0 + \cfrac{c_1}{x^2 + d_0 + \ddots}}} \qquad (x \to \infty). \tag{22}$$

The constants occurring in the right-hand sides of (21) *and* (22) *are given by the following recurrence relations*:

$$\begin{cases} b_0 = -\dfrac{a_2}{a_1}, \quad b_j = -\frac{1}{a_1}\left(a_{j+2} + \sum\limits_{k=1}^{j} a_{k+1} b_{j-k}\right) \\ c_0 = -\dfrac{b_2}{b_1}, \quad c_j = -\dfrac{1}{b_1}\left(b_{j+2} + \sum\limits_{k=1}^{j} b_{k+1} c_{j-k}\right) \\ d_0 = -\dfrac{c_2}{c_1}, \quad d_j = -\dfrac{1}{c_1}\left(c_{j+2} + \sum\limits_{k=1}^{j} c_{k+1} d_{j-k}\right) \\ \qquad \cdots \quad \cdots \end{cases} \tag{23}$$

Remark 1. Clearly, since

$$a_j \Longrightarrow b_j \Longrightarrow c_j \Longrightarrow d_j \Longrightarrow \cdots,$$

the asymptotic expansions (19) and (20) are transformed, respectively, into the continued-fraction approximations (21) and (22). Furthermore, the constants occurring in the right-hand sides of (21) and (22) are determined by the system (23).

Lemma 2. (see [**16**]) *Let* $a_1 \neq 0$ *and suppose that*

$$A_1(x) \sim \sum_{j=1}^{\infty} \frac{a_j}{x^{2j-1}} \qquad (x \to \infty) \tag{24}$$

is a given asymptotic expansion. Then the asymptotic expansion (24) *can be transformed into the continued-fraction approximation of the form*:

$$A_1(x) \approx \cfrac{a_1}{x + \cfrac{b_1}{x + \cfrac{c_1}{x + \cfrac{d_1}{x + \ddots}}}} \qquad (x \to \infty), \tag{25}$$

where the constants in the right-hand side of (25) *are given by the following recur-*

rence relations:

(26)
$$\begin{cases} b_1 = -\frac{a_2}{a_1}, \quad b_j = -\frac{1}{a_1}\left(a_{j+1} + \sum_{k=1}^{j-1} a_{k+1}b_{j-k}\right) \\ c_1 = -\frac{b_2}{b_1}, \quad c_j = -\frac{1}{b_1}\left(b_{j+1} + \sum_{k=1}^{j-1} b_{k+1}c_{j-k}\right) \\ d_1 = -\frac{c_2}{c_1}, \quad d_j = -\frac{1}{c_1}\left(c_{j+1} + \sum_{k=1}^{j-1} c_{k+1}d_{j-k}\right) \\ \quad \cdots \quad \cdots \end{cases}$$

Remark 2. Clearly, since

$$a_j \implies b_j \implies c_j \implies d_j \implies \cdots,$$

the asymptotic expansion (24) is transformed into the continued-fraction approximation (25). Moreover, the constants occurring in the right-hand side of (25) are determined by the system (26).

3. PROPERTIES OF THE BARNES G-FUNCTION

3.1 Integral representation

Using Binet's first formula [**41**, p. 16]:

(27)
$$\ln\Gamma(x) = \left(x - \frac{1}{2}\right)\ln x - x + \ln\sqrt{2\pi} + \int_0^\infty \left(\frac{t}{e^t - 1} - 1 + \frac{t}{2}\right)\frac{e^{-xt}}{t^2}\,\mathrm{d}t \qquad (x > 0),$$

and integrating it by parts, we get

(28)
$$\begin{aligned} x\ln\Gamma(x+1) &= \left(x^2 + \frac{x}{2}\right)\ln x - x^2 + x\ln\sqrt{2\pi} - \int_0^\infty \left(\frac{t}{e^t - 1} - 1 + \frac{t}{2}\right)\frac{1}{t^2}\,\mathrm{d}(e^{-xt}) \\ &= \left(x^2 + \frac{x}{2}\right)\ln x - x^2 + x\ln\sqrt{2\pi} + \frac{1}{12} \\ &\quad + \int_0^\infty \left(2 - \frac{t}{2} - \frac{t}{e^t - 1} - \frac{t^2e^t}{(e^t - 1)^2}\right)\frac{e^{-xt}}{t^3}\,\mathrm{d}t. \end{aligned}$$

The choice $N=1$ in (7) yields

$$\ln G(x+1) = \frac{1}{4}\,x^2 + x\ln\Gamma(x+1) - \left(\frac{1}{2}\,x^2 + \frac{1}{2}\,x + \frac{1}{12}\right)\ln x - \ln A + \int_0^\infty \left(\frac{t}{e^t-1} - 1 + \frac{t}{2} - \frac{t^2}{12}\right)\frac{e^{-xt}}{t^3}\,\mathrm{d}t \qquad (x>0). \tag{29}$$

Upon substituting from (28) into (29), we get the following integral representation for Barnes G-function:

$$\ln G(x+1) = \frac{x}{2}\ln(2\pi) - \ln A + \frac{1}{12} - \frac{3x^2}{4} + \left(\frac{x^2}{2} - \frac{1}{12}\right)\ln x - r_1(x), \tag{30}$$

where, for convenience,

$$r_1(x) = \int_0^\infty \left(\frac{t^2e^t}{(e^t-1)^2} - 1 + \frac{t^2}{12}\right)\frac{e^{-xt}}{t^3}\,\mathrm{d}t \qquad (x>0). \tag{31}$$

We now make use of the formula (30) in order to produce a general result given by Theorem 1 below.

The Bernoulli polynomials $B_k(x)$ are defined by

$$\frac{te^{xt}}{e^t-1} = \sum_{k=0}^\infty B_k(x)\,\frac{x^k}{k!} \qquad (|x|<2\pi)$$

and the *double* Bernoulli polynomials $B_{2,k}(x)$ are defined by

$$\frac{t^2e^{xt}}{(e^t-1)^2} = \sum_{k=0}^\infty B_{2,k}(x)\,\frac{x^k}{k!} \qquad (|x|<2\pi).$$

The relation between $B_{2,k}(x)$ and $B_k(x)$ is given by the following formula (see [**33**, p. 187]):

$$B_{2,k}(x) = k(k-1)\left((x-1)\frac{B_{k-1}(x)}{k-1} - \frac{B_k(x)}{k}\right). \tag{32}$$

We thus find that

$$B_{2,k}(1) = -(k-1)B_k(1) = (-1)^{k+1}(k-1)B_k \qquad (k\in\mathbb{N}_0 := \mathbb{N}\cup\{0\}).$$

The following assertion is known to hold true:

$$\begin{aligned}\frac{t^2e^t}{(e^t-1)^2} &= \sum_{k=0}^\infty B_{2,k}(1)\,\frac{x^k}{k!}\\ &= 1 - \frac{1}{12}\,t^2 + \frac{1}{240}\,t^4 - \frac{1}{6048}\,t^6 + \frac{1}{172800}\,t^8 - \frac{1}{5322240}\,t^{10}\\ &\quad + \frac{691}{118879488000}\,t^{12} - \frac{1}{5748019200}\,t^{14} + \frac{3617}{711374856192000}\,t^{16}\\ &\quad - \frac{43867}{300534953951232000}\,t^{18} + \cdots.\end{aligned} \tag{33}$$

By noting that

$$\frac{t^2 e^t}{(e^t-1)^2} = \left(\frac{\frac{t}{2}}{\sinh(\frac{t}{2})}\right)^2$$

is an even function, only the even terms appear in the right-hand side of (33). In fact, we have

$$B_{2,2k}(1) = -(2k-1)B_{2k} \qquad (k \in \mathbb{N}_0). \tag{34}$$

We thus find from (30) that

$$\begin{aligned} \ln G(x+1) &= \frac{x}{2}\ln(2\pi) - \ln A + \frac{1}{12} - \frac{3x^2}{4} + \left(\frac{x^2}{2} - \frac{1}{12}\right)\ln x \\ &\quad - \int_0^\infty \left(\sum_{k=2}^{N} B_{2,2k}(1)\,\frac{x^{2k}}{(2k)!}\right)\frac{e^{-xt}}{t^3}\,\mathrm{d}t \\ &\quad - \int_0^\infty \left(\frac{t^2 e^t}{(e^t-1)^2} - 1 + \frac{t^2}{12} - \sum_{k=2}^{N} B_{2,2k}(1)\,\frac{t^{2k}}{(2k)!}\right)\frac{e^{-xt}}{t^3}\,\mathrm{d}t \\ &= \frac{x}{2}\ln(2\pi) - \ln A + \frac{1}{12} - \frac{3x^2}{4} + \left(\frac{x^2}{2} - \frac{1}{12}\right)\ln x \\ &\quad - \sum_{k=1}^{N-1} \frac{B_{2,2k+2}(1)}{2k(2k+1)(2k+2)x^{2k}} - r_N(x), \end{aligned} \tag{35}$$

where

$$r_N(x) = \int_0^\infty \left(\frac{t^2 e^t}{(e^t-1)^2} - \sum_{k=0}^{N} B_{2,2k}(1)\,\frac{t^{2k}}{(2k)!}\right)\frac{e^{-xt}}{t^3}\,\mathrm{d}t \qquad (N \in \mathbb{N}). \tag{36}$$

In view of (34), from the formula (35) we obtain the integral representation for Barnes G-function given by Theorem 1 below.

Theorem 1. *For $x > 0$ and $N \in \mathbb{N}$, the following integral representation holds true for the Barnes G-function*:

$$\begin{aligned} \ln G(x+1) &= \frac{x}{2}\ln(2\pi) - \ln A + \frac{1}{12} - \frac{3x^2}{4} + \left(\frac{x^2}{2} - \frac{1}{12}\right)\ln x \\ &\quad + \sum_{k=1}^{N-1} \frac{B_{2k+2}}{2k(2k+2)x^{2k}} - r_N(x), \end{aligned} \tag{37}$$

where

$$r_N(x) = \int_0^\infty \left(\frac{t^2 e^t}{(e^t-1)^2} - 1 + \sum_{k=1}^{N} \frac{B_{2k}}{2k}\,\frac{t^{2k}}{(2k-2)!}\right)\frac{e^{-xt}}{t^3}\,\mathrm{d}t. \tag{38}$$

Remark 3. The formula (37) generalizes the formula (30). In fact, the choice $N = 1$ in (37) yields (30). Remarkably, he formula (37) is different from the formula (7), since the formula (37) avoids the term $x \ln \Gamma(x+1)$ and the remainder in (37) is different from the remainder in (7).

3.2 Complete monotonicity

A function f is said to be completely monotonic on an interval $I \subseteq \mathbb{R}$ if it has derivatives of all orders on I and satisfies the following inequality:

$$(-1)^n f^{(n)}(x) \geq 0 \qquad (x \in I;\ n \in \mathbb{N}_0). \tag{39}$$

Dubourdieu [**24**, p. 98] pointed out that, if a non-constant function f is completely monotonic on $I = (a, \infty)$, then strict inequality holds true in (39). We refer also to [**28**] for a simpler proof of this result.

We first establish the double inequality for $\frac{t^2 e^t}{(e^t-1)^2}$ given by Theorem 2 below. As a consequence of Theorem 2, we then show that, for any $N \in \mathbb{N}$, the function $x \mapsto (-1)^{N-1} r_N(x)$ is completely monotonic on $(0, \infty)$.

Theorem 2. *For $t > 0$ and $N \in \mathbb{N}$, the following two-sided inequality holds true*:

$$1 - \sum_{k=1}^{2N-1} \frac{B_{2k}}{2k} \frac{t^{2k}}{(2k-2)!} < \frac{t^2 e^t}{(e^t-1)^2} < 1 - \sum_{k=1}^{2N} \frac{B_{2k}}{2k} \frac{t^{2k}}{(2k-2)!} \tag{40}$$

or, alternatively,

$$(-1)^{N-1} \left(\frac{t^2 e^t}{(e^t-1)^2} - 1 + \sum_{k=1}^{N} \frac{B_{2k}}{2k} \frac{t^{2k}}{(2k-2)!} \right) > 0. \tag{41}$$

Proof. It is known that (see, for example, [**43**, p. 64])

$$\frac{t}{e^t-1} - 1 + \frac{t}{2} = \sum_{j=1}^{n} \frac{B_{2j}}{(2j)!} t^{2j} + (-1)^n t^{2n+2} \nu_n(t) \qquad (n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}), \tag{42}$$

where

$$\nu_n(t) = \sum_{k=1}^{\infty} \frac{2}{(t^2 + 4\pi^2 k^2)(2\pi k)^{2n}}.$$

By (42), we obtain

$$\begin{aligned} \frac{t^2 e^t}{(e^t-1)^2} &= \frac{t}{e^t-1} - t \left(\frac{t}{e^t-1} \right)' \\ &= 1 - \sum_{k=1}^{N} \frac{B_{2k}}{2k} \frac{t^{2k}}{(2k-2)!} + (-1)^{N-1} \Big(t Q_N'(t) - Q_N(t) \Big) \\ &= 1 - \sum_{k=1}^{N} \frac{B_{2k}}{2k} \frac{t^{2k}}{(2k-2)!} + (-1)^{N-1} t^2 \left(\frac{Q_N(t)}{t} \right)', \end{aligned} \tag{43}$$

where

$$Q_N(t) = t^{2N+2}\nu_N(t) \qquad (\forall\ N \in \mathbb{N};\ t > 0).$$

We now recall from [**29**] that

$$\left(\frac{Q_N(t)}{t}\right)' > 0 \qquad (\forall\ N \in \mathbb{N};\ t > 0).$$

We then find from (43) that

$$(-1)^{N-1}\left(\frac{t^2e^t}{(e^t-1)^2} - 1 + \sum_{k=1}^{N}\frac{B_{2k}}{2k}\,\frac{t^{2k}}{(2k-2)!}\right) = t^2\left(\frac{Q_N(t)}{t}\right)' > 0$$
$$(\forall\ N \in \mathbb{N};\ t > 0).$$

The proof of Theorem 2 is thus completed. □

Corollary 1. *For any* $N \in \mathbb{N}$, *the following function*:

(44)
$$(-1)^{N-1}r_N(x) = \int_0^\infty (-1)^{N-1}\left(\frac{t^2e^t}{(e^t-1)^2} - 1 + \sum_{k=1}^{N}\frac{B_{2k}}{2k}\,\frac{t^{2k}}{(2k-2)!}\right)\frac{e^{-xt}}{t^3}\,\mathrm{d}t$$

is completely monotonic on the interval $(0,\infty)$.

Remark 4. It follows from (37) and (44) that, for all $N \in \mathbb{N}$ and $x > 0$, we have

$$(-1)^N\left[\ln G(x+1) - \left(\frac{x}{2}\ln(2\pi) - \ln A + \frac{1}{12} - \frac{3x^2}{4} + \left(\frac{x^2}{2} - \frac{1}{12}\right)\ln x + \sum_{k=1}^{N-1}\frac{B_{2k+2}}{2k(2k+2)x^{2k}}\right)\right] = (-1)^{N-1}r_N(x) > 0 \tag{45}$$

or, alternatively,

(46)
$$\exp\left[\frac{x}{2}\ln(2\pi) - \ln A + \frac{1}{12} - \frac{3x^2}{4} + \left(\frac{x^2}{2} - \frac{1}{12}\right)\ln x + \sum_{k=1}^{2N-1}\frac{B_{2k+2}}{2k(2k+2)x^{2k}}\right]$$
$$< G(x+1)$$
$$< \exp\left[\frac{x}{2}\ln(2\pi) - \ln A + \frac{1}{12} - \frac{3x^2}{4} + \left(\frac{x^2}{2} - \frac{1}{12}\right)\ln x + \sum_{k=1}^{2N}\frac{B_{2k+2}}{2k(2k+2)x^{2k}}\right],$$

which provides the upper and lower bounds of the Barnes G-function, without the term $x\ln\Gamma(x+1)$.

3.3 Continued-fraction approximation

Theorem 3 below transforms the asymptotic expansion (14) into a continued-fraction approximation.

Theorem 3. *It is asserted that*

(47)
$$\ln G(x+1) - \left[\frac{x}{2}\ \ln(2\pi) - \ln A + \frac{1}{12} - \frac{3x^2}{4} + \left(\frac{x^2}{2} - \frac{1}{12}\right)\ln x\right]$$
$$\approx \cfrac{-\frac{1}{240}}{x^2 + \frac{5}{21} + \cfrac{-\frac{97}{882}}{x^2 + \frac{32885}{22407} + \cfrac{-\frac{213268083}{148003570}}{x^2 + \frac{40158013805}{10836049741} + \ddots}}} \qquad (x \to \infty).$$

Proof. Let us put

$$A^*(x) = \ln G(x+1) - \left[\frac{x}{2}\ \ln(2\pi) - \ln A + \frac{1}{12} - \frac{3x^2}{4} + \left(\frac{x^2}{2} - \frac{1}{12}\right)\ln x\right]. \tag{48}$$

It follows from (14) that

$$\begin{aligned} A^*(x) \sim \sum_{k=1}^{\infty} \frac{a_k}{x^{2k}} &= \frac{x}{2}\ \ln(2\pi) - \ln A + \frac{1}{12} - \frac{3x^2}{4} + \left(\frac{x^2}{2} - \frac{1}{12}\right)\ln x \\ &\quad - \frac{1}{240x^2} + \frac{1}{1008x^4} - \frac{1}{1440x^6} + \frac{1}{1056x^8} \\ &\quad - \frac{691}{327600x^{10}} + \frac{1}{144x^{12}} - \frac{3617}{114240x^{14}} + \cdots \\ &\qquad (x \to \infty), \end{aligned} \tag{49}$$

where

$$a_k := q_{2k} = -\frac{B_{2k+2}}{2k(2k+2)} \qquad (k \in \mathbb{N}).$$

By applying Lemma 1, the asymptotic expansion (49) can be transformed into the following continued-fraction approximation:

$$A^*(x) \approx \cfrac{a_1}{x^2 + b_0 + \cfrac{b_1}{x^2 + c_0 + \cfrac{c_1}{x^2 + d_0 + \ddots}}} \qquad (x \to \infty), \tag{50}$$

where the constants in the right-hand side of (50) can be determined by using the system (23).

We now see from (49) that

$$a_1 = -\frac{1}{240},\ a_2 = \frac{1}{1008},\ a_3 = -\frac{1}{1440},\ a_4 = \frac{1}{1056},\ a_5 = -\frac{691}{327600},\ a_6 = \frac{1}{144},\ \cdots.$$

From the first recurrence relation in (23), we thus find that

$$\begin{aligned}
b_0 &= -\frac{a_2}{a_1} = \frac{5}{21},\\
b_1 &= -\frac{a_3 + a_2 b_0}{a_1} = -\frac{97}{882},\\
b_2 &= -\frac{a_4 + a_2 b_1 + a_3 b_0}{a_1} = \frac{32885}{203742},\\
b_3 &= -\frac{a_5 + a_2 b_2 + a_3 b_1 + a_4 b_0}{a_1} = -\frac{219902759}{556215660},\\
b_4 &= -\frac{a_6 + a_2 b_3 + a_3 b_2 + a_4 b_1 + a_5 b_0}{a_1} = \frac{3270801577}{2336105772},\ \cdots.
\end{aligned}$$

Also, from the second recurrence relation in (23), we get

$$\begin{aligned}
c_0 &= -\frac{b_2}{b_1} = \frac{32885}{22407},\\
c_1 &= -\frac{b_3 + b_2 c_0}{b_1} = -\frac{213268083}{148003570},\\
c_2 &= -\frac{b_4 + b_2 c_1 + b_3 c_0}{b_1} = \frac{12974127537}{2429535526},\ \cdots.
\end{aligned}$$

Continuing the above process, we obtain

$$d_0 = -\frac{c_2}{c_1} = \frac{40158013805}{10836049741},\ \cdots.$$

We thus find for $x \to \infty$ that

$$A^*(x) \approx \cfrac{-\frac{1}{240}}{x^2 + \frac{5}{21} + \cfrac{-\frac{97}{882}}{x^2 + \frac{32885}{22407} + \cfrac{-\frac{213268083}{148003570}}{x^2 + \frac{40158013805}{10836049741} + \ddots}}} \qquad (x \to \infty). \tag{51}$$

Finally, by combining (48) and (51), we obtain the desired result (47). This evidently completes the proof of Theorem 3. □

4. THE GLAISHER-KINKELIN CONSTANT AND THE CHOI-SRIVASTAVA CONSTANTS

Choi and Srivastava (see [**21**, p. 102] and [**22**]) introduced two mathematical constants B and C (analogous to the Glaisher-Kinkelin constant A), which are defined by

$$\ln B = \lim_{n\to\infty}\left\{\sum_{k=1}^{n} k^2 \ln k - \left(\frac{n^3}{3}+\frac{n^2}{2}+\frac{n}{6}\right)\ln n + \frac{n^3}{9} - \frac{n}{12}\right\} \tag{52}$$

and

$$\ln C = \lim_{n\to\infty}\left\{\sum_{k=1}^{n} k^3 \ln k - \left(\frac{n^4}{4}+\frac{n^3}{2}+\frac{n^2}{4}-\frac{1}{120}\right)\ln n + \frac{n^4}{16} - \frac{n^2}{12}\right\}, \tag{53}$$

respectively. The approximate numerical values of the Choi-Srivastava constants B and C are given by

$$B = 1.03091675\cdots \qquad \text{and} \qquad C = 0.97955746\cdots.$$

Analogous to the expression of the Glaisher-Kinkelin constant A in (5), the Choi-Srivastava constants B and C are also known to be expressible in terms of special values of the derivative of the Riemann Zeta function $\zeta(s)$ as follows (see [**22**] and [**23**, Eq. (1.9)]):

$$\ln B = -\zeta'(-2) \qquad \text{and} \qquad \ln C = -\frac{11}{720} - \zeta'(-3). \tag{54}$$

As the Euler-Mascheroni constant γ is involved in the classical Gamma function Γ, the constants A, B and C have appeared naturally in the theory of the multiple Gamma functions Γ_n (see, for details, [**42**, Section 1.4]) and play their respective rôles (see, for example, [**41**, pp. 39 and 247], [**20**, p. 523, Eq. (2.50)] and [**19**]).

Chen [**9**] and [**32**] dealt with the problem of approximating and finding asymptotic expansions related to the Glaisher-Kinkelin constant A and the Choi-Srivastava constants B and C. Subsequently, Cheng and Chen [**17**] as well as Chen and Choi [**12**] established new asymptotic expansions of the Glaisher-Kinkelin A and the Choi-Srivastava constants B and C. on the other hand, by using the Bernoulli numbers B_n, Chen [**9**] established the asymptotic expansions related to the constants A, B and C. More recently, Chen [**11**] presented a recurrence relation for determining the coefficients of the asymptotic expansion related to each of the constants A, B and C, without using the Bernoulli numbers B_n.

In terms of the Bernoulli numbers B_n, the following asymptotic expansion related to the constant A was derived by Chen [**9**, Theorem 1]:

$$\sum_{k=1}^{n} k\ln k - \left(\frac{n^2}{2}+\frac{n}{2}+\frac{1}{12}\right)\ln n + \frac{n^2}{4} \tag{55}$$
$$\sim \ln A - \sum_{k=1}^{\infty} \frac{B_{2k+2}}{2k(2k+1)(2k+2)}\,\frac{1}{n^{2k}}$$
$$= \ln A + \frac{1}{720n^2} - \frac{1}{5040n^4} + \frac{1}{10080n^6} - \frac{1}{9504n^8} + \frac{691}{3603600n^{10}} - \frac{1}{1872n^{12}} + \cdots \qquad (n \to \infty).$$

Theorem 4 below transforms the asymptotic expansion (55) into a continued-fraction approximation.

Theorem 4. *The following continued-fraction approximation holds true for the Glaisher-Kinkelin constant* A:

$$\sum_{k=1}^{n} k\ln k - \left(\frac{n^2}{2}+\frac{n}{2}+\frac{1}{12}\right)\ln n + \frac{n^2}{4} - \ln A \tag{56}$$
$$\approx \cfrac{\frac{1}{720}}{n^2+\frac{1}{7}+\cfrac{-\frac{5}{98}}{n^2+\frac{1319}{1155}+\cfrac{-\frac{676123}{707850}}{n^2+\frac{10013915}{3187437}+\ddots}}} \qquad (n \to \infty).$$

Proof. Let us put

$$A^*(n) = \sum_{k=1}^{n} k\ln k - \left(\frac{n^2}{2}+\frac{n}{2}+\frac{1}{12}\right)\ln n + \frac{n^2}{4} - \ln A. \tag{57}$$

It then follows from (55) that

$$A^*(n) \sim \sum_{k=1}^{\infty} \frac{a_k}{n^{2k}} \tag{58}$$
$$= \frac{1}{720n^2} - \frac{1}{5040n^4} + \frac{1}{10080n^6} - \frac{1}{9504n^8} + \frac{691}{3603600n^{10}} - \frac{1}{1872n^{12}} + \cdots \qquad (n \to \infty),$$

where

$$a_k = -\frac{B_{2k+2}}{2k(2k+1)(2k+2)} \qquad (k \in \mathbb{N}).$$

By applying Lemma 1, the asymptotic expansion (58) can be transformed into the following continued-fraction approximation:

$$A^*(n) \approx \cfrac{a_1}{n^2 + b_0 + \cfrac{b_1}{n^2 + c_0 + \cfrac{c_1}{n^2 + d_0 + \ddots}}} \qquad (n \to \infty), \tag{59}$$

where the constants in the right-hand side of (59) can be determined by using the system (23).

We see from (58) that

$$a_1 = \frac{1}{720}, \quad a_2 = -\frac{1}{5040}, \quad a_3 = \frac{1}{10080},$$
$$a_4 = -\frac{1}{9504}, \quad a_5 = \frac{691}{3603600}, \quad a_6 = -\frac{1}{1872}, \ \cdots,$$

Moreover, from the first recurrence relation in (23), we have

$$b_0 = -\frac{a_2}{a_1} = \frac{1}{7},$$
$$b_1 = -\frac{a_3 + a_2 b_0}{a_1} = -\frac{5}{98},$$
$$b_2 = -\frac{a_4 + a_2 b_1 + a_3 b_0}{a_1} = \frac{1319}{22638},$$
$$b_3 = -\frac{a_5 + a_2 b_2 + a_3 b_1 + a_4 b_0}{a_1} = -\frac{2374661}{20600580},$$
$$b_4 = -\frac{a_6 + a_2 b_3 + a_3 b_2 + a_4 b_1 + a_5 b_0}{a_1} = \frac{4462433}{13109460}, \ \cdots.$$

Also, from the second recurrence relation in (23), we get

$$c_0 = -\frac{b_2}{b_1} = \frac{1319}{1155},$$
$$c_1 = -\frac{b_3 + b_2 c_0}{b_1} = -\frac{676123}{707850},$$
$$c_2 = -\frac{b_4 + b_2 c_1 + b_3 c_0}{b_1} = \frac{14019481}{4671810}, \ \cdots.$$

Continuing the above process, we find that

$$d_0 = -\frac{c_2}{c_1} = \frac{10013915}{3187437}, \ \cdots.$$

We thus obtain

$$A^*(n) \approx \cfrac{\frac{1}{720}}{n^2 + \frac{1}{7} + \cfrac{-\frac{5}{98}}{n^2 + \frac{1319}{1155} + \cfrac{-\frac{676123}{707850}}{n^2 + \frac{10013915}{3187437} + \ddots}}} \qquad (n \to \infty). \tag{60}$$

Finally, by making use of (57) and (60), we are led to the desired result (56). The proof of Theorem 4 is thus completed. □

We now recall the following asymptotic expansions related to the constants B and C, which were given by Chen [**9**, Theorem 2]:

$$\begin{aligned}(61)\qquad &\sum_{k=1}^{n} k^2 \ln k - \left(\frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6}\right)\ln n + \frac{n^3}{9} - \frac{n}{12}\\ &\sim \ln B + \sum_{k=1}^{\infty} \frac{2B_{2k+2}\Gamma(2k-1)}{\Gamma(2k+3)}\,\frac{1}{n^{2k-1}}\\ &= \ln B - \frac{1}{360n} + \frac{1}{7560n^3} - \frac{1}{25200n^5} + \frac{1}{33264n^7} - \frac{691}{16216200n^9}\\ &\quad + \frac{1}{10296n^{11}} - \frac{3617}{11138400n^{13}} + \cdots \qquad (n \to \infty)\end{aligned}$$

and

$$\begin{aligned}(62)\qquad &\sum_{k=1}^{n} k^3 \ln k - \left(\frac{n^4}{4} + \frac{n^3}{2} + \frac{n^2}{4} - \frac{1}{120}\right)\ln n + \frac{n^4}{16} - \frac{n^2}{12}\\ &\sim \ln C - 6\sum_{k=1}^{\infty} \frac{\Gamma(2k)}{\Gamma(2k+5)}\,\frac{B_{2k+4}}{n^{2k}}.\\ &= \ln C - \frac{1}{5040n^2} + \frac{1}{33600n^4} - \frac{1}{66528n^6} + \frac{691}{43243200n^8}\\ &\quad - \frac{1}{34320n^{10}} + \frac{3617}{44553600n^{12}} - \frac{43867}{136745280n^{14}} + \cdots \qquad (n \to \infty)\end{aligned}$$

in terms of the Bernoulli numbers B_n.

Our next result (Theorem 5 below) transforms the asymptotic expansion (61) into a continued-fraction approximation.

Theorem 5. *The following continued-fraction approximation holds true for the Choi-Srivastava constant* B:

$$\begin{aligned}(63)\qquad &\sum_{k=1}^{n} k^2 \ln k - \left(\frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6}\right)\ln n + \frac{n^3}{9} - \frac{n}{12} - \ln B\\ &\approx \cfrac{-\frac{1}{360}}{n + \cfrac{\frac{1}{21}}{n + \cfrac{\frac{53}{210}}{n + \cfrac{\frac{3171}{5830}}{n + \ddots}}}} \qquad (n \to \infty).\end{aligned}$$

Proof. Let us put

$$(64)\qquad A_1(n) = \sum_{k=1}^{n} k^2 \ln k - \left(\frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6}\right)\ln n + \frac{n^3}{9} - \frac{n}{12} - \ln B.$$

It then follows from (61) that

$$
\begin{aligned}
(65)\qquad A_1(n) &\sim \sum_{k=1}^{\infty} \frac{a_k}{n^{2k-1}} \\
&= -\frac{1}{360n} + \frac{1}{7560n^3} - \frac{1}{25200n^5} + \frac{1}{33264n^7} - \frac{691}{16216200n^9} \\
&\quad + \frac{1}{10296n^{11}} - \frac{3617}{11138400n^{13}} + \cdots \qquad (n \to \infty),
\end{aligned}
$$

where

$$a_k = \frac{2B_{2k+2}\Gamma(2k-1)}{\Gamma(2k+3)} \qquad (k \in \mathbb{N}).$$

By appealing to Lemma 2, the asymptotic expansion (65) can be transformed into a continued-fraction approximation of the form given by

$$(66)\qquad A_1(n) \approx \cfrac{a_1}{n + \cfrac{b_1}{n + \cfrac{c_1}{n + \cfrac{d_1}{n + \ddots}}}} \qquad (n \to \infty),$$

where the constants in the right-hand side of (66) can be determined by using the system (26).

We see from (65) that

$$a_1 = -\frac{1}{360}, \quad a_2 = \frac{1}{7560}, \quad a_3 = -\frac{1}{25200}, \quad a_4 = \frac{1}{33264}, \quad a_5 = -\frac{691}{16216200}, \ \cdots .$$

Furthermore, from the first recurrence relation in (26), we have

$$
\begin{aligned}
b_1 &= -\frac{a_2}{a_1} = \frac{1}{21}, \\
b_2 &= -\frac{a_3 + a_2 b_1}{a_1} = -\frac{53}{4410}, \\
b_3 &= -\frac{a_4 + a_2 b_2 + a_3 b_1}{a_1} = \frac{9749}{1018710}, \\
b_4 &= -\frac{a_5 + a_2 b_3 + a_3 b_2 + a_4 b_1}{a_1} = -\frac{39484243}{2781078300}, \ \cdots .
\end{aligned}
$$

Also, from the second recurrence relation in (26), we get

$$
\begin{aligned}
c_1 &= -\frac{b_2}{b_1} = \frac{53}{210}, \\
c_2 &= -\frac{b_3 + b_2 c_1}{b_1} = -\frac{151}{1100}, \ \cdots .
\end{aligned}
$$

Continuing the above process, we find that

$$d_1 = -\frac{c_2}{c_1} = \frac{3171}{5830}, \ \cdots .$$

We thus find for $n \to \infty$ that

$$A_1(n) \approx \cfrac{-\frac{1}{360}}{n + \cfrac{\frac{1}{21}}{n + \cfrac{\frac{53}{210}}{n + \cfrac{\frac{3171}{5830}}{n + \ddots}}}} \qquad (n \to \infty). \tag{67}$$

From (64) and (67), we finally obtain the desired result (63). The proof of Theorem 5 is evidently completed. □

Our last result (Theorem 6 below) transforms the asymptotic expansion (62) into a continued-fraction approximation. Following the same method as that was used in the proof of Theorem 4, we can prove Theorem 6. The details of the proof are, therefore, omitted.

Theorem 6. *The following continued-fraction approximation holds true for the Choi-Srivastava constant C:*

$$\sum_{k=1}^{n} k^3 \ln k - \left(\frac{n^4}{4} + \frac{n^3}{2} + \frac{n^2}{4} - \frac{1}{120}\right) \ln n + \frac{n^4}{16} - \frac{n^2}{12} - \ln C \tag{68}$$

$$\approx \cfrac{-\frac{1}{5040}}{n^2 + \frac{3}{20} + \cfrac{-\frac{703}{13200}}{n^2 + \frac{209983}{182780} + \cfrac{-\frac{2653377200}{2756203593}}{n^2 + \frac{129790553120067}{41223664192360} + \ddots}}} \qquad (n \to \infty).$$

5. CONCLUDING REMARKS AND OBSERVATIONS

In our present investigation, we have derived several potentially useful properties of the Barnes G-function, where $G = 1/\Gamma_2$ in terms of the double Gamma function Γ_2. By applying several lemmas, which we have presented in Section 2, we have successfully obtained such properties of the Barnes G-function as (for example) its integral representation, complete monotonicity, and continued-fraction approximation. We have then given some continued-fraction approximations of the Glaisher-Kinkelin constant and the Choi-Srivastava constants. Our results for the Barnes G-function, the Glaisher-Kinkelin constant A and the Choi-Srivastava constants B and C are motivated by a number of earlier works which we have adequately described and cited in our paper.

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