

ON EXPONENTIALLY ϱ -PREINVEX FUNCTIONS AND ASSOCIATED TRAPEZIUM LIKE INEQUALITIES

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In this paper, authors introduce a new extension of ϱ -convexity called ϱ -preinvexity and generalize the discussed results by Wu et al. in "On a new class of convex functions and integral inequalities". Some special cases are deduced from main results. At the end, a briefly conclusion is given.

1. INTRODUCTION

The class of convex functions is well known in the literature and is usually defined in the following way:

Definition 1. Let I be an interval in \mathbb{R} . A function $f : I \rightarrow \mathbb{R}$ is said to be convex on I , if the inequality

$$(1) \quad f(tu + (1 - t)v) \leq tf(u) + (1 - t)f(v)$$

holds for all $u, v \in I$ and $t \in [0, 1]$. Also, we say that f is concave, if the inequality in (1) holds in the reverse direction.

Convex functions and their variant forms are being used to study a wide class of problems which arises in various branches of pure and applied sciences. This theory provides us a natural, unified and general framework to study a wide class of unrelated problems.

The following inequality, named Hermite–Hadamard inequality, is one of the most famous inequalities in the literature for convex functions.

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Theorem 2. Let $f : I \subset \mathbb{R} \rightarrow \mathbb{R}$ be a convex function and $u, v \in I$ with $u < v$. Then

$$(2) \quad f\left(\frac{u+v}{2}\right) \leq \frac{1}{v-u} \int_u^v f(x)dx \leq \frac{f(u)+f(v)}{2}.$$

This inequality (2) is also known as trapezium inequality.

The trapezium inequality has remained an area of great interest due to its wide applications in the field of mathematical analysis. Authors of recent decades have studied (2) in the premises of newly invented definitions due to motivation of convex function. Interested readers can see the references [1]–[12].

Recently, Wu et al. in [11], introduced the following new classes of ϱ -convex sets and ϱ -convex functions.

Definition 3. A set $\mathcal{B} \subseteq \mathbb{R}$ is said to be ϱ -convex set with respect to strictly monotonic continuous function ϱ , if

$$\mathcal{M}_{[\varrho]}(x, y) = \varrho^{-1}((1-t)\varrho(x) + t\varrho(y)) \in \mathcal{B}, \quad \forall x, y \in \mathcal{B}, t \in [0, 1].$$

Definition 4. A function $\mathcal{B} \subseteq \mathbb{R}$ is said to be ϱ -convex function with respect to strictly monotonic continuous function ϱ , if

$$f(\mathcal{M}_{[\varrho]}(x, y)) \leq (1-t)f(x) + tf(y), \quad \forall x, y \in \mathcal{B}, t \in [0, 1].$$

Note that the function f is called strictly ϱ -convex on \mathcal{B} if the above inequality is true as a strict inequality for each distinct x and y in \mathcal{B} and for each $t \in (0, 1)$. The function f is called ϱ -concave (strictly ϱ -concave) on \mathcal{B} , if $(-f)$ is ϱ -convex (strictly ϱ -convex) on \mathcal{B} .

The main objective of this article is to introduce a new extension of ϱ -convexity called ϱ -preinvexity and to generalize the discussed results by Wu et al. in [11]. Some special cases will be given as well. Interested reader can obtain in a similar way new results for different operators and this results can be applied in different areas of pure and applied sciences.

2. MAIN RESULTS

Now, we are in position to introduce the notions of ϱ -preinvex sets and ϱ -preinvex functions with respect to the mapping τ by using the quasi-arithmetic means which can bring together all the power means \mathcal{M}_p for $p \in \mathbb{R}$.

Definition 5. A set $\mathcal{B} \subseteq \mathbb{R}$ is said to be ϱ -invex with respect to the mapping $\tau : \mathcal{B} \times \mathcal{B} \rightarrow \mathbb{R}$ and strictly monotonic continuous function ϱ , if

$$\mathcal{M}_{[\varrho, \tau]}(x, y) = \varrho^{-1}(\varrho(x) + t\tau(\varrho(y), \varrho(x))) \in \mathcal{B}, \quad \forall x, y \in \mathcal{B}, t \in [0, 1].$$

Definition 6. A function $f : \mathcal{B} \subseteq \mathbb{R}$ is said to be exponentially ϱ -preinvex with respect to the mapping $\tau : \mathcal{B} \times \mathcal{B} \rightarrow \mathbb{R}$, strictly monotonic continuous function ϱ , and non-positive β , if

$$f(\mathcal{M}_{[\varrho, \tau]}(x, y)) \leq (1 - t) \frac{f(x)}{e^{\beta x}} + t \frac{f(y)}{e^{\beta y}}, \quad \forall x, y \in \mathcal{B}, t \in [0, 1].$$

Note that the function f is called strictly exponentially ϱ -preinvex on \mathcal{B} if the above inequality is true as a strict inequality for each distinct x and y in \mathcal{B} and for each t on $(0, 1)$.

If $t = \frac{1}{2}$, then

$$f\left(\varrho^{-1}\left(\varrho(x) + \frac{\tau(\varrho(y), \varrho(x))}{2}\right)\right) \leq \frac{1}{2} \left[\frac{f(x)}{e^{\beta x}} + \frac{f(y)}{e^{\beta y}} \right], \quad \forall x, y \in \mathcal{B}.$$

The function f is called exponentially ϱ -Jensen (or mid)-preinvex function.

Remark 7. It is worth to mention here that every ϱ -convex set is ϱ -preinvex with respect to $\tau(\varrho(y), \varrho(x)) = \varrho(y) - \varrho(x)$, but the converse is not necessarily true.

We now discuss some special cases of Definition 6.

I). If we take $\tau(\varrho(y), \varrho(x)) = \varrho(y) - \varrho(x)$ and $\varrho(x) = \ln x$, then

$$f(x^{1-t}y^t) \leq (1 - t) \frac{f(x)}{e^{\beta x}} + t \frac{f(y)}{e^{\beta y}}, \quad \forall x, y \in [a, b] \subset (0, +\infty), t \in [0, 1],$$

which is the concept of geometrically exponentially convex functions.

II). If we take $\tau(\varrho(y), \varrho(x)) = \varrho(y) - \varrho(x)$ and $\varrho(x) = \frac{1}{x}$, then

$$\begin{aligned} & f\left(\frac{xy}{tx + (1-t)y}\right) \\ & \leq (1 - t) \frac{f(x)}{e^{\beta x}} + t \frac{f(y)}{e^{\beta y}}, \quad \forall x, y \in [a, b] \subset (0, +\infty), t \in [0, 1], \end{aligned}$$

which is the concept of harmonically exponentially convex function.

III). If $p \in \mathbb{R}$ and taking $\tau(\varrho(y), \varrho(x)) = \varrho(y) - \varrho(x)$, $\varrho(x) = x^p$, then

$$\begin{aligned} & f\left([(1-t)x^p + ty^p]^{\frac{1}{p}}\right) \\ & \leq (1 - t) \frac{f(x)}{e^{\beta x}} + t \frac{f(y)}{e^{\beta y}}, \quad \forall x, y \in [a, b] \subset (0, +\infty), t \in [0, 1], \end{aligned}$$

which is the concept of exponentially p -convexity as considered in [6].

IV). If $p \in \mathbb{R}$ and taking $\tau(\varrho(y), \varrho(x)) = \varrho(y) - \varrho(x)$, $\varrho(x) = e^x$, then

$$\begin{aligned} & f(\ln((1-t)e^x + te^y)) \\ & \leq (1-t)\frac{f(x)}{e^{\beta x}} + t\frac{f(y)}{e^{\beta y}}, \quad \forall x, y \in [a, b] \subset (0, +\infty), \quad t \in [0, 1], \end{aligned}$$

which is the concept of log-exponential convex functions.

From the above discussion it is evident that the class of exponentially ϱ -preinvex functions is quite unifying one. We now derive some integral inequalities using this new class. It is worth to mention here that all these results continue to hold for other classes of the convexity under suitable conditions.

Let us establish some new integral inequalities of Hermite–Hadamard type via exponentially ϱ -preinvex functions with respect to the mapping τ and strictly monotonic continuous function ϱ .

Before we proceed further, let us recall the famous Condition C, which was introduced and studied by Mohan and Noegy [8].

Condition C. A set $\mathcal{B} \subset \mathbb{R}$ is said to be an invex set with respect to mapping $\tau(\cdot, \cdot)$ if and only if for any $x, y \in \mathcal{B}$ and $t \in [0, 1]$, we have

1. $\tau(x, x + t\tau(y, x)) = -t\tau(y, x)$,
2. $\tau(y, x + t\tau(y, x)) = (1-t)\tau(y, x)$.

Note that for any $x, y \in \mathcal{B}$, $t_1, t_2 \in [0, 1]$ and from Condition C, we can deduce

$$\tau(x + t_2\tau(y, x), x + t_1\tau(y, x)) = (t_2 - t_1)\tau(y, x).$$

Theorem 8. Let $f : I = [u, v] \rightarrow \mathbb{R}$ be an integrable exponentially ϱ -preinvex function with respect to the mapping $\tau : I \times I \rightarrow \mathbb{R}$ which satisfies Condition C, and strictly monotonic continuous function ϱ , where $\tau(\varrho(v), \varrho(u)) > 0$. Then for non-positive β and $\alpha > 0$, we have

$$\begin{aligned} & f\left(\varrho^{-1}\left(\varrho(u) + \frac{\tau(\varrho(v), \varrho(u))}{2}\right)\right) \leq \frac{\alpha}{2\tau^\alpha(\varrho(v), \varrho(u))} \\ & \times \left[\int_u^{\varrho^{-1}(\varrho(u) + \tau(\varrho(v), \varrho(u)))} (\varrho(x) - \varrho(u))^{\alpha-1} \frac{f(x)}{e^{\beta x}} \varrho'(x) dx \right. \\ & \left. + \int_u^{\varrho^{-1}(\varrho(u) + \tau(\varrho(v), \varrho(u)))} (\varrho(u) + \tau(\varrho(v), \varrho(u)) - \varrho(x))^{\alpha-1} \frac{f(x)}{e^{\beta x}} \varrho'(x) dx \right] \\ (3) \quad & \leq \frac{1}{2} \left[A_1 \frac{f(u)}{e^{\beta u}} + A_2 \frac{f(v)}{e^{\beta v}} \right], \end{aligned}$$

where

$$A_1 = \int_0^1 t^{\alpha-1} \left[\frac{(1-t)}{e^{\beta \varrho^{-1}(\varrho(u)+t\tau(\varrho(v),\varrho(u))}} + \frac{t}{e^{\beta \varrho^{-1}(\varrho(u)+(1-t)\tau(\varrho(v),\varrho(u))}} \right] dt$$

and

$$A_2 = \int_0^1 t^{\alpha-1} \left[\frac{t}{e^{\beta \varrho^{-1}(\varrho(u)+t\tau(\varrho(v),\varrho(u))}} + \frac{(1-t)}{e^{\beta \varrho^{-1}(\varrho(u)+(1-t)\tau(\varrho(v),\varrho(u))}} \right] dt.$$

Proof. Since f is an exponentially ϱ -preinvex function, taking $t = \frac{1}{2}$, we have

$$(4) \quad f \left(\varrho^{-1} \left(\varrho(x) + \frac{\tau(\varrho(y), \varrho(x))}{2} \right) \right) \leq \frac{1}{2} \left[\frac{f(x)}{e^{\beta x}} + \frac{f(y)}{e^{\beta y}} \right].$$

Using $x = \varrho^{-1}(\varrho(u) + t\tau(\varrho(v), \varrho(u)))$ and $y = \varrho^{-1}(\varrho(u) + (1-t)\tau(\varrho(v), \varrho(u)))$ in (4), we get

$$(5) \quad f \left(\varrho^{-1} \left(\varrho(u) + \frac{\tau(\varrho(v), \varrho(u))}{2} \right) \right) \leq \frac{1}{2} \left[\frac{f(\varrho^{-1}(\varrho(u) + t\tau(\varrho(v), \varrho(u))))}{e^{\beta(\varrho^{-1}(\varrho(u)+t\tau(\varrho(v),\varrho(u)))}} + \frac{f(\varrho^{-1}(\varrho(u) + (1-t)\tau(\varrho(v), \varrho(u))))}{e^{\beta(\varrho^{-1}(\varrho(u)+(1-t)\tau(\varrho(v),\varrho(u)))}} \right].$$

Multiplying both sides of (5) with $t^{\alpha-1}$ and integrating with respect to t on $[0, 1]$, we obtain

$$\begin{aligned} & \int_0^1 t^{\alpha-1} f \left(\varrho^{-1} \left(\varrho(u) + \frac{\tau(\varrho(v), \varrho(u))}{2} \right) \right) dt \\ & \leq \frac{1}{2} \left[\int_0^1 t^{\alpha-1} \frac{f(\varrho^{-1}(\varrho(u) + t\tau(\varrho(v), \varrho(u))))}{e^{\beta(\varrho^{-1}(\varrho(u)+t\tau(\varrho(v),\varrho(u)))}} dt \right. \\ & \quad \left. + \int_0^1 t^{\alpha-1} \frac{f(\varrho^{-1}(\varrho(u) + (1-t)\tau(\varrho(v), \varrho(u))))}{e^{\beta(\varrho^{-1}(\varrho(u)+(1-t)\tau(\varrho(v),\varrho(u)))}} dt \right]. \end{aligned}$$

Hence

$$\begin{aligned} & f \left(\varrho^{-1} \left(\varrho(u) + \frac{\tau(\varrho(v), \varrho(u))}{2} \right) \right) \leq \frac{\alpha}{2\tau^\alpha(\varrho(v), \varrho(u))} \\ & \times \left[\int_u^{\varrho^{-1}(\varrho(u)+\tau(\varrho(v),\varrho(u)))} (\varrho(x) - \varrho(u))^{\alpha-1} \frac{f(x)}{e^{\beta x}} \varrho'(x) dx \right. \\ & \left. + \int_u^{\varrho^{-1}(\varrho(u)+\tau(\varrho(v),\varrho(u)))} (\varrho(u) + \tau(\varrho(v), \varrho(u)) - \varrho(x))^{\alpha-1} \frac{f(x)}{e^{\beta x}} \varrho'(x) dx \right]. \end{aligned}$$

So, the left-hand side of (3) is proved. Similarly, since f is a exponentially ϱ -preinvex function, we have

$$(6) \quad \frac{f(\varrho^{-1}(\varrho(u) + t\tau(\varrho(v), \varrho(u))))}{e^{\beta(\varrho^{-1}(\varrho(u) + t\tau(\varrho(v), \varrho(u))))}} \leq \frac{(1-t)\frac{f(u)}{e^{\beta u}} + t\frac{f(v)}{e^{\beta v}}}{e^{\beta(\varrho^{-1}(\varrho(u) + t\tau(\varrho(v), \varrho(u))))}}$$

and

$$(7) \quad \frac{f(\varrho^{-1}(\varrho(u) + (1-t)\tau(\varrho(v), \varrho(u))))}{e^{\beta(\varrho^{-1}(\varrho(u) + (1-t)\tau(\varrho(v), \varrho(u))))}} \leq \frac{t\frac{f(u)}{e^{\beta u}} + (1-t)\frac{f(v)}{e^{\beta v}}}{e^{\beta(\varrho^{-1}(\varrho(u) + (1-t)\tau(\varrho(v), \varrho(u))))}}.$$

Multiplying both sides of (6) and (7) with $t^{\alpha-1}$ and integrating with respect to t on $[0, 1]$, we get

$$\begin{aligned} & \int_0^1 t^{\alpha-1} \left[\frac{f(\varrho^{-1}(\varrho(u) + t\tau(\varrho(v), \varrho(u))))}{e^{\beta(\varrho^{-1}(\varrho(u) + t\tau(\varrho(v), \varrho(u))))}} + \frac{f(\varrho^{-1}(\varrho(u) + (1-t)\tau(\varrho(v), \varrho(u))))}{e^{\beta(\varrho^{-1}(\varrho(u) + (1-t)\tau(\varrho(v), \varrho(u))))}} \right] dt \\ & \leq \int_0^1 t^{\alpha-1} \left[\left[\frac{(1-t)}{e^{\beta\varrho^{-1}(\varrho(u) + t\tau(\varrho(v), \varrho(u)))}} + \frac{t}{e^{\beta\varrho^{-1}(\varrho(u) + (1-t)\tau(\varrho(v), \varrho(u))}} \right] \frac{f(u)}{e^{\beta u}} \right. \\ & \quad \left. + \left[\frac{t}{e^{\beta\varrho^{-1}(\varrho(u) + t\tau(\varrho(v), \varrho(u))}} + \frac{(1-t)}{e^{\beta\varrho^{-1}(\varrho(u) + (1-t)\tau(\varrho(v), \varrho(u))}} \right] \frac{f(v)}{e^{\beta v}} \right] dt. \end{aligned}$$

Hence

$$\begin{aligned} & \frac{\alpha}{2\tau^\alpha(\varrho(v), \varrho(u))} \\ & \quad \times \left[\int_u^{\varrho^{-1}(\varrho(u) + \tau(\varrho(v), \varrho(u)))} (\varrho(x) - \varrho(u))^{\alpha-1} \frac{f(x)}{e^{\beta x}} \varrho'(x) dx \right. \\ & \quad \left. + \int_u^{\varrho^{-1}(\varrho(u) + \tau(\varrho(v), \varrho(u)))} (\varrho(u) + \tau(\varrho(v), \varrho(u)) - \varrho(x))^{\alpha-1} \frac{f(x)}{e^{\beta x}} \varrho'(x) dx \right] \\ & \leq \frac{1}{2} \left[A_1 \frac{f(u)}{e^{\beta u}} + A_2 \frac{f(v)}{e^{\beta v}} \right]. \end{aligned}$$

The proof of Theorem 8 is completed. \square

Corollary 9. In Theorem 8, taking $\tau(\varrho(v), \varrho(u)) = \varrho(v) - \varrho(u)$, we get

$$\begin{aligned} & f\left(\varrho^{-1}\left(\frac{\varrho(u) + \varrho(v)}{2}\right)\right) \leq \frac{\alpha}{2(\varrho(v) - \varrho(u))^\alpha} \\ & \times \left[\int_u^v (\varrho(x) - \varrho(u))^{\alpha-1} f(x) \varrho'(x) dx + \int_u^v (\varrho(v) - \varrho(x))^{\alpha-1} f(x) \varrho'(x) dx \right] \\ & \leq \frac{1}{2} \left[A_3 \frac{f(u)}{e^{\beta u}} + A_4 \frac{f(v)}{e^{\beta v}} \right], \end{aligned}$$

where

$$A_3 = \int_0^1 t^{\alpha-1} \left[\frac{(1-t)}{e^{\beta e^{-1}((1-t)\varrho(u)+t\varrho(v))}} + \frac{t}{e^{\beta e^{-1}(t\varrho(u)+(1-t)\varrho(v))}} \right] dt$$

and

$$A_4 = \int_0^1 t^{\alpha-1} \left[\frac{t}{e^{\beta e^{-1}((1-t)\varrho(u)+t\varrho(v))}} + \frac{(1-t)}{e^{\beta e^{-1}(t\varrho(u)+(1-t)\varrho(v))}} \right] dt.$$

Corollary 10. *In Corollary 9, taking $\alpha = 1$, we get*

$$\begin{aligned} f\left(\varrho^{-1}\left(\frac{\varrho(u) + \varrho(v)}{2}\right)\right) &\leq \frac{1}{(\varrho(v) - \varrho(u))} \int_u^v f(x)\varrho'(x)dx \\ &\leq \frac{1}{2} \left[A_5 \frac{f(u)}{e^{\beta u}} + A_6 \frac{f(v)}{e^{\beta v}} \right], \end{aligned}$$

where

$$A_5 = \int_0^1 \left[\frac{(1-t)}{e^{\beta e^{-1}((1-t)\varrho(u)+t\varrho(v))}} + \frac{t}{e^{\beta e^{-1}(t\varrho(u)+(1-t)\varrho(v))}} \right] dt$$

and

$$A_6 = \int_0^1 \left[\frac{t}{e^{\beta e^{-1}((1-t)\varrho(u)+t\varrho(v))}} + \frac{(1-t)}{e^{\beta e^{-1}(t\varrho(u)+(1-t)\varrho(v))}} \right] dt.$$

Theorem 11. *Let $f : I = [u, v] \rightarrow \mathbb{R}^+$ be an integrable exponentially ϱ -preinvex function with respect to the mapping $\tau : I \times I \rightarrow \mathbb{R}$ and strictly monotonic continuous function ϱ , where $\tau(\varrho(v), \varrho(u)) > 0$. Then for non-positive β and $\alpha > 0$, we have*

$$\begin{aligned} &\frac{2f(u)}{e^{\beta u} \tau^{\alpha+1}(\varrho(v), \varrho(u))} \\ &\times \int_u^{\varrho^{-1}(\varrho(u)+\tau(\varrho(v), \varrho(u)))} (\varrho(x) - \varrho(u))^{\alpha-1} (\varrho(u) + \tau(\varrho(v), \varrho(u)) - \varrho(x)) f(x)\varrho'(x)dx \\ &+ \frac{2f(v)}{e^{\beta v} \tau^{\alpha+1}(\varrho(v), \varrho(u))} \int_u^{\varrho^{-1}(\varrho(u)+\tau(\varrho(v), \varrho(u)))} (\varrho(x) - \varrho(u))^{\alpha-1} f(x)\varrho'(x)dx \\ &\leq \frac{1}{\tau^\alpha(\varrho(v), \varrho(u))} \int_u^{\varrho^{-1}(\varrho(u)+\tau(\varrho(v), \varrho(u)))} (\varrho(x) - \varrho(u))^{\alpha-1} f^2(x)\varrho'(x)dx \\ &\quad + \frac{1}{\alpha(\alpha+1)(\alpha+2)} \left[\frac{2f^2(u)}{e^{2\beta u}} + \frac{\alpha(\alpha+1)f^2(v)}{e^{2\beta v}} + \frac{2\alpha f(u)f(v)}{e^{\beta(u+v)}} \right] \\ (8) \quad &\leq \frac{1}{\alpha(\alpha+1)(\alpha+2)} \left[\frac{4f^2(u)}{e^{2\beta u}} + \frac{2\alpha(\alpha+1)f^2(v)}{e^{2\beta v}} + \frac{4\alpha f(u)f(v)}{e^{\beta(u+v)}} \right]. \end{aligned}$$

Proof. Using the arithmetic–geometric means inequality, we have

$$\begin{aligned}
 & 2f(\varrho^{-1}(\varrho(u) + t\tau(\varrho(v), \varrho(u)))) \left((1-t)\frac{f(u)}{e^{\beta u}} + t\frac{f(v)}{e^{\beta v}} \right) \\
 & \leq [f(\varrho^{-1}(\varrho(u) + t\tau(\varrho(v), \varrho(u))))]^2 + \left[(1-t)\frac{f(u)}{e^{\beta u}} + t\frac{f(v)}{e^{\beta v}} \right]^2 \\
 (9) \quad & = [f(\varrho^{-1}(\varrho(u) + t\tau(\varrho(v), \varrho(u))))]^2 + (1-t)^2\frac{f^2(u)}{e^{2\beta u}} + t^2\frac{f^2(v)}{e^{2\beta v}} + 2t(1-t)\frac{f(u)f(v)}{e^{\beta(u+v)}}.
 \end{aligned}$$

Multiplying both sides of (9) with $t^{\alpha-1}$ and integrating with respect to t on $[0, 1]$, we get

$$\begin{aligned}
 & \frac{2f(u)}{e^{\beta u}} \int_0^1 t^{\alpha-1}(1-t)[f(\varrho^{-1}(\varrho(u) + t\tau(\varrho(v), \varrho(u))))] dt \\
 & \quad + \frac{2f(v)}{e^{\beta v}} \int_0^1 t^{\alpha-1}t[f(\varrho^{-1}(\varrho(u) + t\tau(\varrho(v), \varrho(u))))] dt \\
 & \leq \int_0^1 t^{\alpha-1}[f(\varrho^{-1}(\varrho(u) + t\tau(\varrho(v), \varrho(u))))]^2 dt + \frac{f^2(u)}{e^{2\beta u}} \int_0^1 t^{\alpha-1}(1-t)^2 dt \\
 (10) \quad & \quad + \frac{f^2(v)}{e^{2\beta v}} \int_0^1 t^{\alpha-1}t^2 dt + \frac{2f(u)f(v)}{e^{\beta(u+v)}} \int_0^1 t^{\alpha-1}t(1-t) dt.
 \end{aligned}$$

Substituting $x = \varrho^{-1}(\varrho(u) + t\tau(\varrho(v), \varrho(u)))$ in (10), we get

$$\begin{aligned}
 & \frac{2f(u)}{e^{\beta u}\tau^{\alpha+1}(\varrho(v), \varrho(u))} \\
 & \times \int_u^{\varrho^{-1}(\varrho(u) + \tau(\varrho(v), \varrho(u)))} (\varrho(x) - \varrho(u))^{\alpha-1} (\varrho(u) + \tau(\varrho(v), \varrho(u)) - \varrho(x)) f(x)\varrho'(x) dx \\
 & \quad + \frac{2f(v)}{e^{\beta v}\tau^{\alpha+1}(\varrho(v), \varrho(u))} \int_u^{\varrho^{-1}(\varrho(u) + \tau(\varrho(v), \varrho(u)))} (\varrho(x) - \varrho(u))^{\alpha-1} f(x)\varrho'(x) dx \\
 & \leq \frac{1}{\tau^\alpha(\varrho(v), \varrho(u))} \int_u^{\varrho^{-1}(\varrho(u) + \tau(\varrho(v), \varrho(u)))} (\varrho(x) - \varrho(u))^{\alpha-1} f^2(x)\varrho'(x) dx \\
 (11) \quad & \quad + \frac{1}{\alpha(\alpha+1)(\alpha+2)} \left[\frac{2f^2(u)}{e^{2\beta u}} + \frac{\alpha(\alpha+1)f^2(v)}{e^{2\beta v}} + \frac{2\alpha f(u)f(v)}{e^{\beta(u+v)}} \right].
 \end{aligned}$$

So the left-hand side of (8) is proved. On the other hand, since f is an exponentially ϱ -preinvex function, we have

$$f(\varrho^{-1}(\varrho(u) + t\tau(\varrho(v), \varrho(u)))) \leq (1-t)\frac{f(u)}{e^{\beta u}} + t\frac{f(v)}{e^{\beta v}}.$$

Hence

$$\begin{aligned}
 & \frac{1}{\tau^\alpha(\varrho(v), \varrho(u))} \int_u^{\varrho^{-1}(\varrho(u) + \tau(\varrho(v), \varrho(u)))} (\varrho(x) - \varrho(u))^{\alpha-1} f^2(x) \varrho'(x) dx \\
 &= \int_0^1 t^{\alpha-1} [f(\varrho^{-1}(\varrho(u) + t\tau(\varrho(v), \varrho(u))))]^2 dt \\
 &\leq \int_0^1 t^{\alpha-1} \left[(1-t) \frac{f(u)}{e^{\beta u}} + t \frac{f(v)}{e^{\beta v}} \right]^2 dt \\
 (12) \quad &= \frac{1}{\alpha(\alpha+1)(\alpha+2)} \left[\frac{2f^2(u)}{e^{2\beta u}} + \frac{\alpha(\alpha+1)f^2(v)}{e^{2\beta v}} + \frac{2\alpha f(u)f(v)}{e^{\beta(u+v)}} \right].
 \end{aligned}$$

Combining (11) and (12) we obtain the right-hand side of (8). The proof of Theorem 11 is completed. \square

Corollary 12. *In Theorem 11, taking $\tau(\varrho(v), \varrho(u)) = \varrho(v) - \varrho(u)$, we get*

$$\begin{aligned}
 & \frac{2f(u)}{e^{\beta u}(\varrho(v) - \varrho(u))^{\alpha+1}} \int_u^v (\varrho(x) - \varrho(u))^{\alpha-1} (\varrho(v) - \varrho(x)) f(x) \varrho'(x) dx \\
 &+ \frac{2f(v)}{e^{\beta v}(\varrho(v) - \varrho(u))^{\alpha+1}} \int_u^v (\varrho(x) - \varrho(u))^{\alpha-1} f(x) \varrho'(x) dx \\
 &\leq \frac{1}{(\varrho(v) - \varrho(u))^\alpha} \int_u^v (\varrho(x) - \varrho(u))^{\alpha-1} f^2(x) \varrho'(x) dx \\
 &+ \frac{1}{\alpha(\alpha+1)(\alpha+2)} \left[\frac{2f^2(u)}{e^{2\beta u}} + \frac{\alpha(\alpha+1)f^2(v)}{e^{2\beta v}} + \frac{2\alpha f(u)f(v)}{e^{\beta(u+v)}} \right] \\
 &\leq \frac{1}{\alpha(\alpha+1)(\alpha+2)} \left[\frac{4f^2(u)}{e^{2\beta u}} + \frac{2\alpha(\alpha+1)f^2(v)}{e^{2\beta v}} + \frac{4\alpha f(u)f(v)}{e^{\beta(u+v)}} \right].
 \end{aligned}$$

Corollary 13. *In Corollary 12, taking $\alpha = 1$, we get*

$$\begin{aligned}
 & \frac{2f(u)}{e^{\beta u}(\varrho(v) - \varrho(u))^2} \int_u^v (\varrho(v) - \varrho(x)) f(x) \varrho'(x) dx \\
 &+ \frac{2f(v)}{e^{\beta v}(\varrho(v) - \varrho(u))^2} \int_u^v f(x) \varrho'(x) dx \\
 &\leq \frac{1}{(\varrho(v) - \varrho(u))} \int_u^v f^2(x) \varrho'(x) dx + \frac{1}{3} \left[\frac{f^2(u)}{e^{2\beta u}} + \frac{f^2(v)}{e^{2\beta v}} + \frac{f(u)f(v)}{e^{\beta(u+v)}} \right] \\
 &\leq \frac{2}{3} \left[\frac{f^2(u)}{e^{2\beta u}} + \frac{f^2(v)}{e^{2\beta v}} + \frac{f(u)f(v)}{e^{\beta(u+v)}} \right].
 \end{aligned}$$

Theorem 14. Let $f, g : I = [u, v] \rightarrow \mathbb{R}^+$ be two integrable exponentially ϱ -preinvex functions with respect to the mapping $\tau : I \times I \rightarrow \mathbb{R}$ and strictly monotonic continuous function ϱ , where $\tau(\varrho(v), \varrho(u)) > 0$. Then for non-positive β and $\alpha > 0$, we have

$$\begin{aligned} & \frac{1}{\tau^\alpha(\varrho(v), \varrho(u))} \int_u^{\varrho^{-1}(\varrho(u) + \tau(\varrho(v), \varrho(u)))} (\varrho(x) - \varrho(u))^{\alpha-1} f(x)g(x)\varrho'(x)dx \\ & \leq \frac{2f(u)g(u)}{e^{2\beta u}\alpha(\alpha+1)(\alpha+2)} + \frac{f(v)g(v)}{e^{2\beta v}(\alpha+2)} + \frac{N(u, v)}{(\alpha+1)(\alpha+2)} \\ & \leq \frac{F(u)}{\alpha(\alpha+1)(\alpha+2)} + \frac{F(v)}{2(\alpha+2)} + \frac{P(u, v)}{(\alpha+1)(\alpha+2)}, \end{aligned}$$

where

$$(13) \quad N(u, v) = \frac{f(u)g(v) + f(v)g(u)}{e^{\beta(u+v)}}, \quad P(u, v) = \frac{f(u)f(v) + g(u)g(v)}{e^{\beta(u+v)}}$$

and

$$F(u) = \frac{f^2(u) + g^2(u)}{e^{2\beta u}}, \quad F(v) = \frac{f^2(v) + g^2(v)}{e^{2\beta v}}.$$

Proof. Since f and g are two integrable exponentially ϱ -preinvex functions, we have

$$\begin{aligned} & f(\varrho^{-1}(\varrho(u) + t\tau(\varrho(v), \varrho(u)))) g(\varrho^{-1}(\varrho(u) + t\tau(\varrho(v), \varrho(u)))) \\ (14) \quad & \leq \left[(1-t)\frac{f(u)}{e^{\beta u}} + t\frac{f(v)}{e^{\beta v}} \right] \left[(1-t)\frac{g(u)}{e^{\beta u}} + t\frac{g(v)}{e^{\beta v}} \right]. \end{aligned}$$

Multiplying both sides of (14) with $t^{\alpha-1}$ and integrating with respect to t on $[0, 1]$, we obtain

$$\begin{aligned} & \int_0^1 t^{\alpha-1} f(\varrho^{-1}(\varrho(u) + t\tau(\varrho(v), \varrho(u)))) g(\varrho^{-1}(\varrho(u) + t\tau(\varrho(v), \varrho(u)))) dt \\ (15) \quad & \leq \int_0^1 t^{\alpha-1} \left[(1-t)\frac{f(u)}{e^{\beta u}} + t\frac{f(v)}{e^{\beta v}} \right] \left[(1-t)\frac{g(u)}{e^{\beta v}} + t\frac{g(v)}{e^{\beta v}} \right] dt \\ & = \frac{f(u)g(u)}{e^{2\beta u}} \int_0^1 t^{\alpha-1}(1-t)^2 dt + N(u, v) \int_0^1 t^{\alpha-1}t(1-t)dt + \frac{f(v)g(v)}{e^{2\beta v}} \int_0^1 t^{\alpha-1}t^2 dt. \end{aligned}$$

Substituting $x = \varrho^{-1}(\varrho(u) + t\tau(\varrho(v), \varrho(u)))$ in (15), we obtain the left-hand side of (14). For the right-hand side inequality, using the arithmetic-geometric means inequality in (15), gives

$$\int_0^1 t^{\alpha-1} \left[(1-t)\frac{f(u)}{e^{\beta u}} + t\frac{f(v)}{e^{\beta v}} \right] \left[(1-t)\frac{g(u)}{e^{\beta u}} + t\frac{g(v)}{e^{\beta v}} \right] dt$$

$$\begin{aligned} &\leq \frac{1}{2} \int_0^1 t^{\alpha-1} \left[\left[(1-t) \frac{f(u)}{e^{\beta u}} + t \frac{f(v)}{e^{\beta v}} \right]^2 + \left[(1-t) \frac{g(u)}{e^{\beta u}} + t \frac{g(v)}{e^{\beta v}} \right]^2 \right] dt \\ &= \frac{1}{2} \int_0^1 \left[F(u)t^{\alpha-1}(1-t)^2 + F(v)t^{\alpha-1}t^2 + 2P(u,v)t^{\alpha-1}t(1-t) \right] dt \\ &= \frac{F(u)}{\alpha(\alpha+1)(\alpha+2)} + \frac{F(v)}{2(\alpha+2)} + \frac{P(u,v)}{(\alpha+1)(\alpha+2)}. \end{aligned}$$

The proof of Theorem 14 is completed. □

Corollary 15. *In Theorem 14, taking $\tau(\varrho(v), \varrho(u)) = \varrho(v) - \varrho(u)$, we get*

$$\begin{aligned} &\frac{1}{(\varrho(v) - \varrho(u))^\alpha} \int_u^v (\varrho(x) - \varrho(u))^{\alpha-1} f(x)g(x)\varrho'(x)dx \\ &\leq \frac{2f(u)g(u)}{e^{2\beta u}\alpha(\alpha+1)(\alpha+2)} + \frac{f(v)g(v)}{e^{2\beta v}(\alpha+2)} + \frac{N(u,v)}{(\alpha+1)(\alpha+2)} \\ &\leq \frac{F(u)}{\alpha(\alpha+1)(\alpha+2)} + \frac{F(v)}{2(\alpha+2)} + \frac{P(u,v)}{(\alpha+1)(\alpha+2)}. \end{aligned}$$

Corollary 16. *In Corollary 15, taking $\alpha = 1$, we get*

$$\begin{aligned} \frac{1}{(\varrho(v) - \varrho(u))} \int_u^v f(x)g(x)\varrho'(x)dx &\leq \frac{f(u)g(u)}{3e^{2\beta u}} + \frac{f(v)g(v)}{3e^{2\beta v}} + \frac{N(u,v)}{6} \\ &\leq \frac{F(u) + F(v) + P(u,v)}{6}. \end{aligned}$$

Theorem 17. *Let $f, g : I = [u, v] \rightarrow \mathbb{R}^+$ be two similarly ordered integrable exponentially ϱ -preinvex functions with respect to the mapping $\tau : I \times I \rightarrow \mathbb{R}$ and strictly monotonic continuous function ϱ , where $\tau(\varrho(v), \varrho(u)) > 0$. Then for non-positive β and $\alpha > 0$, we have*

$$\begin{aligned} \frac{1}{\tau^\alpha(\varrho(v), \varrho(u))} \int_u^{\varrho^{-1}(\varrho(u) + \tau(\varrho(v), \varrho(u)))} (\varrho(x) - \varrho(u))^{\alpha-1} f(x)g(x)\varrho'(x)dx \\ \leq \frac{M(\alpha, \beta, u, v)}{\alpha(\alpha+1)}, \end{aligned}$$

where

$$M(\alpha, \beta, u, v) = \frac{f(u)g(u)}{e^{2\beta u}} + \alpha \frac{f(v)g(v)}{e^{2\beta v}}.$$

Proof. Since f and g are two similarly ordered integrable exponentially ϱ -preinvex functions, we have

$$f(\varrho^{-1}(\varrho(u) + t\tau(\varrho(v), \varrho(u)))) g(\varrho^{-1}(\varrho(u) + t\tau(\varrho(v), \varrho(u))))$$

$$\begin{aligned}
&\leq \left[(1-t) \frac{f(u)}{e^{\beta u}} + t \frac{f(v)}{e^{\beta v}} \right] \left[(1-t) \frac{g(u)}{e^{\beta u}} + t \frac{g(v)}{e^{\beta v}} \right] \\
&= (1-t) \frac{f(u)g(u)}{e^{2\beta u}} + t \frac{f(v)g(v)}{e^{2\beta v}} - t(1-t) \left[\left(\frac{f(u)}{e^{\beta u}} - \frac{f(v)}{e^{\beta v}} \right) \left(\frac{g(u)}{e^{\beta u}} - \frac{g(v)}{e^{\beta v}} \right) \right] \\
(16) \quad &\leq (1-t) \frac{f(u)g(u)}{e^{2\beta u}} + t \frac{f(v)g(v)}{e^{2\beta v}},
\end{aligned}$$

where $\left(\frac{f(u)}{e^{\beta u}} - \frac{f(v)}{e^{\beta v}} \right) \left(\frac{g(u)}{e^{\beta u}} - \frac{g(v)}{e^{\beta v}} \right) \geq 0$. Multiplying both sides of (16) with $t^{\alpha-1}$ and integrating with respect to t on $[0, 1]$, we obtain

$$\begin{aligned}
&\int_0^1 t^{\alpha-1} f(\varrho^{-1}(\varrho(u) + t\tau(\varrho(v), \varrho(u)))) g(\varrho^{-1}(\varrho(u) + t\tau(\varrho(v), \varrho(u)))) dt \\
(17) \quad &\leq \int_0^1 t^{\alpha-1} \left[(1-t) \frac{f(u)g(u)}{e^{2\beta u}} + t \frac{f(v)g(v)}{e^{2\beta v}} \right] dt.
\end{aligned}$$

Substituting $x = \varrho^{-1}(\varrho(u) + t\tau(\varrho(v), \varrho(u)))$ in (17), we obtain (17). The proof of Theorem 17 is completed. \square

Corollary 18. In Theorem 17, taking $\tau(\varrho(v), \varrho(u)) = \varrho(v) - \varrho(u)$, we get

$$\frac{1}{(\varrho(v) - \varrho(u))^\alpha} \int_u^v (\varrho(x) - \varrho(u))^{\alpha-1} f(x)g(x)\varrho'(x)dx \leq \frac{M(\alpha, \beta, u, v)}{\alpha(\alpha+1)}.$$

Corollary 19. In Corollary 18, taking $\alpha = 1$, we get

$$\frac{1}{(\varrho(v) - \varrho(u))} \int_u^v f(x)g(x)\varrho'(x)dx \leq \frac{1}{2} \left[\frac{f(u)g(u)}{e^{2\beta u}} + \frac{f(v)g(v)}{e^{2\beta v}} \right],$$

Theorem 20. Let $f, g : I = [u, v] \rightarrow \mathbb{R}^+$ be two similarly ordered integrable exponentially ϱ -preinvex functions with respect to the mapping $\tau : I \times I \rightarrow \mathbb{R}$ and strictly monotonic continuous function ϱ , where $\tau(\varrho(v), \varrho(u)) > 0$. Then for non-positive β and $\alpha > 0$, we have

$$\begin{aligned}
&\frac{1}{\tau^\alpha(\varrho(v), \varrho(u))} \int_u^{\varrho^{-1}(\varrho(u) + \tau(\varrho(v), \varrho(u)))} (\varrho(u) + \tau(\varrho(v), \varrho(u)) - \varrho(x))^{\alpha-1} f(x)g(x)\varrho'(x)dx \\
&\leq \frac{R(\alpha, \beta, u, v)}{\alpha(\alpha+1)},
\end{aligned}$$

where

$$R(\alpha, \beta, u, v) = \alpha \frac{f(u)g(u)}{e^{2\beta u}} + \frac{f(v)g(v)}{e^{2\beta v}}.$$

Proof. The proof is similarly as Theorem 17, so we omit it. \square

Corollary 21. In Theorem 20, taking $\tau(\varrho(v), \varrho(u)) = \varrho(v) - \varrho(u)$, we get

$$\frac{1}{(\varrho(v) - \varrho(u))^\alpha} \int_u^v (\varrho(v) - \varrho(x))^{\alpha-1} f(x)g(x)\varrho'(x)dx \leq \frac{R(\alpha, \beta, u, v)}{\alpha(\alpha + 1)}.$$

Theorem 22. Let $f, g : I = [u, v] \rightarrow \mathbb{R}^+$ be two integrable exponentially ϱ -preinvex functions with respect to the mapping $\tau : I \times I \rightarrow \mathbb{R}$ which satisfies Condition C and strictly monotonic continuous function ϱ , where $\tau(\varrho(v), \varrho(u)) > 0$. Then for non-positive β and $\alpha > 0$, we have

$$\begin{aligned} & 2f\left(\varrho^{-1}\left(\varrho(u) + \frac{\tau(\varrho(v), \varrho(u))}{2}\right)\right)g\left(\varrho^{-1}\left(\varrho(u) + \frac{\tau(\varrho(v), \varrho(u))}{2}\right)\right) \\ & \quad - \frac{\alpha}{2\tau^\alpha(\varrho(v), \varrho(u))} \\ & \quad \times \left[\int_u^{\varrho^{-1}(\varrho(u) + \tau(\varrho(v), \varrho(u)))} \frac{(\varrho(x) - \varrho(u))^{\alpha-1} f(x)g(x)\varrho'(x)}{e^{2\beta x}} dx \right. \\ & \quad \left. + \int_u^{\varrho^{-1}(\varrho(u) + \tau(\varrho(v), \varrho(u)))} \frac{(\varrho(u) + \tau(\varrho(v), \varrho(u)) - \varrho(x))^{\alpha-1} f(x)g(x)\varrho'(x)}{e^{2\beta x}} dx \right] \\ & \leq \frac{\alpha}{(\alpha + 1)(\alpha + 2)} M(1, \beta, u, v) + \left[\frac{2 + \alpha(\alpha + 1)}{2\alpha(\alpha + 1)(\alpha + 2)} \right] N(u, v), \end{aligned}$$

where $N(u, v)$ is defined from (13) and $M(1, \beta, u, v)$ is defined from (17) for value $\alpha = 1$.

Proof. Since f and g are two integrable exponentially ϱ -preinvex functions with respect to the function ϱ , by the same way as in the proof of Theorem 8, we have

$$\begin{aligned} & f\left(\varrho^{-1}\left(\varrho(u) + \frac{\tau(\varrho(v), \varrho(u))}{2}\right)\right)g\left(\varrho^{-1}\left(\varrho(u) + \frac{\tau(\varrho(v), \varrho(u))}{2}\right)\right) \\ & \leq \frac{1}{4} \left[\frac{f\left(\varrho^{-1}(\varrho(u) + t\tau(\varrho(v), \varrho(u)))\right)}{e^{\beta(\varrho^{-1}(\varrho(u) + t\tau(\varrho(v), \varrho(u)))}}} + \frac{f\left(\varrho^{-1}(\varrho(u) + (1-t)\tau(\varrho(v), \varrho(u)))\right)}{e^{\beta(\varrho^{-1}(\varrho(u) + (1-t)\tau(\varrho(v), \varrho(u)))}}} \right] \\ & \quad \times \left[\frac{g\left(\varrho^{-1}(\varrho(u) + t\tau(\varrho(v), \varrho(u)))\right)}{e^{\beta(\varrho^{-1}(\varrho(u) + t\tau(\varrho(v), \varrho(u)))}}} + \frac{g\left(\varrho^{-1}(\varrho(u) + (1-t)\tau(\varrho(v), \varrho(u)))\right)}{e^{\beta(\varrho^{-1}(\varrho(u) + (1-t)\tau(\varrho(v), \varrho(u)))}}} \right] \\ (18) \quad & \leq \frac{1}{4} \left\{ \frac{f\left(\varrho^{-1}(\varrho(u) + t\tau(\varrho(v), \varrho(u)))\right)g\left(\varrho^{-1}(\varrho(u) + t\tau(\varrho(v), \varrho(u)))\right)}{e^{2\beta(\varrho^{-1}(\varrho(u) + t\tau(\varrho(v), \varrho(u)))}}} \right. \\ & \quad \left. + \frac{f\left(\varrho^{-1}(\varrho(u) + (1-t)\tau(\varrho(v), \varrho(u)))\right)g\left(\varrho^{-1}(\varrho(u) + (1-t)\tau(\varrho(v), \varrho(u)))\right)}{e^{2\beta(\varrho^{-1}(\varrho(u) + (1-t)\tau(\varrho(v), \varrho(u)))}}} \right. \\ & \quad \left. + \left[(1-t)\frac{f(u)}{e^{\beta u}} + t\frac{f(v)}{e^{\beta v}} \right] \left[t\frac{g(u)}{e^{\beta u}} + (1-t)\frac{g(v)}{e^{\beta v}} \right] \right\} \end{aligned}$$

$$+ \left[t \frac{f(u)}{e^{\beta u}} + (1-t) \frac{f(v)}{e^{\beta v}} \right] \left[(1-t) \frac{g(u)}{e^{\beta u}} + t \frac{g(v)}{e^{\beta v}} \right] \Bigg\}.$$

Multiplying both sides of (18) with $t^{\alpha-1}$ and integrating with respect to t on $[0, 1]$, we obtain

$$\begin{aligned} & f \left(\varrho^{-1} \left(\varrho(u) + \frac{\tau(\varrho(v), \varrho(u))}{2} \right) \right) g \left(\varrho^{-1} \left(\varrho(u) + \frac{\tau(\varrho(v), \varrho(u))}{2} \right) \right) \\ (19) \quad & \leq \frac{\alpha}{4} \left\{ \int_0^1 t^{\alpha-1} \frac{f \left(\varrho^{-1}(\varrho(u) + t\tau(\varrho(v), \varrho(u))) \right) g \left(\varrho^{-1}(\varrho(u) + t\tau(\varrho(v), \varrho(u))) \right)}{e^{2\beta(\varrho^{-1}(\varrho(u) + t\tau(\varrho(v), \varrho(u))))}} dt \right. \\ & + \int_0^1 t^{\alpha-1} \frac{f \left(\varrho^{-1}(\varrho(u) + (1-t)\tau(\varrho(v), \varrho(u))) \right) g \left(\varrho^{-1}(\varrho(u) + (1-t)\tau(\varrho(v), \varrho(u))) \right)}{e^{2\beta(\varrho^{-1}(\varrho(u) + (1-t)\tau(\varrho(v), \varrho(u))))}} dt \\ & \left. + 2M(1, \beta, u, v) \int_0^1 t^{\alpha-1} t(1-t) dt + N(u, v) \int_0^1 t^{\alpha-1} (t^2 + (1-t)^2) dt \right\}. \end{aligned}$$

Substituting

$$x = \varrho^{-1}(\varrho(u) + t\tau(\varrho(v), \varrho(u)))$$

and

$$y = \varrho^{-1}(\varrho(u) + (1-t)\tau(\varrho(v), \varrho(u)))$$

in (19), we obtain (22). The proof of Theorem 22 is completed. \square

Corollary 23. In Theorem 22, taking $\tau(\varrho(v), \varrho(u)) = \varrho(v) - \varrho(u)$, we get

$$\begin{aligned} & 2f \left(\varrho^{-1} \left(\frac{\varrho(u) + \varrho(v)}{2} \right) \right) g \left(\varrho^{-1} \left(\frac{\varrho(u) + \varrho(v)}{2} \right) \right) \\ & \quad - \frac{\alpha}{2(\varrho(v) - \varrho(u))^\alpha} \\ & \times \left[\int_u^v \frac{(\varrho(x) - \varrho(u))^{\alpha-1} f(x)g(x)\varrho'(x)}{e^{2\beta x}} dx + \int_u^v \frac{(\varrho(v) - \varrho(x))^{\alpha-1} f(x)g(x)\varrho'(x)}{e^{2\beta x}} dx \right] \\ & \leq \frac{\alpha}{(\alpha+1)(\alpha+2)} M(1, \beta, u, v) + \left[\frac{2 + \alpha(\alpha+1)}{2\alpha(\alpha+1)(\alpha+2)} \right] N(u, v). \end{aligned}$$

Corollary 24. In Corollary 23, taking $\alpha = 1$, we get

$$\begin{aligned} & 2f \left(\varrho^{-1} \left(\frac{\varrho(u) + \varrho(v)}{2} \right) \right) g \left(\varrho^{-1} \left(\frac{\varrho(u) + \varrho(v)}{2} \right) \right) \\ & \quad - \frac{1}{(\varrho(v) - \varrho(u))} \int_u^v \frac{f(x)g(x)\varrho'(x)}{e^{2\beta x}} dx \\ & \leq \frac{M(1, \beta, u, v) + 2N(u, v)}{6}. \end{aligned}$$

Theorem 25. Let $f, g : I = [u, v] \rightarrow \mathbb{R}^+$ be two similarly ordered integrable exponentially ϱ -preinvex functions with respect to the mapping $\tau : I \times I \rightarrow \mathbb{R}$ which satisfies Condition C and strictly monotonic continuous function ϱ , where $\tau(\varrho(v), \varrho(u)) > 0$. Then for non-positive β and $\alpha > 0$, we have

$$\begin{aligned}
 & f\left(\varrho^{-1}\left(\varrho(u) + \frac{\tau(\varrho(v), \varrho(u))}{2}\right)\right) g\left(\varrho^{-1}\left(\varrho(u) + \frac{\tau(\varrho(v), \varrho(u))}{2}\right)\right) \\
 (20) \quad & \leq \left[\frac{1}{4} + \frac{\alpha}{(\alpha+1)(\alpha+2)}\right] M(1, \beta, u, v) + \left[\frac{2 + \alpha(\alpha+1)}{4\alpha(\alpha+1)(\alpha+2)}\right] N(u, v),
 \end{aligned}$$

where $N(u, v)$ is defined from (13) and $M(1, \beta, u, v)$ is defined from (17) for value $\alpha = 1$.

Proof. From Theorem 17, Theorem 20 and Theorem 22, we get inequality (20). \square

Corollary 26. In Theorem 25, taking $\tau(\varrho(v), \varrho(u)) = \varrho(v) - \varrho(u)$, we get

$$\begin{aligned}
 & f\left(\varrho^{-1}\left(\frac{\varrho(u) + \varrho(v)}{2}\right)\right) g\left(\varrho^{-1}\left(\frac{\varrho(u) + \varrho(v)}{2}\right)\right) \\
 & \leq \left[\frac{1}{4} + \frac{\alpha}{(\alpha+1)(\alpha+2)}\right] M(1, \beta, u, v) + \left[\frac{2 + \alpha(\alpha+1)}{4\alpha(\alpha+1)(\alpha+2)}\right] N(u, v).
 \end{aligned}$$

Corollary 27. In Corollary 26, taking $\alpha = 1$, we get

$$f\left(\varrho^{-1}\left(\frac{\varrho(u) + \varrho(v)}{2}\right)\right) g\left(\varrho^{-1}\left(\frac{\varrho(u) + \varrho(v)}{2}\right)\right) \leq \frac{5M(1, \beta, u, v) + 2N(u, v)}{12}.$$

Theorem 28. Let $f, g : I = [u, v] \rightarrow \mathbb{R}^+$ be two integrable exponentially ϱ -preinvex functions with respect to the mapping $\tau : I \times I \rightarrow \mathbb{R}$ and strictly monotonic continuous function ϱ , where $\tau(\varrho(v), \varrho(u)) > 0$. Then for non-positive β and $\alpha > 0$, we have

$$\begin{aligned}
 & \int_u^{e^{-1}(\varrho(u) + \tau(\varrho(v), \varrho(u)))} \int_u^{e^{-1}(\varrho(u) + \tau(\varrho(v), \varrho(u)))} \int_0^1 t^{\alpha-1} (\varrho(x) - \varrho(u))^{\alpha-1} (\varrho(y) - \varrho(u))^{\alpha-1} \\
 & \times f\left(\varrho^{-1}(\varrho(x) + t\tau(\varrho(y), \varrho(x)))\right) g\left(\varrho^{-1}(\varrho(x) + t\tau(\varrho(y), \varrho(x)))\right) \varrho'(x)\varrho'(y) dt dy dx \\
 & \leq \frac{(2 + \alpha(\alpha+1))}{\alpha^2(\alpha+1)(\alpha+2)} \tau^{2\alpha}(\varrho(v), \varrho(u)) D(\alpha, \beta, u, v) \\
 & \quad + \frac{1}{(\alpha+1)(\alpha+2)} \\
 & \times \left\{ \int_u^{e^{-1}(\varrho(u) + \tau(\varrho(v), \varrho(u)))} (\varrho(y) - \varrho(u))^{\alpha-1} g(y)\varrho'(y) dy \right. \\
 & \quad \times \left. \int_u^{e^{-1}(\varrho(u) + \tau(\varrho(v), \varrho(u)))} (\varrho(x) - \varrho(u))^{\alpha-1} f(x)\varrho'(x) dx \right.
 \end{aligned}$$

$$\begin{aligned}
& + \int_u^{e^{-1}(\varrho(u)+\tau(\varrho(v),\varrho(u)))} (\varrho(y) - \varrho(u))^{\alpha-1} f(y)\varrho'(y)dy \\
& \times \int_u^{e^{-1}(\varrho(u)+\tau(\varrho(v),\varrho(u)))} (\varrho(x) - \varrho(u))^{\alpha-1} g(x)\varrho'(x)dx \Big\} \\
(21) \quad & \leq \frac{(2 + \alpha(\alpha + 1))}{\alpha^2(\alpha + 1)(\alpha + 2)} \tau^{2\alpha}(\varrho(v), \varrho(u)) D(\alpha, \beta, u, v) \\
& + \frac{1}{2(\alpha + 1)(\alpha + 2)} \left[\frac{f(u)}{e^{\beta u}} + \frac{f(v)}{e^{\beta v}} \right] \left[\frac{g(u)}{e^{\beta u}} + \frac{g(v)}{e^{\beta v}} \right] \tau^{2\alpha}(\varrho(v), \varrho(u)),
\end{aligned}$$

where

$$D(\alpha, \beta, u, v) = \frac{2f(u)g(u)}{e^{2\beta u}\alpha(\alpha + 1)(\alpha + 2)} + \frac{f(v)g(v)}{e^{2\beta v}(\alpha + 2)} + \frac{N(u, v)}{(\alpha + 1)(\alpha + 2)}$$

and $N(u, v)$ is defined from (13).

Proof. Since f and g are two integrable exponentially ϱ -preinvex functions with respect to the function ϱ , we have

$$\begin{aligned}
& f(\varrho^{-1}(\varrho(x) + t\tau(\varrho(y), \varrho(x)))) g(\varrho^{-1}(\varrho(x) + t\tau(\varrho(y), \varrho(x)))) \\
& \leq \left[(1-t)\frac{f(x)}{e^{\beta x}} + t\frac{f(y)}{e^{\beta y}} \right] \left[(1-t)\frac{g(x)}{e^{\beta x}} + t\frac{g(y)}{e^{\beta y}} \right] \\
(22) \quad & = (1-t)^2 \frac{f(x)g(x)}{e^{2\beta x}} + t(1-t) \left[\frac{f(x)g(y) + g(x)f(y)}{e^{\beta(x+y)}} \right] + t^2 \frac{f(y)g(y)}{e^{2\beta y}}.
\end{aligned}$$

Multiplying both sides of (22) with $t^{\alpha-1}$ and integrating with respect to t on $[0, 1]$, we obtain

$$\begin{aligned}
& \int_0^1 t^{\alpha-1} f(\varrho^{-1}(\varrho(x) + t\tau(\varrho(y), \varrho(x)))) g(\varrho^{-1}(\varrho(x) + t\tau(\varrho(y), \varrho(x)))) dt \\
& \leq \frac{f(x)g(x)}{e^{2\beta x}} \int_0^1 t^{\alpha-1}(1-t)^2 dx + \frac{f(y)g(y)}{e^{2\beta y}} \int_0^1 t^{\alpha-1}t^2 dt \\
& \quad + \left[\frac{f(x)g(y) + g(x)f(y)}{e^{\beta(x+y)}} \right] \int_0^1 t^{\alpha-1}t(1-t)dt \\
(23) \quad & = \frac{2f(x)g(x)}{e^{2\beta x}\alpha(\alpha + 1)(\alpha + 2)} + \frac{f(y)g(y)}{e^{2\beta y}(\alpha + 2)} + \frac{f(x)g(y) + g(x)f(y)}{e^{\beta(x+y)}(\alpha + 1)(\alpha + 2)}.
\end{aligned}$$

Again, integrating both sides of (23) over the plane domain $\{(x, y) : x \in [u, v], y \in [u, v]\}$ and then using the left-hand side of Theorem 14, we deduce the left-hand side of (21).

On the other hand, substituting $x = \varrho^{-1}(\varrho(u) + t\tau(\varrho(v), \varrho(u)))$ and using again the fact that f and g are two integrable ϱ -preinvex functions with respect to the function ϱ , we get

$$\begin{aligned}
 & \int_u^{\varrho^{-1}(\varrho(u)+\tau(\varrho(v),\varrho(u)))} (\varrho(x) - \varrho(u))^{\alpha-1} f(x)\varrho'(x)dx \\
 &= \tau^\alpha(\varrho(v), \varrho(u)) \int_0^1 f(\varrho^{-1}(\varrho(u) + t\tau(\varrho(v), \varrho(u))))dt \\
 (24) \quad & \leq \frac{1}{2} \left[\frac{f(u)}{e^{\beta u}} + \frac{f(v)}{e^{\beta v}} \right] \tau^\alpha(\varrho(v), \varrho(u)).
 \end{aligned}$$

Similarly, we have

$$\begin{aligned}
 & \int_u^{\varrho^{-1}(\varrho(u)+\tau(\varrho(v),\varrho(u)))} (\varrho(y) - \varrho(u))^{\alpha-1} f(y)\varrho'(y)dy \\
 (25) \quad & \leq \frac{1}{2} \left[\frac{f(u)}{e^{\beta u}} + \frac{f(v)}{e^{\beta v}} \right] \tau^\alpha(\varrho(v), \varrho(u)).
 \end{aligned}$$

$$\begin{aligned}
 & \int_u^{\varrho^{-1}(\varrho(u)+\tau(\varrho(v),\varrho(u)))} (\varrho(x) - \varrho(u))^{\alpha-1} g(x)\varrho'(x)dx \\
 (26) \quad & \leq \frac{1}{2} \left[\frac{g(u)}{e^{\beta u}} + \frac{g(v)}{e^{\beta v}} \right] \tau^\alpha(\varrho(v), \varrho(u)).
 \end{aligned}$$

$$\begin{aligned}
 & \int_u^{\varrho^{-1}(\varrho(u)+\tau(\varrho(v),\varrho(u)))} (\varrho(y) - \varrho(u))^{\alpha-1} g(y)\varrho'(y)dy \\
 (27) \quad & \leq \frac{1}{2} \left[\frac{g(u)}{e^{\beta u}} + \frac{g(v)}{e^{\beta v}} \right] \tau^\alpha(\varrho(v), \varrho(u)).
 \end{aligned}$$

Combining (24)–(27) we obtain the right-hand side of (21). The proof of Theorem 28 is completed. \square

Corollary 29. *In Theorem 28, taking $\tau(\varrho(v), \varrho(u)) = \varrho(v) - \varrho(u)$, we get*

$$\begin{aligned}
 & \int_u^v \int_u^v \int_0^1 t^{\alpha-1} (\varrho(x) - \varrho(u))^{\alpha-1} (\varrho(y) - \varrho(u))^{\alpha-1} f(\varrho^{-1}((1-t)\varrho(x) + t\varrho(y))) \\
 & \quad \times g(\varrho^{-1}((1-t)\varrho(x) + t\varrho(y))) \varrho'(x)\varrho'(y) dt dy dx \\
 & \leq \frac{(2 + \alpha(\alpha + 1))}{\alpha^2(\alpha + 1)(\alpha + 2)} (\varrho(v) - \varrho(u))^{2\alpha} D(\alpha, \beta, u, v) \\
 & \quad + \frac{1}{(\alpha + 1)(\alpha + 2)}
 \end{aligned}$$

$$\begin{aligned}
& \times \left\{ \int_u^v (\varrho(y) - \varrho(u))^{\alpha-1} g(y) \varrho'(y) dy \int_u^v (\varrho(x) - \varrho(u))^{\alpha-1} f(x) \varrho'(x) dx \right. \\
& \left. + \int_u^v (\varrho(y) - \varrho(u))^{\alpha-1} f(y) \varrho'(y) dy \int_u^v (\varrho(x) - \varrho(u))^{\alpha-1} g(x) \varrho'(x) dx \right\} \\
& \leq \frac{(2 + \alpha(\alpha + 1))}{\alpha^2(\alpha + 1)(\alpha + 2)} (\varrho(v) - \varrho(u))^{2\alpha} D(\alpha, \beta, u, v) \\
& + \frac{1}{2(\alpha + 1)(\alpha + 2)} \left[\frac{f(u)}{e^{\beta u}} + \frac{f(v)}{e^{\beta v}} \right] \left[\frac{g(u)}{e^{\beta u}} + \frac{g(v)}{e^{\beta v}} \right] (\varrho(v) - \varrho(u))^{2\alpha}.
\end{aligned}$$

Corollary 30. *In Corollary 29, taking $\alpha = 1$, we get*

$$\begin{aligned}
& \int_u^v \int_u^v \int_0^1 f(\varrho^{-1}((1-t)\varrho(x) + t\varrho(y))) g(\varrho^{-1}((1-t)\varrho(x) + t\varrho(y))) \\
& \quad \cdot \varrho'(x) \varrho'(y) dt dy dx \\
& \leq \frac{1}{3} \left\{ 2(\varrho(v) - \varrho(u))^2 U(\beta, u, v) + \int_u^v \int_u^v f(x)g(y) \varrho'(x) \varrho'(y) dy dx \right\} \\
& \leq \frac{(\varrho(v) - \varrho(u))^2}{3} \left[2U(\beta, u, v) + \frac{1}{4} \left[\frac{f(u)}{e^{\beta u}} + \frac{f(v)}{e^{\beta v}} \right] \left[\frac{g(u)}{e^{\beta u}} + \frac{g(v)}{e^{\beta v}} \right] \right],
\end{aligned}$$

where

$$U(\beta, u, v) = \frac{1}{3} \left[\frac{f(u)g(u)}{e^{2\beta u}} + \frac{f(v)g(v)}{e^{2\beta v}} \right] + \frac{N(u, v)}{6}.$$

3. CONCLUSION

Interested readers can obtain in a similar way new results for different operators such as the k -Riemann–Liouville fractional integral, Katugampola fractional integrals, the conformable fractional integral, Hadamard fractional integrals, etc., and these results can be applied in different areas of pure and applied sciences. It is also worth to mention here that if we take $\beta = 0$, then we have results for the class of ϱ -preinvex functions, which are also new in the literature.

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