Applicable Analysis and Discrete Mathematics
available online at http://pefmath.etf.rs

Appl. Anal. Discrete Math. **15** (2021), 213–232.
https://doi.org/10.2298/AADM191103006A

ON THE COMPOSITION AND DECOMPOSITION OF POSITIVE LINEAR OPERATORS (VII)

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In the present paper we study the compositions of the piecewise linear interpolation operator S_{Δ_n} and the Beta-type operator $\overline{\mathbb{B}}_n$, namely $\mathbb{A}_n := S_{\Delta_n} \circ \overline{\mathbb{B}}_n$ and $\mathbb{G}_n := \overline{\mathbb{B}}_n \circ S_{\Delta_n}$. Voronovskaya type theorems for the operators $\mathbb{A}_n$ and $\mathbb{G}_n$ are proved, substantially improving some corresponding known results. The rate of convergence for the iterates of the operators $\mathbb{G}_n$ and $\mathbb{A}_n$ is considered. Some estimates of the differences between $\mathbb{A}_n$, $\mathbb{G}_n$, $\overline{\mathbb{B}}_n$ and S_{Δ_n}, respectively, are given. Also, we study the behaviour of the operators $\mathbb{A}_n$ on the subspace of $C[0,1]$ consisting of all polygonal functions with nodes $\left\{0, \frac{1}{2}, \dots, \frac{n-1}{n}, 1\right\}$. Finally, we propose to the readers a conjecture concerning the eigenvalues of the operators $\mathbb{A}_n$ and $\mathbb{G}_n$. If true, this conjecture would emphasize a new and strong relationship between $\mathbb{G}_n$ and the classical Bernstein operator B_n.

1. INTRODUCTION

In 1912, Bernstein [**2**] introduced his famous polynomials in order to prove Weierstrass' fundamental theorem. Many useful properties and their simple structures and advantages in calculations make them an interesting area of research. The present work is motivated by a problem raised by Lupaş and Gonska in 2006. Their question was if there are non-trivial positive linear operators P and Q such that the classical Bernstein operator can be decomposed as $B_n = P \circ Q$. The first candidates for the factors P and Q were a Beta-type operator introduced by

2020 Mathematics Subject Classification. 41A25, 41A36.
Keywords and Phrases. Bernstein operator; Beta operator, Piecewise linear interpolation operator, Eigenstructure of operators, Iterates of operators.

Mühlbach [**17, 18**] and Lupaş [**14, 13**] and piecewise interpolation at equidistant points in $[0, 1]$.

These Beta operators are given for $f \in C[0, 1]$, $x \in [0, 1]$, by

$$\overline{\mathbb{B}}_n(f; x) = \begin{cases} f(0), & x = 0, \\ \dfrac{1}{B(nx, n(1-x))} \displaystyle\int_0^1 t^{nx-1}(1-t)^{n(1-x)-1} f(t)dt, & x \in (0, 1), \\ f(1), & x = 1, \end{cases}$$

where $B(\cdot, \cdot)$ is the Beta function.

The piecewise linear interpolation operators $S_{\Delta_n} : C[0, 1] \to C[0, 1]$ at the points

$$\Delta_n : x_{-1} = 0 = x_0 < x_1 < \cdots < x_k < \cdots < x_n = x_{n+1} = 1,$$

where $x_k = \dfrac{k}{n}$, $k = 0, 1, \dots, n$, are described as

$$S_{\Delta_n}(f; x) = \frac{1}{n} \sum_{k=0}^{n} [x_{k-1}, x_k, x_{k+1}; |u - x|]_u\, f\left(\frac{k}{n}\right),$$

where $[a_1, a_2, a_3; f] = [a_1, a_2, a_3; f(u)]_u$ denotes the divided difference of a function $f : D \to \mathbb{R}$ on the knots $\{a_1, a_2, a_3\} \subset D$, with respect to u.

Let $\mathbb{G}_n : C[0, 1] \to C[0, 1]$, $\mathbb{G}_n := \overline{\mathbb{B}}_n \circ S_{\Delta_n}$. In [**6**] the authors proved that $\mathbb{G}_2 \neq B_2$ and there is no positive linear operator $Q : C[0, 1] \to \Pi_n$ such that $B_n = \overline{\mathbb{B}}_n \circ Q$, where Π_n is the linear space of all real polynomials of degree $\leq n$. Also, it was shown that $L \circ S_{\Delta_n} \neq B_n$, $n \geq 2$, for a large class of positive integral operators $L : C[0, 1] \to C[0, 1]$.

This study aiming to find a decomposition of classical Bernstein operators was continued. In [**6**], Gonska et al. gave a decomposition of the form $B_n = \overline{\mathbb{B}}_n \circ F_n$, where F_n is a nonpositive linear operator. The images of the monomials under F_n and its moments were obtained. Also, an asymptotic formula of Voronovskaya type for polynomials is proved. Several other results concerning the composition and decomposition of positive linear operators (in particular, several results about the operators F_n) can be found in the series [**5**]-[**10**] initiated by Heiner Gonska. For notation, definitions and existing results the reader is referred to these papers.

Motivated by the properties of $\mathbb{G}_n$, in this paper we are interested to study the composition of the piecewise linear interpolation operator and the Beta-type operator, namely $\mathbb{A}_n := S_{\Delta_n} \circ \overline{\mathbb{B}}_n$. Some preliminary results concerning these operators are established in Section 2. In Section 3, using a very recent result of Nasaireh and the second author of this paper (see [**19**]) we prove a Voronovskaya type theorem, substantially improving the corresponding results from [**7**]. The rate of convergence for the iterates of the operators $\overline{\mathbb{B}}_n \circ S_{\Delta_n}$ and $S_{\Delta_n} \circ \overline{\mathbb{B}}_n$ will be considered in Section 4. In order to compare the operator $S_{\Delta_n} \circ \overline{\mathbb{B}}_n$ with $\overline{\mathbb{B}}_n \circ S_{\Delta_n}$, Beta operators $\overline{\mathbb{B}}_n$ and the piecewise linear interpolation operator S_{Δ_n}, we give in Section 5 some estimates of the differences between these operators in terms of moduli of continuity. In Section 6 and Section 7 we study the behaviour of the operators $S_{\Delta_n} \circ \overline{\mathbb{B}}_n$ on the subspace of $C[0,1]$ consisting of all polygonal functions with nodes $\left\{0, \frac{1}{n}, \dots, \frac{n-1}{n}, 1\right\}$. Also, the iterates of the operator $S_{\Delta_n} \circ \overline{\mathbb{B}}_n$ will be studied. Finally, we investigate the eigenvalues of $S_{\Delta_n} \circ \overline{\mathbb{B}}_n$ and $\overline{\mathbb{B}}_n \circ S_{\Delta_n}$, and propose a conjecture concerning these eigenvalues and those of Bernstein operators. In our opinion, the most intriguing outcome of this paper is Conjecture 1, for which we present some experimental evidence. If true, this conjecture would emphasis a new and strong relationship between $\mathbb{G}_n$ and the classical Bernstein operator B_n.

2. THE OPERATOR $\mathbb{A}_N$

In this section we discuss the composition of the piecewise linear interpolation operator S_{Δ_n} and Beta-type operator, namely

$$\mathbb{A}_n := S_{\Delta_n} \circ \overline{\mathbb{B}}_n.$$

The operator $\mathbb{A}_n$ is positive and linear. Since S_{Δ_n} and $\overline{\mathbb{B}}_n$ preserve monotonicity and convexity, the operator $\mathbb{A}_n$ also has these properties.

Throughout the paper we use the notation $e_j(t) = t^j,\ j \in \mathbb{N}_0$.

Lemma 1. *The operators $\mathbb{A}_n$ verify*

$$\frac{x(1-x)}{n+1} \le \mathbb{A}_n(e_1 - xe_0)^2(x) \le \frac{2x(1-x)}{n+1},\ x \in [0,1].$$

Moreover,

$$\mathbb{A}_n(e_1 - xe_0)^2(x) \le \mathcal{K}(n;x),$$

where

$$\begin{aligned} \mathcal{K}(n;x) &:= \min\left\{\frac{2x(1-x)}{n+1}, \frac{x(1-x)}{n+1} + \frac{1}{4n(n+1)}\right\} \\ &= \begin{cases} \dfrac{2x(1-x)}{n+1},\ x \in [0,x_1] \cup [x_2,1], \\ \dfrac{x(1-x)}{n+1} + \dfrac{1}{4n(n+1)},\ x \in [x_1,x_2], \end{cases} \end{aligned}$$

and $x_1 := \dfrac{1}{2} - \dfrac{1}{2}\sqrt{\dfrac{n-1}{n}}$ *and* $x_2 := \dfrac{1}{2} + \dfrac{1}{2}\sqrt{\dfrac{n-1}{n}}$.

Proof. Since $(e_1 - xe_0)^2$ is a convex function, then $\overline{\mathbb{B}}_n(e_1 - xe_0)^2$ is a convex function and

$$\overline{\mathbb{B}}_n(e_1 - xe_0)^2 \le S_{\Delta_n}\left(\overline{\mathbb{B}}_n(e_1 - xe_0)^2\right) = \mathbb{A}_n(e_1 - xe_0)^2.$$

But $\overline{\mathbb{B}}_n(e_1 - xe_0)^2 = \dfrac{x(1-x)}{n+1}$. Therefore $\dfrac{x(1-x)}{n+1} \le \mathbb{A}_n(e_1 - xe_0)^2(x)$, and the lower bound of $\mathbb{A}_n(e_1 - xe_0)^2$ is obtained.

Moreover, we have

$$\mathbb{A}_n(e_1 - xe_0)^2(x) = S_{\Delta_n}\left[\,\overline{\mathbb{B}}_n(e_1 - xe_0)^2\right](x) = S_{\Delta_n}\left(\overline{\mathbb{B}}_n e_2\right)(x) - x^2$$
$$= \frac{n}{n+1}S_{\Delta_n}e_2(x) + \frac{x}{n+1} - x^2.$$

In [**7**, p. 3] it was proved that

$$S_{\Delta_n}(e_1 - xe_0)^2(x) = \left(x - \frac{l}{n}\right)\left(\frac{l+1}{n} - x\right) \tag{1}$$
$$\leq \frac{1}{4n^2},\ x \in \left[\frac{l}{n}, \frac{l+1}{n}\right],\ l \in \{0, 1, ..., n-1\}.$$

From (1) we get

$$S_{\Delta_n}e_2(x) = x\frac{2l+1}{n} - \frac{l(l+1)}{n^2},\ \text{for } x \in \left[\frac{l}{n}, \frac{l+1}{n}\right],\ l \in \{0, 1, ..., n-1\}.$$

Therefore,

$$\mathbb{A}_n(e_1 - xe_0)^2(x) = -x^2 + 2\frac{l+1}{n+1}x - \frac{l(l+1)}{n(n+1)} \tag{2}$$
$$\leq \frac{(l+1)(n-l)}{n(n+1)^2},\ x \in \left[\frac{l}{n}, \frac{l+1}{n}\right].$$

Starting from (2) it is not difficult to prove that

$$\mathbb{A}_n(e_1 - xe_0)^2(x) \leq \frac{x(1-x)}{n+1} + \frac{1}{4n(n+1)},\ x \in [0, 1],$$

and also

$$\mathbb{A}_n(e_1 - xe_0)^2(x) \leq \frac{2x(1-x)}{n+1},\ x \in [0, 1].$$

Therefore, for $x \in [0, 1]$,

$$\mathbb{A}_n(e_1 - xe_0)^2(x) \leq \min\left\{\frac{2x(1-x)}{n+1}, \frac{x(1-x)}{n+1} + \frac{1}{4n(n+1)}\right\} = \mathcal{K}(n; x).$$

□

Remark 1. *The central moment of* $\mathbb{A}_n$ *verifies*

$$\mathbb{A}_n(e_1 - xe_0)^2(x) \leq \frac{1}{4n},\ x \in [0, 1].$$

Remark 2. *Using a well known result* (*see, e.g.,* [**16**]) *the following estimate can be obtained*

$$|\mathbb{A}_n(f; x) - f(x)| \leq 2\omega\left(f, \sqrt{\mathcal{K}(n, x)}\right), \tag{3}$$

where $\omega(f, t) := \sup\{|f(x) - f(y)|, x, y \in [0, 1], |x - y| \leq t\},\ f \in C[0, 1]$.

Gonska et al. [**7**] gave an estimate of the central moment of $\mathbb{G}_n$ as follows

$$\frac{x(1-x)}{n+1} \le j_n(x)\frac{x(1-x)}{n+1} = \mathbb{G}_n(e_1 - xe_0)^2(x) \le 2\frac{x(1-x)}{n+1}, \tag{4}$$

where

$$1 \le j_n(x) \le 2,\ x \in [0,1]. \tag{5}$$

Remark 3. *As in Remark 2 one can obtain the next estimate for the operator* $\mathbb{G}_n$

$$|\mathbb{G}_n(f;x) - f(x)| \le 2\omega\left(f, \sqrt{\frac{2x(1-x)}{n+1}}\right), \tag{6}$$

which can be compared with (3).

Proposition 1. *Let* $f \in C[0,1]$*. If* f *is a convex function, then*

$$S_{\Delta_n} f \le \mathbb{A}_n f \le U_n f,$$

where the positive linear operator $U_n := B_n \circ \overline{\mathbb{B}}_n$ *was introduced by Chen* [**3**] *and Goodman and Sharma* [**11**].

Proof. Let $\mathcal{L} = \{B_n, \overline{\mathbb{B}}_n, S_{\Delta_n}\}$ and $L_n \in \mathcal{L}$. It is well-known that L_n preserves monotonicity and convexity, and for a convex function f, $f \le L_n f$. Therefore,

$$f \le S_{\Delta_n} f \le B_n S_{\Delta_n} f = B_n f.$$

From $S_{\Delta_n} f \le B_n f$, we get $S_{\Delta_n}\overline{\mathbb{B}}_n f \le B_n \overline{\mathbb{B}}_n f$. Thus, $\mathbb{A}_n f \le U_n f$. Moreover, from $f \le \overline{\mathbb{B}}_n f$, it follows $S_{\Delta_n} f \le S_{\Delta_n}\overline{\mathbb{B}}_n f = \mathbb{A}_n f$. This completes the proof. □

3. VORONOVSKAYA TYPE THEOREMS

Lemma 2. *For the functions* j_n *from (4), one has*

$$\lim_{n\to\infty} x(1-x)(j_n(x)-1) = 0 \text{ uniformly on } [0,1].$$

Proof. We get

$$\begin{aligned} x(1-x)j_n(x) &= (n+1)\mathbb{G}_n(e_1 - xe_0)^2(x) \\ &= (n+1)\left(\overline{\mathbb{B}}_n \circ S_{\Delta_n}\right)(e_1 - xe_0)^2(x) = (n+1)\left[\overline{\mathbb{B}}_n S_{\Delta_n} e_2(x) - x^2\right] \\ &= (n+1)\left[\overline{\mathbb{B}}_n\left(S_{\Delta_n} e_2 - e_2\right) + \left(\overline{\mathbb{B}}_n e_2 - e_2\right)\right](x). \end{aligned} \tag{7}$$

From relation (1), one has

$$\left(S_{\Delta_n} e_2 - e_2\right)(x) = S_{\Delta_n}\left((e_1 - xe_0)^2\right)(x) \le \frac{1}{4n^2}, \text{ for all } x \in [0,1],$$

therefore

$$\|\overline{\mathbb{B}}_n\left(S_{\Delta_n}e_2-e_2\right)\|\leq\frac{1}{4n^2}. \tag{8}$$

From (5), (7) and (8), we get

$$0\leq x(1-x)(j_n(x)-1)\leq\frac{n+1}{4n^2},$$

thus

$$\lim_{n\to\infty}x(1-x)(j_n(x)-1)=0 \text{ uniformly on } [0,1].$$

□

Theorem 1. *If $f\in C^2[0,1]$, $x\in[0,1]$, then*

$$\left|\mathbb{G}_nf(x)-f(x)-\frac{1}{2}\frac{x(1-x)}{n}f''(x)\right|\leq\frac{x(1-x)}{n}\left[\frac{j_n(x)-1}{2}|f''(x)|+\tilde{\omega}\left(f'',\frac{2}{\sqrt{n+1}}\right)\right],$$

where $\tilde{\omega}(f;t)=\sup\left\{\dfrac{(t-x)\omega(f;y)+(y-t)\omega(f;x)}{y-x};0\leq x<t\leq y\leq b-a\right\}$ *is the least concave majorant of the usual modulus of continuity ω.*

Proof. In [**7**, Proposition 3] the next Voronovskaya type result that depends on the functions j_n was obtained

$$\left|\mathbb{G}_nf(x)-f(x)-\frac{1}{2}j_n(x)\frac{x(1-x)}{n}f''(x)\right|\leq\frac{x(1-x)}{n+1}\tilde{\omega}\left(f'',\frac{2}{\sqrt{n+1}}\right).$$

Moreover,

$$\begin{aligned}&\left|\mathbb{G}_nf(x)-f(x)-\frac{1}{2}\frac{x(1-x)}{n}f''(x)\right|\leq\left|\mathbb{G}_nf(x)-f(x)-\frac{1}{2}j_n(x)\frac{x(1-x)}{n}f''(x)\right|\\&+\left|\left(\mathbb{G}_nf(x)-f(x)-\frac{1}{2}\frac{x(1-x)}{n}f''(x)\right)-\left(\mathbb{G}_nf(x)-f(x)-\frac{1}{2}j_n(x)\frac{x(1-x)}{n}f''(x)\right)\right|\\&\leq\frac{1}{2}\frac{x(1-x)}{n}|f''(x)||1-j_n(x)|+\frac{x(1-x)}{n+1}\tilde{\omega}\left(f'',\frac{2}{\sqrt{n+1}}\right).\end{aligned}$$

The proof is complete. □

From Lemma 2 and Theorem 1 the following result is obtained

Corollary 1. *If $f\in C^2[0,1]$, $x\in[0,1]$, then*

$$\lim_{n\to\infty}n\left(\mathbb{G}_nf(x)-f(x)\right)=\frac{x(1-x)}{2}f''(x), \tag{9}$$

uniformly on $[0,1]$.

In [**19**], Nasaireh and Raşa extended Voronovskaya type formulas for operators which are not necessarily positive. We recall this result.

Let X be a Banach space and $W \subset Z \subset Y$ linear subspaces of X. Let $A, B : Y \to X$; $U, V : Z \to X$ be linear operators. Consider also two sequences of linear operators $P_n : X \to X$, $Q_n : Y \to X$, $n \geq 1$, and suppose that each P_n is bounded.

Theorem 2. [**19**] *(i) Suppose that*

$$\lim_{n\to\infty} P_n x = x,\ x \in X \tag{10}$$

and

$$\lim_{n\to\infty} n(P_n y - y) = Ay, \quad \lim_{n\to\infty} n(Q_n y - y) = By,\ y \in Y. \tag{11}$$

Then

$$\lim_{n\to\infty} n(P_n Q_n y - y) = Ay + By,\ y \in Y. \tag{12}$$

ii) In addition to (10) and (11), suppose that

$$Bz \in Y, z \in Z,$$

$$\lim_{n\to\infty} n[n(P_n z - z) - Az] = Uz; \ \lim_{n\to\infty} n[n(Q_n z - z) - Bz] = Vz,\ z \in Z.$$

Then

$$\lim_{n\to\infty} n\left[n(P_n Q_n z - z) - Az - Bz\right] = Uz + Vz + ABz,\ z \in Z.$$

Corollary 2. *If* $f \in C^2[0,1]$, $x \in [0,1]$, *then*

$$\lim_{n\to\infty} n(\mathbb{A}_n f(x) - f(x)) = \frac{x(1-x)}{2} f''(x),\ f \in C^2[0,1] \text{ uniformly on } [0,1].$$

Proof. In [**7**, Theorem 2] the following estimate in terms of second modulus of continuity was obtained

$$|S_{\Delta_n} f(x) - f(x)| \leq \omega_2\left(f, \frac{1}{2n}\right) \leq \frac{1}{4n^2}\|f''\|,\ f \in C^2[0,1].$$

Therefore, $\lim_{n\to\infty} n\left(S_{\Delta_n} f(x) - f(x)\right) = 0$. Let $P_n := S_{\Delta_n}$ and $Q_n := \overline{\mathbb{B}}_n$ in Theorem 2. Then, $Af(x) = 0$ and $Bf(x) = \dfrac{x(1-x)}{2} f''(x)$, and

$$\lim_{n\to\infty} n\left(S_{\Delta_n} \circ \overline{\mathbb{B}}_n f(x) - f(x)\right) = Af(x) + Bf(x) = \frac{x(1-x)}{2} f''(x).$$

□

Remark 4. *The Voronovskaya type formula (9) for the operator $\mathbb{G}_n$ can be obtained considering $P_n := \overline{\mathbb{B}}_n$ and $Q_n := S_{\Delta_n}$ in Theorem 2.*

4. RATE OF CONVERGENCE OF THE ITERATES

Let $L_n : C[0,1] \to C[0,1]$, $n \geq 1$, be positive linear operators reproducing the affine functions and $P : C[0,1] \to C[0,1]$ defined as

$$Pf(x) = (1-x)f(0) + xf(1),\ x \in [0,1].$$

For each $n \geq 1$ and $s \in (0, +\infty)$ let

$$a_n(s) := \max\{x - x^2 - ns(L_n e_2(x) - e_2(x)) : x \in [0,1]\}.$$

Theorem 3. [**20**] *Let $0 < \alpha < 1$. Then*

$$|L_n^m f(x) - Pf(x)| \leq 2\omega\Big(f, \sqrt{\Big(1 - \frac{1}{nm^\alpha}\Big)^m x(1-x) + a_n(m^\alpha)}\Big)$$

for all $f \in C[0,1]$, $x \in [0,1]$, $m \geq 1$, $n \geq 1$.

In the following we consider $L_n \in \{\mathbb{A}_n, \mathbb{G}_n\}$. Using (4) and Lemma 1, we can write

$$\frac{x(1-x)}{n+1} \leq L_n e_2(x) - e_2(x) \leq \frac{2x(1-x)}{n+1},$$

therefore

$$x(1-x) - 2ns\frac{x(1-x)}{n+1} \leq x - x^2 - ns\left(L_n e_2(x) - e_2(x)\right) \leq x(1-x) - ns\frac{x(1-x)}{n+1},\ s > 0,$$

and $a_n(s) := \max\limits_{x\in[0,1]} \left\{x - x^2 - ns\left(L_n e_2(x) - e_2(x)\right)\right\} \leq \max\limits_{x\in[0,1]} x(1-x)\left(1 - \dfrac{ns}{n+1}\right)$.

For $s > \dfrac{n+1}{n}$, we get $a_n(s) \leq 0$. On the other hand, if we consider $x = 0$ in the definition of $a_n(s)$, it follows $a_n(s) \geq 0$. Thus, for $L_n \in \{\mathbb{A}_n, \mathbb{G}_n\}$, we have $a_n(s) = 0, \forall s > \dfrac{n+1}{n}$.

As a direct application of Theorem 3 the rate of convergence for the iterates of $L_n \in \{\mathbb{A}_n, \mathbb{G}_n\}$ can be obtained

Theorem 4. *Let $0 < \alpha < 1$. Then*

$$|L_n^m f(x) - Pf(x)| \leq 2\omega\left(f, \sqrt{\left(1 - \frac{1}{nm^\alpha}\right)^m x(1-x)}\right),$$

for all $f \in C[0,1]$, $x \in [0,1]$, $n \geq 1$ and $m^\alpha > \dfrac{n+1}{n}$.

5. DIFFERENCES OF CERTAIN OPERATORS

In the following we will compare the operators $\mathbb{A}_n$ and $\mathbb{G}_n$ with Beta operators $\overline{\mathbb{B}}_n$ and the piecewise linear interpolation operators S_{Δ_n}. In order to give some estimates of the differences between these operators we recall the result of Gonska et al. [**1**, Corollary 6].

Let T be a compact metric space. Let $C(T)$ be the Banach lattice of all real-valued continuous functions on T endowed with the max norm $\|\cdot\|_T$ and

$$Lip(T) := \bigcup_{M>0} Lip_M(1;T),$$

where by $Lip_M(\alpha;T), 0<\alpha\le 1$, we denote the set of all $f\in C(T)$ satisfying

$$\sup_{\|x-y\|\le t} |f(x)-f(y)| \le Mt^{\alpha},\ t\ge 0.$$

Corollary 3. (see [**1**, Corollary 6]) *Let T be a compact metric space, and $L: C(T)\to C(T)$, $L\neq 0$ a bounded linear operator mapping $Lip(T)$ to $Lip(T)$ such that for all $g\in Lip(T)$,*

$$|Lg|_{Lip} \le c\cdot |g|_{Lip},$$

with a constant c possibly depending on L, but independent of g. Suppose that for all $\xi, t>0$, all $f\in C(T)$, and some fixed $\eta=\eta(T)>0$ the inequality

$$\omega(f;\xi t) \le (1+\eta\xi)\cdot\omega(f;t)$$

is satisfied. Then,

$$\omega(Lf;t) \le \|L\|\cdot\tilde{\omega}\left(f;\frac{ct}{\|L\|}\right) \le (\|L\|+c\eta)\,\omega(f;t).$$

Let $Lip_M 1 := \{f\in C[0,1]:\ |f(x)-f(y)|\le M|x-y|,\ x,y\in[0,1]\}$, $M>0$ and

$$|f|_{Lip} := \sup\left\{\frac{|f(x)-f(y)|}{|x-y|}\ :\ x\neq y\right\}.$$

Lemma 3. *Let $L: C[0,1]\to C[0,1]$ be a positive linear operator mapping increasing functions into increasing functions. If $Le_0=e_0$ and $Le_1=e_1$, then*

i) $L(Lip_M 1)\subset Lip_M 1$, *for all* $M>0$;

ii) $|Lg|_{Lip}\le |g|_{Lip}$ *for all* $g\in Lip[0,1]$;

iii) $\omega(Lf,\delta)\le 2\omega(f,\delta)$, $f\in C[0,1], \delta>0$.

Proof. i) First, we show that $f \in Lip_1 1$ if and only if $e_1 \pm f$ is increasing.

Indeed, $f \in Lip_1 1 \iff (\forall x > y)\, |f(x) - f(y)| \leq x - y \iff (\forall x > y)\, y - x \leq f(x) - f(y) \leq x - y \iff e_1 + f$ and $e_1 - f$ are increasing.

Now, let $f \in Lip_1 1$. Then, $e_1 + f$ and $e_1 - f$ are increasing. Thus, $Le_1 + Lf$ and $Le_1 - Lf$ are increasing. Since $e_1 \pm Lf$ are increasing, it follows $Lf \in Lip_1 1$. Therefore,

$$L(Lip_1 1) \subset Lip_1 1. \tag{13}$$

From (13), we get $L(Lip_M 1) \subset Lip_M 1$, for all $M > 0$.

ii) Suppose that $|Lg|_{Lip} > |g|_{Lip}$ for a certain $g \in Lip[0,1]$. Choose M such that $|g|_{Lip} < M < |Lg|_{Lip}$. Then $g \in Lip_M 1$, and i) shows that $Lg \in Lip_M 1$. Consequently, $M < |Lg|_{Lip} \leq M$, a contradiction, and so (ii) is proved.

iii) Using i), ii) and Corollary 3, we get $\omega(Lf, \delta) \leq 2\omega(f, \delta), f \in C[0,1], \delta > 0$. □

Using Lemma 3 the next estimates of the differences of certain operators can be obtained

Proposition 2. *If $f \in C[0,1]$, then*

i) $\|\mathbb{A}_n f - \overline{\mathbb{B}}_n f\| \leq 2\omega\left(f, \dfrac{1}{n}\right)$;

ii) $\|\mathbb{G}_n f - \overline{\mathbb{B}}_n f\| \leq \omega\left(f, \dfrac{1}{n}\right)$;

iii) $\|\mathbb{A}_n f - \mathbb{G}_n f\| \leq 3\omega\left(f, \dfrac{1}{n}\right)$;

iv) $\|\mathbb{G}_n f - S_{\Delta_n} f\| \leq 4\omega\left(f, \dfrac{1}{2\sqrt{n+1}}\right)$;

v) $\|\mathbb{A}_n f - S_{\Delta_n} f\| \leq 2\omega\left(f, \dfrac{1}{2\sqrt{n+1}}\right)$.

Proof. i) Using [**7**, Remark 1, p.161], we get

$$\|\mathbb{A}_n f - \overline{\mathbb{B}}_n f\| = \|S_{\Delta_n} \overline{\mathbb{B}}_n f - \overline{\mathbb{B}}_n f\| \leq \omega\left(\overline{\mathbb{B}}_n f, \frac{1}{n}\right) \leq 2\omega\left(f, \frac{1}{n}\right).$$

ii) $\|\mathbb{G}_n f - \overline{\mathbb{B}}_n f\| = \|\overline{\mathbb{B}}_n S_{\Delta_n} f - \overline{\mathbb{B}}_n f\| \leq \|S_{\Delta_n} f - f\| \leq \omega\left(f, \dfrac{1}{n}\right)$.

iii) It follows immediately using i) and ii).

iv) Since $\overline{\mathbb{B}}_n\left((e_1 - x)^2; x\right) = \dfrac{x(1-x)}{n+1}$ and using [**16**], we have

$$\|\mathbb{G}_n f - S_{\Delta_n} f\| = \|\overline{\mathbb{B}}_n S_{\Delta_n} f - S_{\Delta_n} f\| \leq 2\omega\left(S_{\Delta_n} f, \frac{1}{2\sqrt{n+1}}\right) \leq 4\omega\left(f, \frac{1}{2\sqrt{n+1}}\right).$$

v) $\|\mathbb{A}_n f - S_{\Delta_n} f\| = \|S_{\Delta_n}\overline{\mathbb{B}}_n f - S_{\Delta_n} f\| \le \|\overline{\mathbb{B}}_n f - f\| \le 2\omega\left(f, \dfrac{1}{2\sqrt{n+1}}\right)$. □

Following [**15**] let us denote

$$\omega_1^*(f;h) := \inf\{\omega(f - ce_1, h) : c \in \mathbb{R}\},\ f \in C[0,1],\ h > 0.$$

Obviously, $\omega_1^*(f,h) \le \omega(f,h)$. The relationship between ω_1^* and ω_2 is investigated in [**4**], where it is shown that there is no constant $c > 0$ such that $\omega_1^*(f,\delta) \le c\omega_2(f,\delta)$ for all $f \in C[0,1]$, $\delta > 0$, but a constant $d > 0$ exists such that $\omega_1^*(f,\delta) \le d\omega_2(f,\sqrt{\delta})$ for all $f \in C[0,1]$, $\delta > 0$.

Since all the operators appearing in Proposition 2 reproduce the linear functions, we have

Corollary 4. *In all the inequalities i)-v) from Proposition 2, ω can be replaced by ω_1^*.*

6. THE MOMENTS OF $\mathbb{A}_N$

Let $\mathcal{P}_n$ be the subspace of $C[0,1]$ consisting of all polygonal functions with nodes $\left\{0, \dfrac{1}{n}, \ldots, \dfrac{n-1}{n}, 1\right\}$ and $\Phi_n := \{\varphi_{0,n}, \varphi_{n,n}, \varphi_{1,n}, \ldots, \varphi_{n-1,n}\}$ be a basis of $\mathcal{P}_n$. Considering $\varphi_{0,n}(0) = \varphi_{n,n}(1) = 1$, the graphs of functions $\varphi_{l,n}, l = 0, 1, \ldots, n$ are given in Figure 1.

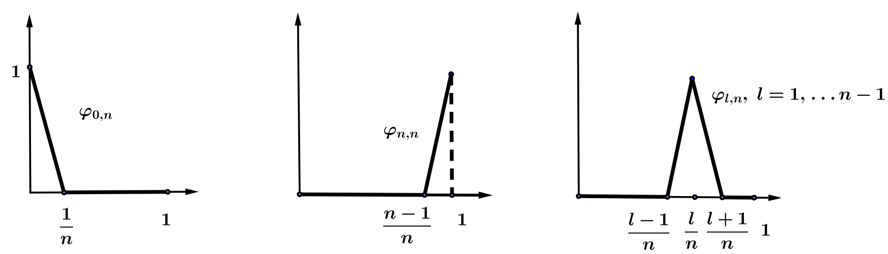

Figure 1: Basis of $\mathcal{P}_n$

Let $f \in C[0,1]$. Since $S_{\Delta_n} f \in \mathcal{P}_n$, we have the next representation of $S_{\Delta_n} f$ in the basis $\{\varphi_{l,n}\}$, $l \in \{0, 1, ..., n\}$

$$S_{\Delta_n} f = f(0)\varphi_{0,n} + f\left(\frac{1}{n}\right)\varphi_{1,n} + \cdots + f\left(\frac{j}{n}\right)\varphi_{j,n} + \cdots + f(1)\varphi_{n,n}.$$

By elementary calculations, we get $\overline{\mathbb{B}}_n e_k(x) = \dfrac{nx(nx+1)(nx+k-1)}{n(n+1)\ldots(n+k-1)}$, $k \ge 1$. Thus,

$$(\overline{\mathbb{B}}_n e_k)\left(\frac{j}{n}\right) = \frac{(n-1)!}{(n+k-1)!}\frac{(j+k-1)!}{(j-1)!}.$$

Using the above relations, the moments of $\mathbb{A}_n$ have the following representation

$$\mathbb{A}_n e_k(x) = \left(S_{\Delta_n}\overline{\mathbb{B}}_n e_k\right)(x) = \frac{(n-1)!}{(n+k-1)!}\sum_{j=1}^{n}\frac{(j+k-1)!}{(j-1)!}\varphi_{j,n}(x),\ k\geq 1,\ \mathbb{A}_n e_0 = e_0.$$

Proposition 3. *The operators $\mathbb{A}_n$ and the piecewise linear interpolation operators S_{Δ_n} verify*

$$(14)\qquad \begin{aligned} 0 &\leq \mathbb{A}_n e_k(x) - S_{\Delta_n} e_k(x) \\ &\leq \left\{\frac{k(k-1)}{2n} - \frac{k(k-1)(2k-1)}{6n^2} + \mathcal{R}(n,k)\right\}\left(1-\varphi_{0,n}(x)-x\right), \end{aligned}$$

where $\mathcal{R}(n,0)=\mathcal{R}(n,1)=0$ and for $k\geq 2$,

$$\lim_{n\to\infty} n^3\mathcal{R}(n,k) = 3c(k,k-3) - 3c(k,k-1)c(k,k-2) + c(k,k-1)^3.$$

Here $c(k,i), i=0,\dots,k$ are unsigned Stirling numbers of the first kind.

Proof. We have

$$\mathbb{A}_n e_k(x) - S_{\Delta_n} e_k(x) = \sum_{j=0}^{n}\left[\frac{j(j+1)\dots(j+k-1)}{n(n+1)\dots(n+k-1)} - \left(\frac{j}{n}\right)^k\right]\varphi_{j,n}(x).$$

In the following we give an estimate of the coefficient of $\varphi_{j,n}$ from the above sum

$$\begin{aligned} &\frac{j(j+1)\dots(j+k-1)}{n(n+1)\dots(n+k-1)} - \left(\frac{j}{n}\right)^k \\ &= \frac{\sum_{i=0}^k c(k,i)j^i}{\sum_{i=0}^k c(k,i)n^i} - \frac{j^k}{n^k} = \frac{\sum_{i=0}^k c(k,i)j^i n^i(n^{k-i}-j^{k-i})}{n^k\sum_{i=0}^k c(k,i)n^i} \\ &= (n-j)\frac{\sum_{i=0}^{k-1} c(k,i)j^i n^i(n^{k-i-1}+n^{k-i-2}j+\dots+nj^{k-i-2}+j^{k-i-1})}{n^k\sum_{i=0}^k c(k,i)n^i} \\ &\leq (n-j)\frac{\sum_{i=0}^{k-1} c(k,i)(k-i)n^{k+i-1}}{n^k\sum_{i=0}^k c(k,i)n^i} = \frac{n-j}{n}\frac{\sum_{i=0}^{k-1} c(k,k-1-i)(i+1)n^{2k-i-2}}{\sum_{i=0}^k c(k,k-i)n^{2k-i-1}} \\ &= \frac{n-j}{n}\left\{\frac{c(k,k-1)}{n} + \frac{1}{n^2}\left[2c(k,k-2) - c(k,k-1)^2\right] + \mathcal{R}(n,k)\right\}, \end{aligned}$$

where $\mathcal{R}(n,0)=\mathcal{R}(n,1)=0$ and for $k\geq 2$,

$$\lim_{n\to\infty} n^3\mathcal{R}(n,k) = 3c(k,k-3) - 3c(k,k-1)c(k,k-2) + c(k,k-1)^3.$$

Since $c(k,k-1) = \dfrac{k(k-1)}{2}$, $c(k,k-2) = \dfrac{k(k-1)(k-2)(3k-1)}{24}$, we get

$$\begin{aligned} 0 &\leq \frac{j(j+1)\dots(j+k-1)}{n(n+1)\dots(n+k-1)} - \left(\frac{j}{n}\right)^k \\ &\leq \left(1-\frac{j}{n}\right)\left\{\frac{k(k-1)}{2n} - \frac{k(k-1)(2k-1)}{6n^2} + \mathcal{R}(n,k)\right\}. \end{aligned}$$

Therefore,

$$\begin{aligned}0 &\le \mathbb{A}_n e_k(x) - S_{\Delta_n} e_k(x)\\ &\le \left\{ \frac{k(k-1)}{2n} - \frac{k(k-1)(2k-1)}{6n^2} + \mathcal{R}(n,k) \right\} \sum_{j=1}^{n} \left(1 - \frac{j}{n}\right) \varphi_{j,n}(x),\end{aligned}$$

and the relation (14) is obtained.

□

7. ITERATES AND EIGENSTRUCTURE OF $\mathbb{A}_N$

Let $f \in \mathcal{P}_n$. Using the definition of $\overline{\mathbb{B}}_n$ it is not difficult to see that there exist $c_{n,l,i} \in \mathbb{R}$, $i, l = \{0, \ldots, n\}$ such that

$$\overline{\mathbb{B}}_n f\left(\frac{l}{n}\right) = \sum_{i=0}^{n} c_{n,l,i} f\left(\frac{i}{n}\right),\ l \in \{0, \ldots, n\}. \tag{15}$$

(For particular cases see Examples 1-3).

Then

$$(\mathbb{A}_n f)\left(\frac{l}{n}\right) = \left(S_{\Delta_n} \overline{\mathbb{B}}_n f\right)\left(\frac{l}{n}\right) = \overline{\mathbb{B}}_n f\left(\frac{l}{n}\right),$$

and hence (15) leads to

$$(\mathbb{A}_n f)\left(\frac{l}{n}\right) = \sum_{i=0}^{n} c_{n,l,i} f\left(\frac{i}{n}\right),\ l \in \{0, \ldots, n\}. \tag{16}$$

Let $\{\varphi_{0,n}, \varphi_{n,n}, \varphi_{1,n}, \ldots, \varphi_{n-1,n}\}$ be the basis of $\mathcal{P}_n$ described at the beginning of Section . Consider the restriction $\mathbb{A}_n : \mathcal{P}_n \to \mathcal{P}_n$ of $\mathbb{A}_n$, and let $\mathcal{M}_n$ be the matrix of $\mathbb{A}_n$ with regard to this basis.

Using (16) we find that

$$\mathcal{M}_n = \begin{pmatrix} 1 & 0 & 0 & \ldots & 0 \\ 0 & 1 & 0 & \ldots & 0 \\ c_{n,1,0} & c_{n,1,n} & c_{n,1,1} & \cdots & c_{n,1,n-1} \\ c_{n,2,0} & c_{n,2,n} & c_{n,2,1} & \cdots & c_{n,2,n-1} \\ \vdots & \vdots & \vdots & \ldots & \vdots \\ c_{n,l,0} & c_{n,l,n} & c_{n,l,1} & \cdots & c_{n,l,n-1} \\ \vdots & \vdots & \vdots & \ldots & \vdots \\ c_{n,n-1,0} & c_{n,n-1,n} & c_{n,n-1,1} & \cdots & c_{n,n-1,n-1} \end{pmatrix}.$$

The matrix $\mathcal{M}_n$ can be written as $\mathcal{M}_n := \begin{pmatrix} I_{2\times 2} & O_{2\times(n-1)} \\ R_{(n-1)\times 2} & Q_{(n-1)\times(n-1)} \end{pmatrix}$.

Example 1. Let $n=2$ and $f\in\mathcal{P}_2$. Then

$$f(t)=\begin{cases} 2\left(f\left(\frac{1}{2}\right)-f(0)\right)t+f(0), & t\in\left[0,\frac{1}{2}\right),\\ 2\left(f(1)-f\left(\frac{1}{2}\right)\right)t+2f\left(\frac{1}{2}\right)-f(1), & t\in\left[\frac{1}{2},1\right].\end{cases}$$

Remember that $\overline{\mathbb{B}}_2f(x)=\dfrac{1}{B(2x,2(1-x))}\displaystyle\int_0^1 t^{2x-1}(1-t)^{2(1-x)-1}f(t)dt$. We get,

$$\overline{\mathbb{B}}_2f(0)=f(0),\ \overline{\mathbb{B}}_2f(1)=f(1),\ \overline{\mathbb{B}}_2f\left(\frac{1}{2}\right)=\frac{1}{4}f(0)+\frac{1}{4}f(1)+\frac{1}{2}f\left(\frac{1}{2}\right).$$

Therefore,

$$\mathcal{M}_2=\begin{pmatrix}1&0&0\\0&1&0\\\frac{1}{4}&\frac{1}{4}&\frac{1}{2}\end{pmatrix}.$$

By elementary techniques it can be proved that

$$\mathcal{M}_2^m=\begin{pmatrix}1&0&0\\0&1&0\\\frac{1}{2}-\frac{1}{2^{m+1}}&\frac{1}{2}-\frac{1}{2^{m+1}}&\frac{1}{2^m}\end{pmatrix}.$$

Therefore,

$$\lim_{m\to\infty}\mathcal{M}_2^m=\begin{pmatrix}1&0&0\\0&1&0\\\frac{1}{2}&\frac{1}{2}&0\end{pmatrix},\text{ and}$$

$$\lim_{m\to\infty}\mathbb{A}_2^m f=f(0)\varphi_{0,2}+f(1)\varphi_{2,2}+\frac{1}{2}\left(f(0)+f(1)\right)\varphi_{1,2}. \tag{17}$$

The limit (17) of the iterates can be obtained using [**20**, Example 6.3].

The eigenvalues and the eigenvectors corresponding to the matrix $\mathcal{M}_2$ are calculated below

$$\lambda_1=\frac{1}{2},\ \lambda_2=1,\ \lambda_3=1,$$
$$v_1=(0\ 0\ 1)^t,\ v_2=(-1\ 1\ 0)^t,\ v_3=(2\ 0\ 1)^t.$$

Example 2. Let $n=3$ and $f\in\mathcal{P}_3$. Then

$$f(t)=\begin{cases} 3\left[f\left(\frac{1}{3}\right)-f(0)\right]t+f(0), & t\in\left[0,\frac{1}{3}\right),\\ \left[f\left(\frac{2}{3}\right)-f\left(\frac{1}{3}\right)\right](3t-1)+f\left(\frac{1}{3}\right), & t\in\left[\frac{1}{3},\frac{2}{3}\right),\\ \left[f(1)-f\left(\frac{2}{3}\right)\right](3t-2)+f\left(\frac{2}{3}\right), & t\in\left[\frac{2}{3},1\right].\end{cases}$$

We obtain,

$$\mathcal{M}_3=\begin{pmatrix}1&0&0&0\\0&1&0&0\\\frac{8}{27}&\frac{1}{27}&\frac{4}{9}&\frac{2}{9}\\\frac{1}{27}&\frac{8}{27}&\frac{2}{9}&\frac{4}{9}\end{pmatrix}.$$

Therefore,

$$\mathcal{M}_3^m = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \frac{2}{3}-\frac{1}{2}\left(\frac{2}{3}\right)^m-\frac{1}{6}\left(\frac{2}{9}\right)^m & \frac{1}{3}-\frac{1}{2}\left(\frac{2}{3}\right)^m+\frac{1}{6}\left(\frac{2}{9}\right)^m & \frac{1}{2}\left(\frac{2}{3}\right)^m+\frac{1}{2}\left(\frac{2}{9}\right)^m & \frac{1}{2}\left(\frac{2}{3}\right)^m-\frac{1}{2}\left(\frac{2}{9}\right)^m \\ \frac{1}{3}-\frac{1}{2}\left(\frac{2}{3}\right)^m+\frac{1}{6}\left(\frac{2}{9}\right)^m & \frac{2}{3}-\frac{1}{2}\left(\frac{2}{3}\right)^m-\frac{1}{6}\left(\frac{2}{9}\right)^m & \frac{1}{2}\left(\frac{2}{3}\right)^m-\frac{1}{2}\left(\frac{2}{9}\right)^m & \frac{1}{2}\left(\frac{2}{3}\right)^m+\frac{1}{2}\left(\frac{2}{9}\right)^m \end{pmatrix}$$

and

$$\lim_{m\to\infty} \mathcal{M}_3^m = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \frac{2}{3} & \frac{1}{3} & 0 & 0 \\ \frac{1}{3} & \frac{2}{3} & 0 & 0 \end{pmatrix}.$$

Thus,

$$\lim_{m\to\infty} \mathbb{A}_3^m f = f(0)\varphi_{0,3}+f(1)\varphi_{3,3}+\left(\frac{2}{3}f(0)+\frac{1}{3}f(1)\right)\varphi_{1,3}+\left(\frac{1}{3}f(0)+\frac{2}{3}f(1)\right)\varphi_{2,3}.$$

The eigenvalues and the eigenvectors corresponding to the matrix $\mathcal{M}_3$ are given below

$$\lambda_1 = \frac{2}{9},\ \lambda_2 = \frac{2}{3},\ \lambda_3 = 1,\ \lambda_4 = 1,$$

$$v_1 = (0\ 0\ -1\ 1)^t,\ v_2 = (0\ 0\ 1\ 1)^t,\ v_3 = (-1\ 2\ 0\ 1)^t,\ v_4 = (2\ -1\ 1\ 0)^t.$$

Example 3. Let $n = 4$ and $f \in \mathcal{P}_4$. Then

$$\mathcal{M}_4 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ \frac{81}{256} & \frac{1}{256} & \frac{55}{128} & \frac{25}{128} & \frac{7}{128} \\ \frac{7}{128} & \frac{7}{128} & \frac{17}{64} & \frac{23}{64} & \frac{17}{64} \\ \frac{1}{256} & \frac{81}{256} & \frac{7}{128} & \frac{25}{128} & \frac{55}{128} \end{pmatrix}.$$

Using [**20**, Example 6.3], we get

$$\lim_{m\to\infty} \mathcal{M}_4^m = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ \frac{3}{4} & \frac{1}{4} & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ \frac{1}{4} & \frac{3}{4} & 0 & 0 & 0 \end{pmatrix}.$$

Thus,

$$\lim_{m\to\infty} \mathbb{A}_4^m f = f(0)\varphi_{0,4}+f(1)\varphi_{4,4}+\left(\frac{3}{4}f(0)+\frac{1}{4}f(1)\right)\varphi_{1,4}+\left(\frac{1}{2}f(0)+\frac{1}{2}f(1)\right)\varphi_{2,4}$$
$$+\left(\frac{1}{4}f(0)+\frac{3}{4}f(1)\right)\varphi_{3,4}.$$

The eigenvalues and the eigenvectors corresponding to the matrix $\mathcal{M}_4$ are calculated below

$$\lambda_1 = 1,\ \lambda_2 = 1,\ \lambda_3 = \frac{3}{8},\ \ \lambda_4 = \frac{3}{32},\ \ \lambda_5 = \frac{3}{4}$$

$$v_1 = (-2\ 2\ \ -1\ 0\ 1)^t,\ \ v_2 = (3\ \ -1\ 2\ 1\ 0)^t,\ \ v_3 = (0\ 0\ \ -1\ 0\ 1)^t,$$

$$v_4 = (0\ 0\ 1\ \ -2\ 1)^t,\ \ v_5 = (0\ 0\ 25\ 34\ 25)^t.$$

From Theorem 4 we infer that

$$\lim_{m\to\infty} \mathbb{A}_n^m f = f(0)\varphi_{0,n} + f(1)\varphi_{n,n} + \left(\frac{n-1}{n}f(0) + \frac{1}{n}f(1)\right)\varphi_{1,n} + \cdots + \left(\frac{1}{n}f(0) + \frac{n-1}{n}f(1)\right)\varphi_{n-1,n}.$$

This can be written as

$$\lim_{m\to\infty} \mathcal{M}_n^m \left(f(0) \quad f(1) \quad f\left(\frac{1}{n}\right) \quad \ldots \quad f\left(\frac{n-1}{n}\right)\right)^t = \left(f(0) \quad f(1) \quad \frac{n-1}{n}f(0)+\frac{1}{n}f(1) \quad \cdots \quad \frac{1}{n}f(0)+\frac{n-1}{n}f(1)\right)^t.$$

Therefore,

$$\lim_{m\to\infty} \mathcal{M}_n^m = \begin{pmatrix} 1 & 0 & 0 & \ldots & 0 \\ 0 & 1 & 0 & \ldots & 0 \\ \frac{n-1}{n} & \frac{1}{n} & 0 & \ldots & 0 \\ \frac{n-2}{n} & \frac{2}{n} & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{1}{n} & \frac{n-1}{n} & 0 & \ldots & 0 \end{pmatrix}$$

Remark 5. *The above result gives us the limit* $\lim_{m\to\infty} \mathcal{M}_n^m$*, even if we do not know the explicit form of the matrix* $\mathcal{M}_n^m$ *(only in the case* $n \in \{2, 3, 4\}$*).*

Let $\psi_{in} := \overline{\mathbb{B}}_n \varphi_{i,n}$, $i = 0, 1, \ldots, n$, and let V_n be the linear subspace of $C[0, 1]$ spanned by the basis $\Psi_n = \{\psi_{0n}, \psi_{nn}, \psi_{1n} \ldots, \psi_{n-1,n}\}$. Consider the restriction $\mathbb{G}_n : V_n \to V_n$. We have, for $j = 0, 1, \ldots, n$,

$$\mathbb{G}_n \psi_{jn} = \overline{\mathbb{B}}_n S_{\Delta_n} \psi_{jn} = \overline{\mathbb{B}}_n \sum_{i=0}^{n} \psi_{jn}\left(\frac{i}{n}\right)\varphi_{in} = \sum_{i=0}^{n} \psi_{jn}\left(\frac{i}{n}\right)\psi_{in}.$$

Therefore the matrix $\mathcal{N}_n$ of $\mathbb{G}_n : V_n \to V_n$ with regard to the above basis is
(18)

$$\mathcal{N}_n = \begin{pmatrix} \psi_{0n}(0) & \psi_{nn}(0) & \psi_{1n}(0) & \dots & \psi_{n-1,n}(0) \\ \psi_{0n}(1) & \psi_{nn}(1) & \psi_{1n}(1) & \dots & \psi_{n-1,n}(1) \\ \psi_{0n}\left(\frac{1}{n}\right) & \psi_{nn}\left(\frac{1}{n}\right) & \psi_{1n}\left(\frac{1}{n}\right) & \dots & \psi_{n-1,n}\left(\frac{1}{n}\right) \\ \vdots & \vdots & \vdots & \dots & \vdots \\ \psi_{0n}\left(\frac{n-1}{n}\right) & \psi_{nn}\left(\frac{n-1}{n}\right) & \psi_{1n}\left(\frac{n-1}{n}\right) & \dots & \psi_{n-1,n}\left(\frac{n-1}{n}\right) \end{pmatrix}.$$

According to (15) we have for $j, l = 0, 1, \dots, n$,

(19)

$$\mathbb{G}_n \psi_{jn}\left(\frac{l}{n}\right) = \left(\overline{\mathbb{B}}_n S_{\Delta_n} \psi_{jn}\right)\left(\frac{l}{n}\right) = \sum_{i=0}^{n} c_{n,l,i}\left(S_{\Delta_n}\psi_{jn}\right)\left(\frac{i}{n}\right) = \sum_{i=0}^{n} c_{n,l,i}\psi_{jn}\left(\frac{i}{n}\right).$$

From (18) and (19) it follows that

$$\mathcal{N}_n = \mathcal{M}_n^{-1} \begin{pmatrix} \mathbb{G}_n\psi_{0n}(0) & \mathbb{G}_n\psi_{nn}(0) & \mathbb{G}_n\psi_{1n}(0) & \dots & \mathbb{G}_n\psi_{n-1,n}(0) \\ \mathbb{G}_n\psi_{0n}(1) & \mathbb{G}_n\psi_{nn}(1) & \mathbb{G}_n\psi_{1n}(1) & \dots & \mathbb{G}_n\psi_{n-1,n}(1) \\ \mathbb{G}_n\psi_{0n}\left(\frac{1}{n}\right) & \mathbb{G}_n\psi_{nn}\left(\frac{1}{n}\right) & \mathbb{G}_n\psi_{1n}\left(\frac{1}{n}\right) & \dots & \mathbb{G}_n\psi_{n-1,n}\left(\frac{1}{n}\right) \\ \vdots & \vdots & \vdots & \dots & \vdots \\ \mathbb{G}_n\psi_{0n}\left(\frac{n-1}{n}\right) & \mathbb{G}_n\psi_{nn}\left(\frac{n-1}{n}\right) & \mathbb{G}_n\psi_{1n}\left(\frac{n-1}{n}\right) & \dots & \mathbb{G}_n\psi_{n-1,n}\left(\frac{n-1}{n}\right) \end{pmatrix}$$

$$= \mathcal{M}_n^{-1}\mathcal{N}_n^2.$$

Consequently, $\mathcal{N}_n = \mathcal{M}_n$. So we have proved

Theorem 5. *The matrix of $\mathbb{G}_n : V_n \to V_n$ with regard to the basis Ψ_n is the same as the matrix of $\mathbb{A}_n : \mathcal{P}_n \to \mathcal{P}_n$ with regard to the basis Φ_n.*

Remark 6. *The iterates of $\mathbb{G}_n : V_n \to V_n$ can be investigated using the same method as the iterates of $\mathbb{A}_n : \mathcal{P}_n \to \mathcal{P}_n$. We omit the details.*

Remark 7. *Let $B_n : \Pi_n \to \Pi_n$ be the restriction of the Bernstein operator. With regard to the basis $\{b_{0n}, b_{nn}, b_{1n} \dots, b_{n-1,n}\}$, where $b_{jn}(x) := \binom{n}{j}x^j(1-x)^{n-j}$, the matrix of B_n is*

$$\mathcal{T}_n = \left(\binom{n}{j}\left(\frac{i}{n}\right)^j\left(1-\frac{i}{n}\right)^{n-j}\right)_{i,j\in\{0,1,\dots,n-1\}}.$$

It is well known that the eigenvalues of B_n are the numbers 1 and $\dfrac{n(n-1)\dots(n-k+1)}{n^k}$, $k \in \{1, ..., n\}$. We have

$$\mathcal{T}_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \end{pmatrix}, \textit{ with eigenvalues } \lambda_1 = \tfrac{1}{2}, \lambda_2 = 1, \lambda_3 = 1,$$

$$\mathcal{T}_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \frac{8}{27} & \frac{1}{27} & \frac{4}{9} & \frac{2}{9} \\ \frac{1}{27} & \frac{8}{27} & \frac{2}{9} & \frac{4}{9} \end{pmatrix}, \textit{ with eigenvalues } \lambda_1 = \frac{2}{9}, \lambda_2 = \tfrac{2}{3}, \lambda_3 = 1, \lambda_4 = 1,$$

$$\mathcal{T}_4 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ \frac{81}{256} & \frac{1}{256} & \frac{54}{128} & \frac{27}{128} & \frac{6}{128} \\ \frac{8}{128} & \frac{8}{128} & \frac{16}{64} & \frac{24}{64} & \frac{16}{64} \\ \frac{1}{256} & \frac{81}{256} & \frac{6}{128} & \frac{27}{128} & \frac{54}{128} \end{pmatrix}, \textit{ with eigenvalues } \lambda_1 = 1, \lambda_2 = 1, \lambda_3 = \frac{3}{8},\ \lambda_4 = \frac{3}{32},\ \lambda_5 = \frac{3}{4}.$$

Compare with $\mathcal{M}_2$, $\mathcal{M}_3$, $\mathcal{M}_4$ and their eigenvalues from Examples 1-3. So we are lead to the

Conjecture 1. *$\mathbb{G}_n : V_n \to V_n$ and $\mathbb{A}_n : \mathcal{P}_n \to \mathcal{P}_n$ have the same eigenvalues, with the same algebraic and geometric multiplicities, as $B_n : \Pi_n \to \Pi_n$. Consequently, these operators are similar.*

If true, this conjecture would show a new and strong relationship between $\mathbb{G}_n$ and the classical Bernstein operator B_n. Other facets of this relationship are presented in [**6**], [**7**], [**12**].

Acknowledgements. This work was supported by a Hasso Plattner Excellence Research Grant (LBUS-HPI-ERG-2020-04), financed by the Knowledge Transfer Center of the Lucian Blaga University of Sibiu.

REFERENCES

1. G. Anastassiou, C. Cottin, H. Gonska: *Global smoothness of approximating functions.* Analysis **11** (1991), 43-57.
2. S.N. Bernstein: *Démonstration du théorème de Weierstrass fondée sur le calcul des probabilités.* Communications de la Société Mathematique de Kharkov, **13** (1913), 1-2.
3. W. Chen: *On the modified Durrmeyer-Bernstein operator.* Report of the Fifth Chinese Conference on Approximation Theory, Zhen Zhou, China, 81-83, 1987 (in handwritten Chinese).

4. H. Gonska: *On approximation by linear operators: Improved estimates.* J. Numer. Anal. Approx. Theory, **14**(1), (1985), 7-32.

5. H. Gonska: *On the composition and decomposition of positive linear operators.* In: Approximation Theory and its Applications (Ukrainian), pp. 161–180. Pr. Inst. Mat. Nats. Akad. Nauk Ukr. Mat. Zastos, Kiev (1999).

6. H. Gonska, M. Heilmann, A. Lupaş, I. Raşa: *On the composition and decomposition of positive linear operators III: A non-trivial decomposition of the Bernstein operator.* arXiv:1204.2723v1 [math.CA], (2012).

7. H. Gonska, D. Kacsó, I. Raşa: *On the Composition and Decomposition of Positive Linear Operators (VI).* Constructive Theory of Functions, Sozopol 2016 (K. Ivanov, G. Nikolov and R. Uluchev, Eds.), Prof. Marin Drinov Academic Publishing House, Sofia, 2018, pp. 1-15.

8. H. Gonska, I. Raşa: *On the composition and decomposition of positive linear operators (II).* Studia Sci. Math. Hungar. **47** (2010), 448–461.

9. H. Gonska, I. Raşa: *On the Composition and Decomposition of Positive Linear Operators (V).* Results Math **72** (2017), 1033–1040.

10. H. Gonska, G. Tachev: *On the composition and decomposition of positive linear operators IV: Favard-Bernstein operators revisited.* Gen. Math. **20**(5) (2012), 37-46.

11. T.N.T. Goodman, A. Sharma: *A modified Bernstein-Schoenberg operator.* Proc. Conf. Constructive Theory of Functions, Varna 1987 (Bl. Sendov et al., eds.). Publ. House Bulg. Acad. Sci., Sofia, 1988, 166-173.

12. M. Heilmann, I. Raşa: *On the decomposition of Bernstein operators.* Numer. Funct. Anal. Optimiz. **36**(1) (2015), 72-85.

13. A. Lupaş: *Die Folge der Beta-Operatoren.* Dissertation Universität Stuttgart 1972.

14. A. Lupaş, L. Lupaş: *Polynomials of binomial type and approximation operators.* Stud. Univ. Babeş-Bolyai Math. **32**(4) (1987), 61–69.

15. M.J. Marsden, I.J. Schoenberg: *On variation diminishing spline approximation methods.* Mathematica (Cluj), **8** (1966), 61-82.

16. B. Mond: *On the degree of approximation by linear positive operators.* J. Approximation Theory, **18**(3) (1976), 304-306.

17. G. Mühlbach: *Verallgemeinerungen der Bernstein- und der Lagrangepolynome.* Bemerkungen zu einer Klasse linearer Polynomoperatoren von D.D. Stancu, Rev. Roumaine Math. Pure Appl. **15** (1970), 1235-1252.

18. G. Mühlbach: *Rekursionsformeln für die zentralen Momente der Polya- und der Beta-Verteilung.* Metrika **19** (1972), 171-177.

19. F. Nasaireh, I. Raşa: *Another Look at Voronovskaja Type Formulas.* Journal of Mathematical Inequalities, **12**(1) (2018), 95-105.

20. I. Raşa: *C_0 - semigroups and iterates of positive linear operators: asymptotic behaviour.* Rendiconti del Circolo Mathematico di Palermo, Serie II, Suppl. **82** (2010), 1-20.

(Received 03. 11. 2019.)
(Revised 28. 02. 2021.)

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