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THE TOTAL TORSION OF KNOTS UNDER SECOND ORDER INFINITESIMAL BENDING

Marija S. Najdanović, Ljubica S. Velimirović and Svetozar R. Rančić*

In this paper we consider infinitesimal bending of the second order of curves and knots. The total torsion of the knot during the second order infinitesimal bending is discussed and expressions for the first and the second variation of the total torsion are given. Some examples aimed to illustrate infinitesimal bending of knots are shown using figures. Colors are used to illustrate torsion values at different points of bent knots and the total torsion is numerically calculated.

1. INTRODUCTION

One of the global properties of a curve C is its total torsion defined by the integral $\mathcal{T} = \int_C \tau(s)\, ds$, where s and τ are the arc length and the torsion of C, respectively. Geometrically, the total torsion is a measure of the binormal indicatrix. It is well known that for any real number r there is a closed curve C such that its total torsion is equal to r. Also, a classical result in differential geometry assures that the total torsion of a closed spherical curve in the three-dimensional space vanishes. Besides, if a surface is such that the total torsion vanishes for all closed curves, it is part of a sphere or a plane. There are many investigations about the total torsion (see, for instance [**9**]).

A knot is a simple closed curve in 3-space. The knowledge of curvature and torsion profiles of a knot is essential, since it allows one to understand the details of its shape. If, for instance, in a certain part of a knot its curvature and

*Corresponding author. Ljubica S. Velimirović
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torsion are both constant, one can conclude that this part of the knot has a helical shape [**1**] . The total curvature and the total torsion have many applications, for instance, they provide important information about how much, and in what ways, the knotted polymers turn in the space [**8**].

Infinitesimal bending of a curve or a surface in Euclidean 3-space, is a type of deformations characterized with the rigidity of the arc length with a given precision. In this case one observes changes of others magnitudes, and then we say that they are rigid or flexible. Theory of infinitesimal bending is in close connection with thin elastic shell theory and has a huge application from the mechanical point of view. Infinitesimal bending of curves and surfaces is studied, for instance, in [**2, 6, 12, 13, 14**]. Some results on infinitesimal transformations can be found in [**3, 5, 11**]. Infinitesimal bending of knots is considered in [**4, 7, 10**].

2. SECOND ORDER INFINITESIMAL BENDING OF A CURVE IN $\mathcal{R}^3$

Let us consider a regular curve

$$C : \mathbf{r} = \mathbf{r}(u), \quad u \in \mathcal{J} \subseteq \mathcal{R}$$

of a class C^α, $\alpha \geq 3$, included in a family of the curves

$$C_\epsilon : \tilde{\mathbf{r}}(u, \epsilon) = \mathbf{r}_\epsilon(u) = \mathbf{r}(u) + \epsilon \overset{(1)}{\mathbf{z}}(u) + \epsilon^2 \overset{(2)}{\mathbf{z}}(u), \tag{1}$$

where $\epsilon \geq 0$, $\epsilon \to 0$ and we get C for $\epsilon = 0$ $(C = C_0)$. The fields $\overset{(j)}{\mathbf{z}}(u) \in C^\alpha$, $\alpha \geq 3$, $j = 1, 2$, are continuous vector functions defined in the points of C.

Definition 1. [**2**] *A family of curves C_ϵ is an* **infinitesimal bending of the second order of the curve** *C if*

$$ds_\epsilon^2 - ds^2 = o(\epsilon^2).$$

The field $\overset{(j)}{\mathbf{z}} = \overset{(j)}{\mathbf{z}}(u)$ is the **infinitesimal bending field of the order** *j, $j = 1, 2$, of the curve C.*

The previous condition is equivalent to the system of equations [**2**]:

$$d\mathbf{r} \cdot d\overset{(1)}{\mathbf{z}} = 0, \quad 2d\mathbf{r} \cdot d\overset{(2)}{\mathbf{z}} + d\overset{(1)}{\mathbf{z}} \cdot d\overset{(1)}{\mathbf{z}} = 0,$$

where $\cdot$ stands for the scalar product in $\mathcal{R}^3$.

Let $\{\mathbf{t}, \mathbf{n_1}, \mathbf{n_2}\}$ be an orthonormal basis along the curve C, where $\mathbf{t}$ is the unit tangent, $\mathbf{n}_1$ and $\mathbf{n}_2$ are unit principal normal and binormal vector field of the curve, respectively. We choose an orientation with $\mathbf{n}_2 = \mathbf{t} \times \mathbf{n}_1$. The next theorem is related to determination of the infinitesimal bending field of a curve C.

Theorem 1. [**7**] *Infinitesimal bending fields of the first and the second order for the curve C are respectively*

$$\overset{(1)}{\mathbf{z}} = \int [p(u)\mathbf{n}_1 + q(u)\mathbf{n}_2]\, du + C_1, \tag{2}$$

$$\overset{(2)}{\mathbf{z}} = \int [-\frac{p^2(u)+q^2(u)}{2\|\dot{\mathbf{r}}\|}\mathbf{t} + r(u)\mathbf{n}_1 + g(u)\mathbf{n}_2]\, du + C_2, \tag{3}$$

where $p(u)$, $q(u)$, $r(u)$, $g(u)$ are arbitrary integrable functions and vectors $\mathbf{t}$, $\mathbf{n}_1$, $\mathbf{n}_2$ are unit tangent, principal normal and binormal vector fields, respectively, of the curve C. C_1 and C_2 are constants.

Under an infinitesimal bending geometric magnitudes of the curve are changed which is described with variations of these geometric magnitudes.

Definition 2. [**12**] *Let $\mathcal{A} = \mathcal{A}(u)$ be a magnitude which characterizes a geometric property on the curve C and $\mathcal{A}_\epsilon = \mathcal{A}_\epsilon(u)$ the corresponding magnitude on the curve C_ϵ being an infinitesimal bending of the curve C and let the equation*

$$\Delta\mathcal{A} = \mathcal{A}_\epsilon - \mathcal{A} = \epsilon\,\delta\mathcal{A} + \epsilon^2\,\delta^2\mathcal{A} + \ldots + \epsilon^n\,\delta^n\mathcal{A} + \ldots$$

be a valid one. The coefficients $\delta\mathcal{A}, \delta^2\mathcal{A}, \ldots, \delta^n\mathcal{A}, \ldots$ are **the first, the second, ..., the n-th variation** *of the geometric magnitude $\mathcal{A}$, respectively under the infinitesimal bending C_ϵ of the curve C.*

In this paper we will only consider the first and the second variation of a magnitude under infinitesimal bending of the second order. For this reason, we can represent the magnitude $\mathcal{A}_\epsilon$ as

$$\mathcal{A}_\epsilon = \mathcal{A} + \epsilon\,\delta\mathcal{A} + \epsilon^2\delta^2\mathcal{A},$$

by neglecting the terms in ϵ^n, $n \geq 3$. It is easy to see that under a second order infinitesimal bending of a curve, the first and the second variation of a line element ds are equal to zero, i. e. $\delta(ds) = \delta^2(ds) = 0$. Also, some other properties of variations are valid [**6**]:

I. $\delta\mathcal{A}\mathcal{B} = \mathcal{A}\,\delta\mathcal{B} + \mathcal{B}\,\delta\mathcal{A}, \quad \delta^2\mathcal{A}\mathcal{B} = \mathcal{A}\,\delta^2\mathcal{B} + \mathcal{B}\,\delta^2\mathcal{A} + \delta\mathcal{A}\,\delta\mathcal{B}$

II. $\delta(\frac{d\mathcal{A}}{du}) = \frac{d(\delta\mathcal{A})}{du}, \quad \delta^2(\frac{d\mathcal{A}}{du}) = \frac{d(\delta^2\mathcal{A})}{du}$

III. $\delta(d\mathcal{A}) = d(\delta\mathcal{A}), \quad \delta^2(d\mathcal{A}) = d(\delta^2\mathcal{A})$

Below we will consider a regular curve

$$C : \mathbf{r} = \mathbf{r}(s) = \mathbf{r}[u(s)], \quad s \in \mathcal{I} = [0, L] \subseteq \mathcal{R}, \tag{4}$$

parameterized by the arc length s. Let us consider an infinitesimal bending of the second order of the curve (4):

$$C_\epsilon : \tilde{\mathbf{r}}(s,\epsilon) = \mathbf{r}_\epsilon(s) = \mathbf{r}(s) + \epsilon\overset{(1)}{\mathbf{z}}(s) + \epsilon^2\overset{(2)}{\mathbf{z}}(s).$$

Since the vector fields $\overset{(1)}{\mathbf{z}}$ and $\overset{(2)}{\mathbf{z}}$ are defined in the points of the curve (4), they can be presented in the form

$$\overset{(j)}{\mathbf{z}} = \overset{(j)}{z}\,\mathbf{t} + \overset{(j)}{z}_1\mathbf{n}_1 + \overset{(j)}{z}_2\mathbf{n}_2, \quad j = 1, 2, \tag{5}$$

where $\overset{(j)}{z}\,\mathbf{t}$ is a tangential and $\overset{(j)}{z}_1\mathbf{n}_1 + \overset{(j)}{z}_2\mathbf{n}_2$ is a normal component, $\overset{(j)}{z}, \overset{(j)}{z}_1, \overset{(j)}{z}_2$ are the functions of s.

The necessary and sufficient conditions for the second order infinitesimal bending are given by the next theorem.

Theorem 2. [**6**] *Necessary and sufficient conditions for the fields* $\overset{(j)}{\mathbf{z}}$*,* $j = 1, 2$*, (5) to be infinitesimal bending fields of the corresponding order of a curve* C *(4) are*

$$\begin{aligned} &\overset{(1)}{z}{}' - k\overset{(1)}{z}_1 = 0, \\ &\overset{(2)}{z}{}' - k\overset{(2)}{z}_1 = -\frac{1}{2}\big\{[k\overset{(1)}{z} + \overset{(1)}{z}_1{}' - \tau\overset{(1)}{z}_2]^2 + [\tau\overset{(1)}{z}_1 + \overset{(1)}{z}_2{}']^2\big\} \end{aligned} \tag{6}$$

where k *is the curvature and* τ *is the torsion of* C.

The first and the second variations of some geometric magnitudes of curves are determined in the paper [**6**]. Some of them will be used in this paper, like $\delta\mathbf{t}$, $\delta^2\mathbf{t}$, $\delta\mathbf{n}_1$, $\delta\mathbf{n_2}$, δk.

3. CHANGE OF THE TOTAL TORSION UNDER INFINITESIMAL BENDING

Let us consider the total torsion of the curve C (4):

$$\mathcal{T} = \int_{\mathcal{I}} \tau(s)\,ds.$$

The total torsion of deformed curve is

$$\mathcal{T}_\epsilon = \int_{\mathcal{I}} \tau_\epsilon\,ds_\epsilon = \int_{\mathcal{I}} (\tau + \epsilon\,\delta\tau + \epsilon^2\,\delta^2\tau)(ds + \epsilon\,\delta\,ds + \epsilon^2\,\delta^2\,ds).$$

Since $\delta\,ds = \delta^2 ds = 0$, we have

$$\mathcal{T}_\epsilon = \mathcal{T} + \epsilon\int_{\mathcal{T}} \tau\,\delta\tau\,ds + \epsilon^2\int_{\mathcal{I}} \delta^2\tau\,ds,$$

which is obtained after neglecting the terms in ϵ^3 and ϵ^4. Therefore,

$$\delta\mathcal{T} = \int_{\mathcal{I}} \delta\tau\,ds, \quad \delta^2\mathcal{T} = \int_{\mathcal{I}} \delta^2\tau\,ds.$$

Let us take the first variation of the Frenet equation for $\mathbf{n}_1'$ and dot with $\mathbf{n}_2$. We have

$$\delta\tau = k\mathbf{n}_2 \cdot \delta\mathbf{t} + \mathbf{n}_2 \cdot \delta\mathbf{n}_1'.$$

We now rewrite the second term on the right hand side as

$$\mathbf{n}_2 \cdot \delta\mathbf{n}_1' = (\mathbf{n}_2 \cdot \delta\mathbf{n}_1)' - \mathbf{n}_2' \cdot \delta\mathbf{n}_1 = (\mathbf{n}_2 \cdot \delta\mathbf{n}_1)',$$

after using the third Frenet equation. Using the expressions for δt and δn_1 [**6**], as well as the first equation in (6), we obtain the first variation of the torsion:

$$\delta\tau = \tau'\overset{(1)}{z} + 2k\tau\overset{(1)}{z_1} - k'\overset{(1)}{z_2} + \Big\{\frac{1}{k}\big[2\tau\overset{(1)}{z_1}{}' + \tau'\overset{(1)}{z_1} + \overset{(1)}{z_2}{}'' + (k^2-\tau^2)\overset{(1)}{z_2}\big]\Big\}'.$$

We used the next transformations:

$$[\frac{1}{k}(k\tau\overset{(1)}{z})]' = (\tau\overset{(1)}{z})' = \tau'\overset{(1)}{z} + \tau\overset{(1)}{z}{}' = \tau'\overset{(1)}{z} + \tau k\overset{(1)}{z_1}, \quad (\overset{(1)}{z}{}' - k\overset{(1)}{z_1} = 0),$$

$$k\overset{(1)}{z_2}{}' = (k\overset{(1)}{z_2})' - k'\overset{(1)}{z_2}.$$

The first variation of the total torsion is now

$$\begin{aligned}\delta\mathcal{T} = &\int_{\mathcal{I}}(\tau'\overset{(1)}{z} + 2k\tau\overset{(1)}{z_1} - k'\overset{(1)}{z_2})\,ds \\ &+ \int_{\mathcal{I}}\Big\{\frac{1}{k}\big[2\tau\overset{(1)}{z_1}{}' + \tau'\overset{(1)}{z_1} + \overset{(1)}{z_2}{}'' + (k^2-\tau^2)\overset{(1)}{z_2}\big]\Big\}'\,ds.\end{aligned}$$

In the case of the infinitesimal bending of knots we specify the condition $\mathbf{z}(0) = \mathbf{z}(L)$ for the infinitesimal bending field in order to get a family of closed curves. Also, we suppose that the knot, as well as the infinitesimal bending field are sufficiently smooth. Keeping this in mind we have the following theorem.

Theorem 3. *Under a second order infinitesimal bending of a knot C, the first variation of its total torsion is*

$$\delta\mathcal{T} = \int_{\mathcal{I}}(\tau'\overset{(1)}{z} + 2k\tau\overset{(1)}{z_1} - k'\overset{(1)}{z_2})\,ds,$$

where k is the curvature and τ is the torsion of C.

Before we begin to examine the second variation of the total torsion, let us determine $\delta^2\mathbf{n}_1$, which will be used below. Starting from the equation $\mathbf{n}_1 \cdot \mathbf{n}_1 = 1$ and using the second variation of its, we obtain

$$\mathbf{n}_1 \cdot \delta^2\mathbf{n}_1 = -\frac{1}{2}\delta\mathbf{n}_1 \cdot \delta\mathbf{n}_1.$$

Using the expression for $\delta\mathbf{n}_1$ [**6**], we have

$$\text{(7)}\quad \mathbf{n}_1 \cdot \delta^2\mathbf{n}_1 = -\frac{1}{2}[(k\overset{(1)}{z} + \overset{(1)}{z_1}{}' - \tau\overset{(1)}{z_2})^2 + \frac{1}{k^2}(k\tau\overset{(1)}{z} + 2\tau\overset{(1)}{z_1}{}' + \tau'\overset{(1)}{z_1} + \overset{(1)}{z_2}{}'' - \tau^2\overset{(1)}{z_2})^2].$$

From the equation $\mathbf{n}_1 \cdot \mathbf{t} = 0$, using the second variation, we obtain

$$\mathbf{t} \cdot \delta^2 \mathbf{n}_1 = -\mathbf{n}_1 \cdot \delta^2 \mathbf{t} - \delta \mathbf{n}_1 \cdot \delta \mathbf{t}.$$

If we make use of $\delta\mathbf{t}$, $\delta^2\mathbf{t}$, and $\delta\mathbf{n}_1$ [**6**], we get

$$\begin{aligned} \mathbf{t} \cdot \delta^2 \mathbf{n}_1 = &-(k\overset{(2)}{z} + \overset{(2)}{z}_1{}' - \tau \overset{(2)}{z}_2 - \frac{1}{k}(\overset{(1)}{z}_2{}' + \tau \overset{(1)}{z}_1)(k\tau \overset{(1)}{z} + 2\tau \overset{(1)}{z}_1{}' \\ &+ \tau' \overset{(1)}{z}_1 + \overset{(1)}{z}_2{}'' - \tau^2 \overset{(1)}{z}_2). \end{aligned} \tag{8}$$

Let us take the second variation of the equation $\mathbf{t}' = k\mathbf{n}_1$ and dot with $\mathbf{n}_2$. We obtain

$$\mathbf{n}_2 \cdot \delta^2 \mathbf{n}_1 = \frac{1}{k}(\mathbf{n}_2 \cdot \delta^2 \mathbf{t}' - \delta k \mathbf{n}_2 \cdot \delta \mathbf{n}_1).$$

To evaluate $\delta^2\mathbf{t}'$ we use $\delta^2\mathbf{t}' = (\delta^2\mathbf{t})'$, the expression for $\delta^2\mathbf{t}$ [**6**] and Frenet equations. Dotting with $\mathbf{n}_2$ we obtain

$$\mathbf{n}_2 \cdot \delta^2 \mathbf{t}' = \tau(k\overset{(2)}{z} + \overset{(2)}{z}_1{}' - \tau \overset{(2)}{z}_2) + (\overset{(2)}{z}_2{}' + \tau \overset{(2)}{z}_1)'.$$

Finally, we have

$$\begin{aligned} \mathbf{n}_2 \cdot \delta^2 \mathbf{n}_1 = &\frac{1}{k}\big[\tau(k\overset{(2)}{z} + \overset{(2)}{z}_1{}' - \tau \overset{(2)}{z}_2) + (\overset{(2)}{z}_2{}' + \tau \overset{(2)}{z}_1)' - \frac{1}{k}(k'\overset{(1)}{z} + \overset{(1)}{z}_1{}'' + (k^2 - \tau^2)\overset{(1)}{z}_1 \\ &- 2\tau \overset{(1)}{z}_2{}' - \tau' \overset{(1)}{z}_2)(k\tau \overset{(1)}{z} + 2\tau \overset{(1)}{z}_1{}' + \tau' \overset{(1)}{z}_1 + \overset{(1)}{z}_2{}'' - \tau^2 \overset{(1)}{z}_2)\big]. \end{aligned} \tag{9}$$

The equations (7), (8) and (9) gives the normal, the tangent and the binormal component of $\delta^2\mathbf{n}_1$, respectively.

Let us go back to the torsion. For the second variation of the torsion let us take the second variation of the second Frenet equation. We have

$$\delta^2 \mathbf{n}_1' = -\delta^2 k \mathbf{t} - k\delta^2 \mathbf{t} - \delta k \delta t + \delta^2 \tau \mathbf{n}_2 + \tau \delta^2 \mathbf{n}_2 + \delta\tau\delta\mathbf{n}_2.$$

Dotting with $\mathbf{n}_2$ we obtain

$$\delta^2 \tau = (\mathbf{n}_2 \cdot \delta^2 \mathbf{n}_1)' + \tau \mathbf{n}_1 \cdot \delta^2 \mathbf{n}_1 + k\mathbf{n}_2 \cdot \delta^2 \mathbf{t} + \delta k\, \mathbf{n}_2 \cdot \delta \mathbf{t} - \tau \mathbf{n}_2 \cdot \delta^2 \mathbf{n}_2.$$

We used $\mathbf{n}_2 \cdot \delta \mathbf{n}_2 = 0$ and also

$$\mathbf{n}_2 \cdot \delta^2 \mathbf{n}_1' = \mathbf{n}_2 \cdot (\delta^2 \mathbf{n}_1)' = (\mathbf{n}_2 \cdot \delta^2 \mathbf{n}_1)' - \mathbf{n}_2' \cdot \delta^2 \mathbf{n}_1 = (\mathbf{n}_2 \cdot \delta^2 \mathbf{n}_1)' + \tau \mathbf{n}_1 \cdot \delta^2 \mathbf{n}_1,$$

wherefrom we have

$$\begin{aligned} \delta^2 \tau = &\Big\{\frac{1}{k}\big[\tau(k\overset{(2)}{z} + \overset{(2)}{z}_1{}' - \tau \overset{(2)}{z}_2) + (\overset{(2)}{z}_2{}' + \tau \overset{(2)}{z}_1)' - \frac{1}{k}(k'\overset{(1)}{z} + \overset{(1)}{z}_1{}'' + (k^2 - \tau^2)\overset{(1)}{z}_1 \\ &- 2\tau \overset{(1)}{z}_2{}' - \tau' \overset{(1)}{z}_2)(k\tau \overset{(1)}{z} + 2\tau \overset{(1)}{z}_1{}' + \tau' \overset{(1)}{z}_1 + \overset{(1)}{z}_2{}'' - \tau^2 \overset{(1)}{z}_2)\big]\Big\}' \\ &- \frac{\tau}{2}\big[(k\overset{(1)}{z} + \overset{(1)}{z}_1{}' - \tau \overset{(1)}{z}_2)^2 - (\overset{(1)}{z}_2{}' + \tau \overset{(1)}{z}_1)^2\big] + k(\overset{(2)}{z}_2{}' + \tau \overset{(2)}{z}_1) \\ &+ (k'\overset{(1)}{z} + \overset{(1)}{z}_1{}'' + (k^2 - \tau^2)\overset{(1)}{z}_1 - 2\tau \overset{(1)}{z}_2{}' - \tau' \overset{(1)}{z}_2)(\overset{(1)}{z}_2{}' + \tau \overset{(1)}{z}_1). \end{aligned}$$

Based on the necessary and sufficient conditions for the second order infinitesimal bending (6), we have

$$(\tau \overset{(1)}{z}_1 + \overset{(1)}{z}_2{}')^2 = -2(\overset{(2)}{z}{}' - k\overset{(2)}{z}_1) - (k\overset{(1)}{z} + \overset{(1)}{z}_1{}' - \tau \overset{(1)}{z}_2)^2.$$

Let us use this in the expression for $\delta^2\tau$, as well as $\tau \overset{(2)}{z}{}' = (\tau \overset{(2)}{z})' - \tau' \overset{(2)}{z}$ and $k\overset{(2)}{z}_2{}' = (k\overset{(2)}{z}_2)' - k'\overset{(2)}{z}_2$. Also, let us use the next notation

$$\begin{aligned} f(\overset{(1)}{\mathbf{z}}) &= -\tau(k\overset{(1)}{z} + \overset{(1)}{z}_1{}' - \tau \overset{(1)}{z}_2)^2 + (k'\overset{(1)}{z} + \overset{(1)}{z}_1{}'' + (k^2-\tau^2)\overset{(1)}{z}_1 \\ &\quad - 2\tau \overset{(1)}{z}_2{}' - \tau' \overset{(1)}{z}_2)(\overset{(1)}{z}_2{}' + \tau \overset{(1)}{z}_1). \end{aligned} \tag{10}$$

Now we have

$$\begin{aligned} \delta^2\tau = \Big\{ & \frac{1}{k}\big[\tau(k\overset{(2)}{z} + \overset{(2)}{z}_1{}' - \tau \overset{(2)}{z}_2) + (\overset{(2)}{z}_2{}' + \tau \overset{(2)}{z}_1)' - \frac{1}{k}(k'\overset{(1)}{z} + \overset{(1)}{z}_1{}'' + (k^2-\tau^2)\overset{(1)}{z}_1 \\ & - 2\tau \overset{(1)}{z}_2{}' - \tau' \overset{(1)}{z}_2)(k\tau \overset{(1)}{z} + 2\tau \overset{(1)}{z}_1{}' + \tau' \overset{(1)}{z}_1 + \overset{(1)}{z}_2{}'' - \tau^2 \overset{(1)}{z}_2)\big] - \tau \overset{(2)}{z} + k\overset{(2)}{z}_2 \Big\}' \\ & + \tau' \overset{(2)}{z} + 2k\tau \overset{(2)}{z}_1 - k'\overset{(2)}{z}_2 + f(\overset{(1)}{\mathbf{z}}). \end{aligned} \tag{11}$$

The second variation of the total torsion will be

$$\begin{aligned} \delta^2\mathcal{T} = \int_{\mathcal{I}} \Big\{ & \frac{1}{k}\big[\tau(k\overset{(2)}{z} + \overset{(2)}{z}_1{}' - \tau \overset{(2)}{z}_2) + (\overset{(2)}{z}_2{}' + \tau \overset{(2)}{z}_1)' - \frac{1}{k}(k'\overset{(1)}{z} + \overset{(1)}{z}_1{}'' + (k^2-\tau^2)\overset{(1)}{z}_1 \\ & - 2\tau \overset{(1)}{z}_2{}' - \tau' \overset{(1)}{z}_2)(k\tau \overset{(1)}{z} + 2\tau \overset{(1)}{z}_1{}' + \tau' \overset{(1)}{z}_1 + \overset{(1)}{z}_2{}'' - \tau^2 \overset{(1)}{z}_2)\big] - \tau \overset{(2)}{z} + k\overset{(2)}{z}_2 \Big\}' \, ds \\ & + \int_{\mathcal{I}} (\tau' \overset{(2)}{z} + 2k\tau \overset{(2)}{z}_1 - k'\overset{(2)}{z}_2)\, ds + \int_{\mathcal{I}} f(\overset{(1)}{\mathbf{z}})\, ds. \end{aligned} \tag{12}$$

Finally we have the next theorem.

Theorem 4. *Under a second order infinitesimal bending of a knot C, the second variation of its total torsion is*

$$\delta^2\mathcal{T} = \int_{\mathcal{I}} (\tau' \overset{(2)}{z} + 2k\tau \overset{(2)}{z}_1 - k'\overset{(2)}{z}_2)\, ds + \int_{\mathcal{I}} f(\overset{(1)}{\mathbf{z}})\, ds, \tag{13}$$

where k is the curvature, τ is the torsion of C, and $f(\overset{(1)}{\mathbf{z}})$ is given in (10).

4. TORSION OF INFINITESIMALLY BENT KNOTS

Here we will consider influence of infinitesimal bending on knots defined by a simple parametric representation. Infinitesimal bending has an influence on the knotted curve, on its shape and geometrical magnitudes and we will give some examples. Our aim is to visualise changes in shape and torsion. In following figures colors are used to indicate the values of torsion at different points of the knots, together with color-values scale. In addition the total torsion is also calculated.

We start from knot representations as a curve in $\mathcal{R}^3$. Then, according to (1) we apply the bending of the first and second order given by (2) and (3). Bending fields are defined by integrals whose sub integral function includes arbitrary functions: p, q, r and g. The curve is visualized as polygonal line which connect points on curve. At every such point, as well as, every subdivision point for the purpose of numerically integral calculation we should calculate functions: the curve, first, second and third derivative, both normals of the curve, p, q, r and g, also local values of torsion. This is necessary to obtain transformed shape of curve and for aimed curve coloring. Instead of using existing software packages capable of symbolic and numeric calculations also with some visualization features, we decided to develop our own software tool using*Microsoft* Visual C++. We are dealing, according to (1), (2) and (3), with arbitrary mathematical functions, so tool is developed for manipulating explicitly defined functions. It starts from usual symbolic definitions as a string, then parsing it to obtain an internal, tree like, function form. For the purpose of efficiency the function is parsed once, then calculated many times. We also have some additional important benefits of the tree like form: make derivatives, combine more function to obtain a compound function like sub integral function for infinitesimal bending fields. Our tool has not possibility to calculate integral symbolically, instead we are using ability for fast calculation of sub integral function. Those values are used to calculate needed integral numerically, according to $F(x) = \int_0^x f(x)dx$.

Knot visualization and obtaining 3D model is done by using *OpenGL*. In the following examples the knot is represented as a tube around a curve. It looks like a rope, but without examination any physical characteristics of the rope.

4.1 Trefoil knot

A trefoil knot is given by the parametric equations: $x = \sin(u) + 2\sin(2u)$, $y = \cos(u) - 2\cos(2u)$, $z = -\sin(3u)$. The basic and infinitesimally bent trefoil knots are given in Figs. 1 and 2. The bending fields are defined by: $p(u) = 0$, $q(u) = 0$, $r(u) = \cos(3u)$ and $g(u) = \sin(6u)$.

The numerically calculated total torsion of the trefoil knot curve is 2.2250, 2.2354 and 2.3708 for $\epsilon = 0$, $\epsilon = 0.55$ and $\epsilon = 1.1$ respectively.

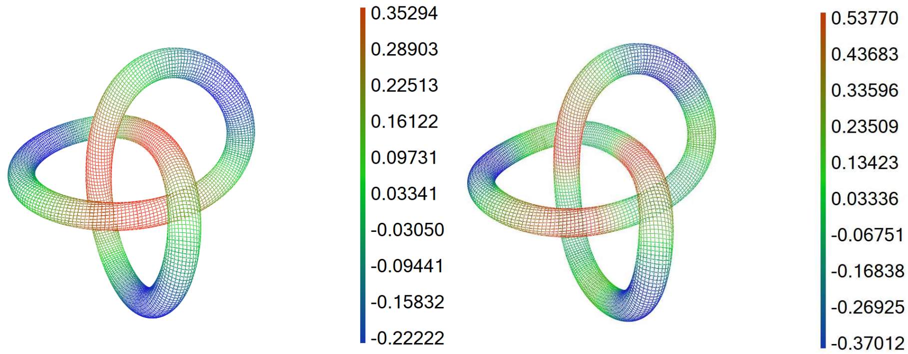

Figure 1: Trefoil knot: basic and infinitesimally bent with $\epsilon = 0.55$.

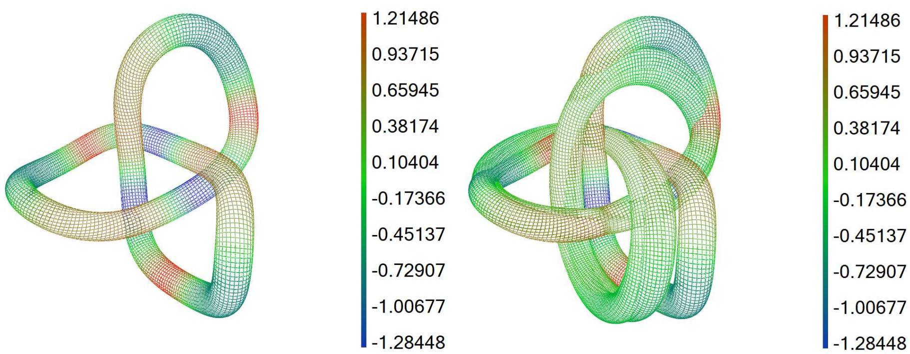

Figure 2: Trefoil knot: infinitesimally bent with $\epsilon = 1.1$ and all knots together.

A second example of the trefoil knot is knot given by the parametric equations: $x = (\cos(2u) + 2) * \cos(3u)$, $y = (\cos(2u) + 2) * \sin(3u)$, $z = -\sin(2u)$, which is a kind of torus knot, obtained for $p = 3$, $q = 2$, topologically equivalent to the trefoil knot. The basic and infinitesimally bent knots are given in Figs. 3 and 4. The bending fields are defined by: $p(u) = 0$, $q(u) = 0$, $r(u) = 0$ and $g(u) = \sin(2u)$.

The numerically calculated total torsion of the $p = 3$, $q = 2$ torus knot is 0.6234, 0.6128 and 0.4725 for $\epsilon = 0$, $\epsilon = 0.8$ and $\epsilon = 1.6$ respectively.

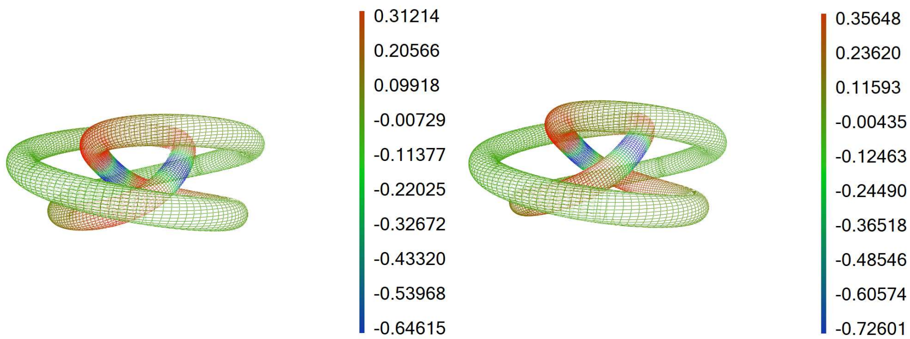

Figure 3: $p = 3$, $q = 2$ torus knot: basic and infinitesimally bent with $\epsilon = 0.8$.

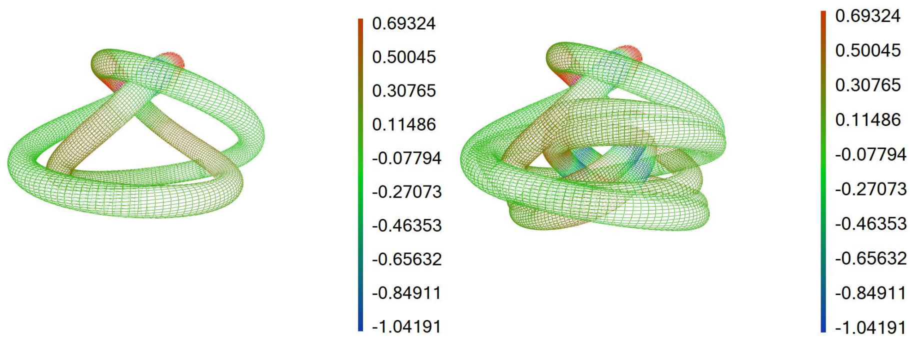

Figure 4: $p = 3$, $q = 2$ torus knot: infinitesimally bent with $\epsilon = 1.6$ and all knots together.

4.2 Figure eight knot

A figure eight knot is given by the parametric equations: $x = (2 + \cos(2u)) * \cos(3u)$, $y = (2+\cos(2u)) * \sin(3u)$, $z = \sin(4u)$. The basic and infinitesimally bent figure eight knot are given in Figs. 5 and 6.

The bending fields are defined by: $p(u) = 0$, $q(u) = 0$, $r(u) = -\cos(2u)$ and $g(u) = \sin(2u)$.

The numerically calculated total torsion of the figure eight knot curve is -0.5423, -0.4843 and -0.3033 for $\epsilon = 0$, $\epsilon = 0.76$ and $\epsilon = 1.52$ respectively.

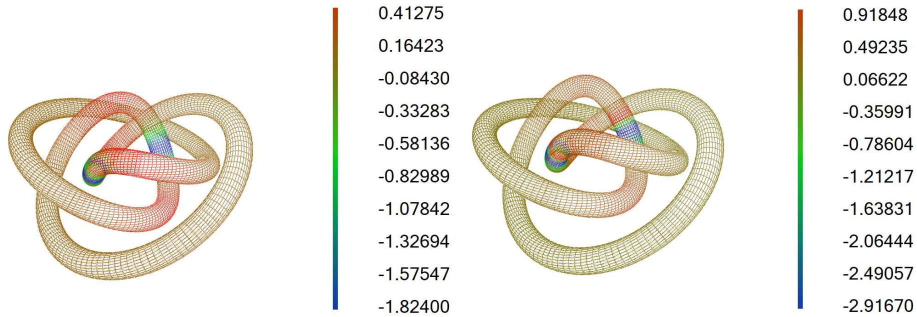

Figure 5: Figure eight knot: basic and infinitesimally bent with $\epsilon = 0.76$.

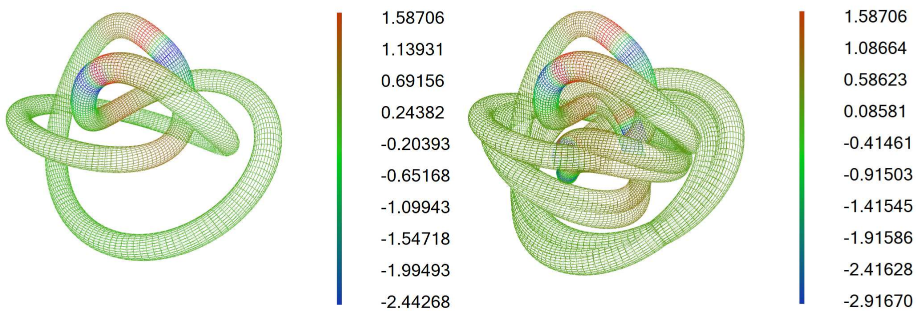

Figure 6: Figure eight knot: infinitesimally bent with $\epsilon = 1.52$ and all knots together.

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Marija S. Najdanović
University of Priština-Kosovska Mitrovica,
Faculty of Sciences, Department of Mathematics,
38220 Kosovska Mitrovica, Serbia,
E-mail: *marija.najdanovic@pr.ac.rs*

Ljubica S, Velimirović
Faculty of Sciences and Mathematics,
University of Niš, Serbia,
E-mail: *vljubica@pmf.ni.ac.rs*

Svetozar R. Rančić
Faculty of Sciences and Mathematics,
University of Niš, Serbia,
E-mail: *rancicsv@yahoo.com*