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NECESSARY AND SUFFICIENT CONDITIONS FOR COMPLETE MONOTONICITY AND MONOTONICITY OF TWO FUNCTIONS DEFINED BY TWO DERIVATIVES OF A FUNCTION INVOLVING TRIGAMMA FUNCTION

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Dedicated to Professor Dr. Gradimir V. Milovanović at the Mathematical Institute of the Serbian Academy of Sciences and Arts

In the paper, by virtue of convolution theorem for the Laplace transforms, Bernstein's theorem for completely monotonic functions, some properties of a function involving exponential function, and other analytic techniques, the author finds necessary and sufficient conditions for two functions defined by two derivatives of a function involving trigamma function to be completely monotonic or monotonic. These results generalize corresponding known ones.

1. INTRODUCTION

In the literature [**1**, Section 6.4], the function $\Gamma(z)=\int_0^\infty t^{z-1}e^{-t}\mathrm{d}t$ for $\Re(z)>0$ and its logarithmic derivative $\psi(z)=[\ln\Gamma(z)]'=\frac{\Gamma'(z)}{\Gamma(z)}$ are called Euler's gamma function and digamma function respectively. Further, the functions $\psi'(z)$, $\psi''(z)$, $\psi'''(z)$, and $\psi^{(4)}(z)$ are known as the trigamma, tetragamma, pentagamma, and hexagamma functions respectively. As a whole, all the derivatives $\psi^{(k)}(z)$ for $k\ge0$ are known as polygamma functions.

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Recall from Chapter XIII in [**4**], Chapter 1 in [**20**], and Chapter IV in [**21**] that, if a function $f(x)$ on an interval I has derivatives of all orders on I and satisfies $(-1)^n f^{(n)}(x) \ge 0$ for $x \in I$ and $n \in \{0\} \cup \mathbb{N}$, where $\mathbb{N}$ denotes the set of all positive integers, then we call $f(x)$ a completely monotonic function on I.

In [**13**, Section 4] and [**15**, Theorem 4], the author turned out that,

1. if and only if $\alpha \ge 2$, the function
$$\mathfrak{H}_\alpha(x) = \psi'(x) + x\psi''(x) + \alpha\big[x\psi'(x) - 1\big]^2 \tag{1}$$
is completely monotonic on $(0, \infty)$;
2. if and only if $\alpha \le 1$, the function $-\mathfrak{H}_\alpha(x)$ is completely monotonic on $(0, \infty)$;
3. the double inequality
$$-2 < \frac{\psi'(x) + x\psi''(x)}{[x\psi'(x) - 1]^2} < -1 \tag{2}$$
is valid on $(0, \infty)$ and sharp in the sense that the lower and upper bounds -2 and -1 cannot be replaced by any bigger and smaller ones respectively.

For $\beta \in \mathbb{R}$, let
$$H_\beta(x) = \frac{\psi'(x) + x\psi''(x)}{[x\psi'(x) - 1]^\beta} \tag{3}$$
on $(0, \infty)$. In [**13**, Theorem 1.1], the author generalized the double inequality (2) by finding the following necessary and sufficient conditions:

1. if and only if $\beta \ge 2$, the function $H_\beta(x)$ is decreasing on $(0, \infty)$, with the limits
$$\lim_{x \to 0^+} H_\beta(x) = \begin{cases} -1, & \beta = 2 \\ 0, & \beta > 2 \end{cases} \quad \text{and} \quad \lim_{x \to \infty} H_\beta(x) = \begin{cases} -2, & \beta = 2 \\ -\infty, & \beta > 2; \end{cases}$$
2. if $\beta \le 1$, the function $H_\beta(x)$ is increasing on $(0, \infty)$, with the limits
$$H_\beta(x) \to \begin{cases} -\infty, & x \to 0^+ \\ 0, & x \to \infty. \end{cases}$$

Let $\Phi(x) = x\psi'(x) - 1$ on $(0, \infty)$. It is easy to see that
$$\Phi^{(k)}(x) = k\psi^{(k)}(x) + x\psi^{(k+1)}(x), \quad k \in \mathbb{N}. \tag{4}$$
The functions $\mathfrak{H}_\alpha(x)$ and $H_\beta(x)$ in (1) and (3) and the double inequality (2) can be reformulated in terms of $\Phi(x)$ and its first derivative as
$$\mathfrak{H}_\alpha(x) = \Phi'(x) + \alpha\Phi^2(x), \quad H_\beta(x) = \frac{\Phi'(x)}{\Phi^\beta(x)}, \quad -2 < \frac{\Phi'(x)}{\Phi^2(x)} < -1.$$

For $k \in \{0\} \cup \mathbb{N}$ and $\lambda_k, \mu_k \in \mathbb{R}$, let

$$\mathfrak{J}_{k,\lambda_k}(x) = \Phi^{(2k+1)}(x) + \lambda_k \big[\Phi^{(k)}(x)\big]^2 \tag{5}$$

and

$$J_{k,\mu_k}(x) = \frac{\Phi^{(2k+1)}(x)}{\big[(-1)^k \Phi^{(k)}(x)\big]^{\mu_k}} \tag{6}$$

on $(0,\infty)$. It is clear that $\mathfrak{J}_{0,\lambda_0}(x) = \mathfrak{H}_{\lambda_0}(x)$ and $J_{0,\mu_0}(x) = H_{\mu_0}(x)$. These functions are analogues of some functions surveyed in the expository article [**16**].

In this paper, we mainly find necessary and sufficient conditions on λ_k and μ_k such that

1. the functions $\pm\mathfrak{J}_{k,\lambda_k}(x)$ for $k \in \mathbb{N}$ are completely monotonic on $(0,\infty)$;
2. the function $J_{k,\mu_k}(x)$ for $k \in \mathbb{N}$ is monotonic on $(0,\infty)$.

These results generalize corresponding ones in [**13, 15**] mentioned above.

In the last section of this paper, we pose several guesses related to our main results in this paper.

2. LEMMAS

The following lemmas are necessary in this paper.

Lemma 1 ([**13**, Lemma 2.3])**.** *Let*

$$h(t) = \begin{cases} \dfrac{e^t(e^t - 1 - t)}{(e^t - 1)^2}, & t \ne 0 \\ \dfrac{1}{2}, & t = 0 \end{cases}$$

on $(-\infty,\infty)$. Then the following conclusions are valid:

1. *the function $h(t)$ is increasing from $(-\infty,\infty)$ onto $(0,1)$, convex on $(-\infty,0)$, concave on $(0,\infty)$, and logarithmically concave on $(-\infty,\infty)$;*
2. *the function $\frac{h(2t)}{h^2(t)}$ is increasing from $(-\infty,0)$ onto $(0,2)$ and decreasing from $(0,\infty)$ onto $(1,2)$;*
3. *the double inequality*

$$1 < \frac{h(2t)}{h^2(t)} < 2 \tag{7}$$

is valid on $(0,\infty)$ and sharp in the sense that the lower bound 1 and the upper bound 2 cannot be replaced by any larger scalar and any smaller scalar respectively;

4. for any fixed $t > 0$, the function $h(st)h((1-s)t)$ is increasing in $s \in \left(0, \frac{1}{2}\right)$.

Lemma 2. *For $k \geq 0$, the function $(-1)^k \Phi^{(k)}(x)$ is completely monotonic on $(0, \infty)$, with the limits*

$$(-1)^k x^{k+1} \Phi^{(k)}(x) \to \begin{cases} k!, & x \to 0^+; \\ \dfrac{k!}{2}, & x \to \infty. \end{cases} \tag{8}$$

Proof. In the proof of [**15**, Theorem 4], the author established that

$$\Phi(x) = \int_0^\infty h(t) e^{-xt} \mathrm{d}t.$$

This means

$$(-1)^k \Phi^{(k)}(x) = \int_0^\infty h(t) t^k e^{-xt} \mathrm{d}t \tag{9}$$

which is completely monotonic on $(0, \infty)$.

For $\Re(z) > 0$ and $k \geq 1$, we have

$$\psi^{(k-1)}(z+1) = \psi^{(k-1)}(z) + (-1)^{k-1} \frac{(k-1)!}{z^k}.$$

See [**1**, p. 260, 6.4.6]. Considering (4), we have

$$\begin{aligned} x^{k+1} \Phi^{(k)}(x) = x^{k+1} \biggl(k \biggl[\psi^{(k)}(x+1) - (-1)^k \frac{k!}{x^{k+1}} \biggr] \\ + x \biggl[\psi^{(k+1)}(x+1) - (-1)^{k+1} \frac{(k+1)!}{x^{k+2}} \biggr] \biggr) \\ \to (-1)^k k! \end{aligned}$$

as $x \to 0^+$. The first limit in (8) follows.

In [**1**, p. 260, 6.4.11], it was given that, for $|\arg z| < \pi$, as $z \to \infty$,

$$\psi^{(n)}(z) \sim (-1)^{n-1} \left[\frac{(n-1)!}{z^n} + \frac{n!}{2z^{n+1}} + \sum_{k=1}^\infty B_{2k} \frac{(2k+n-1)!}{(2k)! z^{2k+n}} \right],$$

where B_{2k} for $k \geq 1$ stands for the Bernoulli numbers which are generated [**5**] by

$$\frac{z}{e^z - 1} = 1 - \frac{z}{2} + \sum_{k=1}^\infty B_{2k} \frac{z^{2k}}{(2k)!}, \quad |z| < 2\pi.$$

Considering (4), we have

$$\begin{aligned} x^{k+1} \Phi^{(k)}(x) \sim x^{k+1} \biggl(k \biggl[(-1)^{k-1} \biggl[\frac{(k-1)!}{x^k} + \frac{k!}{2x^{k+1}} + \cdots \biggr] \biggr] \\ + x \biggl[(-1)^k \biggl[\frac{k!}{x^{k+1}} + \frac{(k+1)!}{2x^{k+2}} + \cdots \biggr] \biggr] \biggr) \\ \to (-1)^k \frac{k!}{2} \end{aligned}$$

as $x \to \infty$. The second limit in (8) is thus proved. The proof of Lemma 2 is complete. □

Lemma 3 (Convolution theorem for the Laplace transforms [**21**, pp. 91–92])**.** *Let $f_k(t)$ for $k=1,2$ be piecewise continuous in arbitrary finite intervals included in $(0,\infty)$. If there exist some constants $M_k>0$ and $c_k\ge0$ such that $|f_k(t)|\le M_ke^{c_kt}$ for $k=1,2$, then*

$$\int_0^\infty\biggl[\int_0^t f_1(u)f_2(t-u)\,\mathrm{d}u\biggr]e^{-st}\,\mathrm{d}t=\int_0^\infty f_1(u)e^{-su}\,\mathrm{d}u\int_0^\infty f_2(v)e^{-sv}\,\mathrm{d}v.$$

Lemma 4 ([**8**, Theorem 6.1])**.** *If $f(x)$ is differentiable and logarithmically concave on $(-\infty,\infty)$, then the product $f(x)f(x_0-x)$ for any fixed number $x_0\in\mathbb{R}$ is increasing in $x\in\bigl(-\infty,\frac{x_0}{2}\bigr)$ and decreasing in $x\in\bigl(\frac{x_0}{2},\infty\bigr)$.*

Lemma 5 (Bernstein's theorem [**21**, p. 161, Theorem 12b])**.** *A function $f(x)$ is completely monotonic on $(0,\infty)$ if and only if*

$$f(x)=\int_0^\infty e^{-xt}\,\mathrm{d}\sigma(t),\quad x\in(0,\infty), \tag{10}$$

where $\sigma(s)$ is non-decreasing and the integral in (10) converges for $x\in(0,\infty)$.

Lemma 6 ([**6**, Lemma 2.6] and [**14**, Lemma 2.5])**.** *For $k,m\in\mathbb{N}$, the function*

$$U_{k,m}(x)=\frac{1}{(x+1)^m}\frac{x^{k+m}+(x+2)^{k+m}}{x^k+(x+2)^k} \tag{11}$$

is decreasing on $[0,\infty)$, with $U_{k,m}(0)=2^m$ and $\lim_{x\to\infty}U_{k,m}(x)=1$. Equivalently, the function

$$V_{k,m}(x)=\frac{(1-x)^{k+m}+(1+x)^{k+m}}{(1-x)^k+(1+x)^k}$$

is increasing in $x\in[0,1]$, with $V_{k,m}(0)=1$ and $V_{k,m}(1)=2^m$.

3. NECESSARY AND SUFFICIENT CONDITIONS FOR COMPLETE MONOTONICITY

In this section, we find necessary and sufficient conditions on λ_k such that the functions $\pm\mathfrak{J}_{k,\lambda_k}(x)$ defined in (5) are completely monotonic on $(0,\infty)$.

Theorem 1. *For $k\in\{0\}\cup\mathbb{N}$ and $\lambda_k\in\mathbb{R}$,*

1. *if and only if $\lambda_k\ge\frac{(2k+2)!}{k!(k+1)!}$, the function $\mathfrak{J}_{k,\lambda_k}(x)$ is completely monotonic on $(0,\infty)$;*
2. *if and only if $\lambda_k\le\frac12\frac{(2k+2)!}{k!(k+1)!}$, the function $-\mathfrak{J}_{k,\lambda_k}(x)$ is completely monotonic on $(0,\infty)$.*

First proof. If $\mathfrak{J}_{k,\lambda_k}(x)$ is completely monotonic on $(0,\infty)$, then its first derivative

$$\mathfrak{J}'_{k,\lambda_k}(x) = \Phi^{(2k+2)}(x) + 2\lambda_k \Phi^{(k)}(x)\Phi^{(k+1)}(x) \le 0$$

on $(0,\infty)$. Hence, we have

$$\begin{aligned}
\lambda_k &\ge -\frac{1}{2}\frac{\Phi^{(2k+2)}(x)}{\Phi^{(k)}(x)\Phi^{(k+1)}(x)}\\
&= -\frac{1}{2}\frac{(2k+2)\psi^{(2k+2)}(x)+x\psi^{(2k+3)}(x)}{[k\psi^{(k)}(x)+x\psi^{(k+1)}(x)][(k+1)\psi^{(k+1)}(x)+x\psi^{(k+2)}(x)]}\\
&= -\frac{1}{2}\frac{x^{2k+3}\big[(2k+2)\psi^{(2k+2)}(x)+x\psi^{(2k+3)}(x)\big]}{x^{k+1}[k\psi^{(k)}(x)+x\psi^{(k+1)}(x)]x^{k+2}[(k+1)\psi^{(k+1)}(x)+x\psi^{(k+2)}(x)]}\\
&\to -\frac{1}{2}\frac{(-1)^{2k+2}\frac{(2k+2)!}{2}}{(-1)^k\frac{k!}{2}(-1)^{k+1}\frac{(k+1)!}{2}}\\
&= \frac{(2k+2)!}{k!(k+1)!}
\end{aligned}$$

as $x \to \infty$, where we used the second limit in (8). Consequently, the necessary condition for $\mathfrak{J}_{k,\lambda_k}(x)$ to be completely monotonic on $(0,\infty)$ is $\lambda_k \ge \frac{(2k+2)!}{k!(k+1)!}$.

Similarly, if $-\mathfrak{J}_{k,\lambda_k}(x)$ is completely monotonic on $(0,\infty)$, then $\mathfrak{J}'_{k,\lambda_k}(x) \ge 0$, that is,

$$\begin{aligned}
\lambda_k &\le -\frac{1}{2}\frac{x^{2k+3}\big[(2k+2)\psi^{(2k+2)}(x)+x\psi^{(2k+3)}(x)\big]}{x^{k+1}[k\psi^{(k)}(x)+x\psi^{(k+1)}(x)]x^{k+2}[(k+1)\psi^{(k+1)}(x)+x\psi^{(k+2)}(x)]}\\
&\to -\frac{1}{2}\frac{(-1)^{2k+2}(2k+2)!}{(-1)^k k!(-1)^{k+1}(k+1)!}\\
&= \frac{1}{2}\frac{(2k+2)!}{k!(k+1)!}
\end{aligned}$$

as $x \to 0^+$, where we used the first limit in (8). Consequently, the necessary condition for $-\mathfrak{J}_{k,\lambda_k}(x)$ to be completely monotonic on $(0,\infty)$ is $\lambda_k \le \frac{1}{2}\frac{(2k+2)!}{k!(k+1)!}$.

By virtue of the integral representation (9), we arrive at

$$\mathfrak{J}_{k,\lambda_k}(x)(x) = \lambda_k\left[\int_0^\infty t^k h(t)e^{-xt}\mathrm{d}t\right]^2 - \int_0^\infty t^{2k+1}h(t)e^{-xt}\mathrm{d}t.$$

By Lemma 3, we obtain

(12)

$$\begin{aligned}
\mathfrak{J}_{k,\lambda_k}(x) &= \lambda_k\int_0^\infty\left[\int_0^t u^k(t-u)^k h(u)h(t-u)\mathrm{d}u\right]e^{-xt}\mathrm{d}t - \int_0^\infty t^{2k+1}h(t)e^{-xt}\mathrm{d}t\\
&= \int_0^\infty\left[\lambda_k\int_0^t u^k(t-u)^k h(u)h(t-u)\mathrm{d}u - t^{2k+1}h(t)\right]e^{-xt}\mathrm{d}t.
\end{aligned}$$

By logarithmic concavity of $h(t)$ in Lemma 1 and by Lemma 4, we acquire

$$
\begin{aligned}
&\lambda_k \int_0^t u^k(t-u)^k h(u)h(t-u)\mathrm{d}u - t^{2k+1}h(t)\\
&\le \lambda_k \int_0^t u^k(t-u)^k h\left(\frac{t}{2}\right) h\left(t-\frac{t}{2}\right)\mathrm{d}u - t^{2k+1}h(t)\\
&= \lambda_k \frac{(k!)^2}{(2k+1)!} t^{2k+1}\left[h\left(\frac{t}{2}\right)\right]^2 - t^{2k+1}h(t)\\
&= \left[h\left(\frac{t}{2}\right)\right]^2 \left(\lambda_k \frac{(k!)^2}{(2k+1)!} - \frac{h(t)}{\left[h\left(\frac{t}{2}\right)\right]^2}\right) t^{2k+1}
\end{aligned}
$$

and

$$
\begin{aligned}
&\lambda_k \int_0^t u^k(t-u)^k h(u)h(t-u)\mathrm{d}u - t^{2k+1}h(t)\\
&\ge \lambda_k \int_0^t u^k(t-u)^k h(0)h(t)\mathrm{d}u - t^{2k+1}h(t)\\
&= \lambda_k \frac{(k!)^2}{(2k+1)!} t^{2k+1}h(0)h(t) - t^{2k+1}h(t)\\
&= \left[\frac{\lambda_k}{2}\frac{(k!)^2}{(2k+1)!} - 1\right] t^{2k+1}h(t),
\end{aligned}
$$

where we used the computation

$$
\begin{aligned}
\int_0^t u^k(t-u)^k \mathrm{d}u &= t^{2k+1}\int_0^1 s^k(1-s)^k \mathrm{d}s\\
&= B(k+1,k+1)t^{2k+1}\\
&= \frac{(k!)^2}{(2k+1)!} t^{2k+1}.
\end{aligned} \tag{13}
$$

By the double inequality (7) in Lemma 1, when $\lambda_k \le \frac{(2k+1)!}{(k!)^2}$, we deduce

$$
\lambda_k \int_0^t u^k(t-u)^k h(u)h(t-u)\mathrm{d}u - t^{2k+1}h(t) < 0, \quad t \in (0,\infty);
$$

when $\lambda_k \ge 2\frac{(2k+1)!}{(k!)^2}$, we have

$$
\lambda_k \int_0^t u^k(t-u)^k h(u)h(t-u)\mathrm{d}u - t^{2k+1}h(t) > 0, \quad t \in (0,\infty).
$$

Consequently, when $\lambda_k \ge 2\frac{(2k+1)!}{(k!)^2} = \frac{(2k+2)!}{k!(k+1)!}$, the function $\mathfrak{J}_{k,\lambda_k}(x)(x)$ is completely monotonic on $(0,\infty)$; when $\lambda_k \le \frac{(2k+1)!}{(k!)^2} = \frac{1}{2}\frac{(2k+2)!}{k!(k+1)!}$, the function $-\mathfrak{J}_{k,\lambda_k}(x)(x)$ is completely monotonic on $(0,\infty)$. The proof of Theorem 1 is complete. □

Second proof. The integral representation (12) can be alternatively reformulated as

$$\begin{aligned}\mathfrak{I}_{k,\lambda_k}(x)&=\int_0^\infty\left[\lambda_k\frac{\int_0^t u^k(t-u)^kh(u)h(t-u)\mathrm{d}u}{t^{2k+1}h(t)}-1\right]t^{2k+1}h(t)e^{-xt}\mathrm{d}t\\&=\int_0^\infty\left[\lambda_k\frac{\int_0^1 v^k(1-v)^kh(vt)h((1-v)t)\mathrm{d}v}{h(t)}-1\right]t^{2k+1}h(t)e^{-xt}\mathrm{d}t.\end{aligned}$$

By the last conclusion in Lemma 1, the sharp lower bound in (7), and the equation (13) in sequence, we obtain the sharp inequalities

$$\frac{\int_0^1 v^k(1-v)^kh(vt)h((1-v)t)\mathrm{d}v}{h(t)}>\frac{h(0)h(t)\int_0^1 v^k(1-v)^k\mathrm{d}v}{h(t)}=\frac{1}{2}\frac{(k!)^2}{(2k+1)!}$$

and

$$\frac{\int_0^1 v^k(1-v)^kh(vt)h((1-v)t)\mathrm{d}v}{h(t)}<\frac{\left[h\left(\frac{1}{2}t\right)\right]^2\int_0^1 v^k(1-v)^k\mathrm{d}v}{h(t)}<\frac{(k!)^2}{(2k+1)!}$$

for $t\in(0,\infty)$. Due to the sharpness of these inequalities, making use of Lemma 5 immediately leads to necessary and sufficient conditions on λ_k in Theorem 1. The proof of Theorem 1 is complete. □

4. NECESSARY AND SUFFICIENT CONDITIONS FOR MONOTONICITY

In this section, we find necessary and sufficient conditions on μ_k such that the function $J_{k,\mu_k}(x)$ defined in (6) is monotonic on $(0,\infty)$.

Theorem 2. *For $k\in\{0\}\cup\mathbb{N}$ and $\mu_k\in\mathbb{R}$,*

1. *if and only if $\mu_k\ge 2$, the function $J_{k,\mu_k}(x)$ is decreasing on $(0,\infty)$, with the limits*

$$\lim_{x\to0^+}J_{k,\mu_k}(x)=\begin{cases}-\dfrac{1}{2}\dfrac{(2k+2)!}{k!(k+1)!}, & \mu_k=2\\ 0, & \mu_k>2\end{cases}\tag{14}$$

and

$$\lim_{x\to\infty}J_{k,\mu_k}(x)=\begin{cases}-\dfrac{(2k+2)!}{k!(k+1)!}, & \mu_k=2\\ -\infty, & \mu_k>2;\end{cases}\tag{15}$$

2. *if $\mu_k\le 1$, the function $J_{k,\mu_k}(x)$ is increasing on $(0,\infty)$, with the limits*

$$J_{k,\mu_k}(x)\to\begin{cases}-\infty, & x\to0^+\\ 0, & x\to\infty;\end{cases}\tag{16}$$

3. *the double inequality*

$$-\frac{(2k+2)!}{k!(k+1)!} < \frac{\Phi^{(2k+1)}(x)}{\left[\Phi^{(k)}(x)\right]^2} < -\frac{1}{2}\frac{(2k+2)!}{k!(k+1)!} \tag{17}$$

is valid on $(0,\infty)$ *and sharp in the sense that the lower and upper bounds cannot be replaced by any larger and smaller numbers respectively.*

Proof. If the function $J_{k,\mu_k}(x)$ is decreasing on $(0,\infty)$, then its first derivative

$$J'_{k,\mu_k}(x) = \frac{\Phi^{(2k+2)}(x)\left[(-1)^k\Phi^{(k)}(x)\right] - \mu_k(-1)^k\Phi^{(k+1)}(x)\Phi^{(2k+1)}(x)}{\left[(-1)^k\Phi^{(k)}(x)\right]^{\mu_k+1}} \le 0,$$

that is,

$$\begin{aligned}\mu_k &\ge \frac{\Phi^{(k)}(x)\Phi^{(2k+2)}(x)}{\Phi^{(k+1)}(x)\Phi^{(2k+1)}(x)}\\ &= \frac{\left[(-1)^k x^{k+1}\Phi^{(k)}(x)\right]\left[(-1)^{2k+2}x^{2k+3}\Phi^{(2k+2)}(x)\right]}{[(-1)^{k+1}x^{k+2}\Phi^{(k+1)}(x)][(-1)^{2k+1}x^{2k+2}\Phi^{(2k+1)}(x)]}\\ &\to \frac{k!(2k+2)!}{(k+1)!(2k+1)!}\\ &= 2\end{aligned}$$

as $x \to 0^+$ or $x \to \infty$, where we used the limits in (8). Hence, the necessary condition for $J_{k,\mu_k}(x)$ to be decreasing on $(0,\infty)$ is $\mu_k \ge 2$.

By the integral representation (9), the function $J_{k,\mu_k}(x)$ can be rewritten as

$$J_{k,\mu_k}(x) = -\frac{\int_0^\infty t^{2k+1}h(t)e^{-xt}\,\mathrm{d}t}{\left[\int_0^\infty t^k h(t)e^{-xt}\,\mathrm{d}t\right]^{\mu_k}}.$$

Since

$$\frac{\mathrm{d}J_{k,\mu_k}(x)}{\mathrm{d}x} = \frac{\begin{bmatrix}\int_0^\infty t^{2k+2}h(t)e^{-xt}\,\mathrm{d}t\int_0^\infty t^k h(t)e^{-xt}\,\mathrm{d}t\\ -\mu_k\int_0^\infty t^{2k+1}h(t)e^{-xt}\,\mathrm{d}t\int_0^\infty t^{k+1}h(t)e^{-xt}\,\mathrm{d}t\end{bmatrix}}{\left[\int_0^\infty t^k h(t)e^{-xt}\,\mathrm{d}t\right]^{\mu_k+1}},$$

in order to prove that the function $J_{k,\mu_k}(x)$ is decreasing on $(0,\infty)$, it is sufficient to show the inequality

$$\begin{aligned}\mu_k\int_0^\infty t^{2k+1}h(t)e^{-xt}\,\mathrm{d}t\int_0^\infty t^{k+1}h(t)e^{-xt}\,\mathrm{d}t\\ \ge \int_0^\infty t^{2k+2}h(t)e^{-xt}\,\mathrm{d}t\int_0^\infty t^k h(t)e^{-xt}\,\mathrm{d}t.\end{aligned} \tag{18}$$

By Lemma 3, the inequality (18) can be reformulated as

$$\begin{aligned}(19)\quad \mu_k \int_0^\infty \left[\int_0^t u^{2k+1}(t-u)^{k+1}h(u)h(t-u)\mathrm{d}u\right]e^{-xt}\mathrm{d}t \\ \ge \int_0^\infty \left[\int_0^t u^{2k+2}(t-u)^{k}h(u)h(t-u)\mathrm{d}u\right]e^{-xt}\mathrm{d}t.\end{aligned}$$

Let

$$P_k(t) = \int_0^t u^{2k+1}(t-u)^{k+1}h(u)h(t-u)\mathrm{d}u$$

and

$$Q_k(t) = \int_0^t u^{2k+2}(t-u)^{k}h(u)h(t-u)\mathrm{d}u.$$

Then the inequality (19) can be rewritten as

$$(20)\qquad \int_0^\infty Q_k(t)\left[\frac{P_k(t)}{Q_k(t)} - \frac{1}{\mu_k}\right]e^{-xt}\mathrm{d}t \ge 0.$$

Changing the variable $u = \frac{(1+v)t}{2}$ results in

$$\begin{aligned}\frac{P_k(t)}{Q_k(t)} &= \frac{\int_0^1 \left[(1-v)^k + (1+v)^k\right]\left(1-v^2\right)^{k+1} h\left(\frac{1+v}{2}t\right)h\left(\frac{1-v}{2}t\right)\mathrm{d}v}{\int_0^1 [(1-v)^{k+2} + (1+v)^{k+2}](1-v^2)^k h\left(\frac{1+v}{2}t\right)h\left(\frac{1-v}{2}t\right)\mathrm{d}v} \\ &\to \frac{\int_0^1 \left[(1-v)^k + (1+v)^k\right]\left(1-v^2\right)^{k+1}\mathrm{d}v}{\int_0^1 [(1-v)^{k+2} + (1+v)^{k+2}](1-v^2)^k\mathrm{d}v} \\ &= \frac{2^{3k+3}B(2k+2,k+2)}{2^{3k+3}B(2k+3,k+1)} \\ &= \frac{1}{2}\end{aligned}\qquad (21)$$

as $t \to 0^+$ or $t \to \infty$, where we used the fact in Lemma 1 that the function $h(t)$ is increasing from $(0,\infty)$ onto $\left(\frac{1}{2},1\right)$ and used the formula

$$\begin{aligned}\int_0^1 \left[(1+x)^{\mu-1}(1-x)^{\nu-1} + (1+x)^{\nu-1}(1-x)^{\mu-1}\right]\mathrm{d}x &= 2^{\mu+\nu-1}B(\mu,\nu) \\ &= 2^{\mu+\nu-1}\frac{\Gamma(\mu)\Gamma(\nu)}{\Gamma(\mu+\nu)}\end{aligned}\qquad (22)$$

for $\Re(\mu), \Re(\nu) > 0$ in [**2**, p. 321, 3.214].

Let

$$\begin{aligned}S_k(t)&=\int_0^1\big[(1-v)^k+(1+v)^k\big]\big(1-v^2\big)^{k+1}h\bigg(\frac{1+v}{2}t\bigg)h\bigg(\frac{1-v}{2}t\bigg)\mathrm{d}v\\&\quad-\frac12\int_0^1\big[(1-v)^{k+2}+(1+v)^{k+2}\big]\big(1-v^2\big)^kh\bigg(\frac{1+v}{2}t\bigg)h\bigg(\frac{1-v}{2}t\bigg)\mathrm{d}v\\&=\int_0^1T_k(v)\big[(1-v)^k+(1+v)^k\big]\big(1-v^2\big)^kh\bigg(\frac{1+v}{2}t\bigg)h\bigg(\frac{1-v}{2}t\bigg)\mathrm{d}v,\end{aligned}$$

where

$$T_k(v)=1-v^2-\frac12\frac{(1-v)^{k+2}+(1+v)^{k+2}}{(1-v)^k+(1+v)^k}$$

with $T_k(0)=\frac12$ and $T_k(1)=-2$. By Lemma 6 for $m=2$, we see that the function $T_k(v)$ is decreasing on $[0,1]$ and has only one zero $v_0\in(0,1)$. As a result, by the fourth conclusion in Lemma 1, we obtain

$$\begin{aligned}S_k(t)&=\int_0^{v_0}+\int_{v_0}^1T_k(v)\big[(1-v)^k+(1+v)^k\big]\big(1-v^2\big)^kh\bigg(\frac{1+v}{2}t\bigg)h\bigg(\frac{1-v}{2}t\bigg)\mathrm{d}v\\&>h\bigg(\frac{1+v_0}{2}t\bigg)h\bigg(\frac{1-v_0}{2}t\bigg)\int_0^{v_0}T_k(v)\big[(1-v)^k+(1+v)^k\big]\big(1-v^2\big)^k\mathrm{d}v\\&\quad+h\bigg(\frac{1+v_0}{2}t\bigg)h\bigg(\frac{1-v_0}{2}t\bigg)\int_{v_0}^1T_k(v)\big[(1-v)^k+(1+v)^k\big]\big(1-v^2\big)^k\mathrm{d}v\\&=h\bigg(\frac{1+v_0}{2}t\bigg)h\bigg(\frac{1-v_0}{2}t\bigg)\int_0^1T_k(v)\big[(1-v)^k+(1+v)^k\big]\big(1-v^2\big)^k\mathrm{d}v\\&=0,\end{aligned}$$

where we used the formula (22) to obtain

$$\begin{aligned}&\int_0^1T_k(v)\big[(1-v)^k+(1+v)^k\big]\big(1-v^2\big)^k\mathrm{d}v\\&=\int_0^1\big[(1+v)^{k+1}(1-v)^{2k+1}+(1+v)^{2k+1}(1-v)^{k+1}\big]\mathrm{d}v\\&\quad-\frac12\int_0^1\big[(1+v)^k(1-v)^{2k+2}+(1+v)^{2k+2}(1-v)^k\big]\mathrm{d}v\\&=2^{3k+3}B(k+2,2k+2)-2^{3k+2}B(k+1,2k+3)\\&=0.\end{aligned}$$

Consequently, considering the limit in (21), we conclude an inequality $\frac{P_k(t)}{Q_k(t)}>\frac12$ for $t>0$, which is sharp in the sense that the lower bound $\frac12$ cannot be replaced by any larger number. This sharp inequality shows that the inequality (20) is valid for all $\mu_k\ge2$. Accordingly, the condition $\mu_k\ge2$ is sufficient for $J_{k,\mu_k}(x)$ to be decreasing on $(0,\infty)$.

It is easy to verify that

$$\begin{aligned}\big[(1-v)^k+(1+v)^k\big]\big(1-v^2\big)^{k+1}-\big[(1-v)^{k+2}+(1+v)^{k+2}\big]\big(1-v^2\big)^k\\ =-2v\big(1-v^2\big)^k\big[(1+v)^k-(1-v)^k+v\big((1-v)^k+(1+v)^k\big)\big]<0\end{aligned}$$

for $v\in(0,1)$. Combining this negativity with the positivity of $h(t)$ on $(0,\infty)$, we deduce an inequality $0<\frac{P_k(t)}{Q_k(t)}<1$ on $(0,\infty)$. This means that, when $\mu_k\le 1$, the function $J_{k,\mu_k}(x)$ is increasing on $(0,\infty)$.

The limits in (14), (15), and (16) follow from applying the limits in (8).

The double inequality (17) and its sharpness follow from monotonicity of $J_{k,\mu_k}(x)$ and the limits (14) and (15) for $\mu_k=2$. The proof of Theorem 2 is complete. □

Corollary 1. *For $k\in\{0\}\cup\mathbb{N}$ and $\mu_k\in\mathbb{R}$, the function*

$$(-1)^k\big[\mu_k\Phi^{(k+1)}(x)\Phi^{(2k+1)}(x)-\Phi^{(2k+2)}(x)\Phi^{(k)}(x)\big]$$

is completely monotonic on $(0,\infty)$ if and only if $\mu_k\ge 2$, while its negativity is completely monotonic on $(0,\infty)$ if $\mu_k\le 1$.

Proof. This follows from the proof of Theorem 2. □

5. SEVERAL REMARKS AND GUESSES

Finally, we list several guesses related to main results in this paper in the form of remarks.

Remark 1. Corollary 1 in this paper generalizes [**13**, Corollary 3.1].

Remark 2. For $k,m\in\mathbb{N}$, we guess that the function $U_{k,m}(x)$ defined in (11) should be completely monotonic on $(0,\infty)$.

Remark 3. For $k\ge m\ge 0$, let

$$\mathcal{J}_{k,m}(x)=\frac{\Phi^{(2k+2)}(x)}{\Phi^{(k-m)}(x)\Phi^{(k+m+1)}(x)}$$

on $(0,\infty)$. Motivated by the proof of necessary conditions in Theorem 1, we guess that the function $\mathcal{J}_{k,m}(x)$ for $k\ge m\ge 0$ should be decreasing on $(0,\infty)$. Consequently, the inequality

$$-\frac{2(2k+2)!}{k!(k+1)!}<\mathcal{J}_{k,0}(x)<-\frac{(2k+2)!}{k!(k+1)!}$$

for $k\ge 0$ should be valid on $(0,\infty)$ and sharp in the sense that the lower and upper bounds cannot be replaced by any larger and smaller numbers respectively.

Remark 4. For $k \geq 0$, we guess that the function of $(-1)^k x^k \Phi^{(k)}(x)$ should be completely monotonic on $(0, \infty)$, but the function $(-1)^k x^{k+1} \Phi^{(k)}(x)$ should not be completely monotonic on $(0, \infty)$. In other words, the completely monotonic degree of $(-1)^k \Phi^{(k)}(x)$ with respect to $x \in (0, \infty)$ should be $k \geq 0$. For the concept and new results of completely monotonic degrees, please refer to the papers [**3, 8, 17, 19, 20**] and closely related references therein.

We also guess that the function $(-1)^k x^{k+1} \Phi^{(k)}(x)$ for $k \geq 0$ should be decreasing on $(0, \infty)$. Consequently, considering the limits in (8), the double inequality

$$\frac{k!}{2} \frac{1}{x^{k+1}} < (-1)^k \Phi^{(k)}(x) < k! \frac{1}{x^{k+1}}$$

for $k \geq 0$ should be valid on $(0, \infty)$ and sharp in the sense that the scalars $\frac{k!}{2}$ and $k!$ in the lower and upper bounds cannot be replaced by any bigger and smaller ones respectively.

Remark 5. By virtue of the integral representation (9), integrating by parts yields

$$\begin{aligned} x^k (-1)^k \Phi^{(k)}(x) &= -x^{k-1} \int_0^\infty t^k h(t) \frac{\mathrm{d} e^{-xt}}{\mathrm{d} t} \mathrm{d} t \\ &= -x^{k-1} \left(\left[t^k h(t) e^{-xt} \right] \Big|_{t \to 0^+}^{t \to \infty} - \int_0^\infty \left[t^k h(t) \right]' e^{-xt} \mathrm{d} t \right) \\ &= x^{k-1} \int_0^\infty \left[t^k h(t) \right]' e^{-xt} \mathrm{d} t. \end{aligned}$$

By induction, consecutively integrating by parts results in

$$\begin{aligned} x^k (-1)^k \Phi^{(k)}(x) &= x \int_0^\infty \left[t^k h(t) \right]^{(k-1)} e^{-xt} \mathrm{d} t \\ &= -\int_0^\infty \left[t^k h(t) \right]^{(k-1)} \frac{\mathrm{d} e^{-xt}}{\mathrm{d} t} \mathrm{d} t \\ &= -\left[\left(\left[t^k h(t) \right]^{(k-1)} e^{-xt} \right) \Big|_{t \to 0^+}^{t \to \infty} - \int_0^\infty \left[t^k h(t) \right]^{(k)} e^{-xt} \mathrm{d} t \right] \\ &= \int_0^\infty \left[t^k h(t) \right]^{(k)} e^{-xt} \mathrm{d} t \end{aligned}$$

and

$$x^{k+1} (-1)^k \Phi^{(k)}(x) = \frac{k!}{2} + \int_0^\infty \left[t^k h(t) \right]^{(k+1)} e^{-xt} \mathrm{d} t.$$

Utilizing the last two integral representations, considering the necessary and sufficient condition expressed in (10), and basing on those guesses in Remark 4 above, we guess that, for given $k \in \mathbb{N}$, all the derivatives $\left[t^k h(t) \right]^{(\ell)}$ for $0 \leq \ell \leq k$ should be positive on $(0, \infty)$, but $\left[t^k h(t) \right]^{(k+1)}$ should change sign on $(0, \infty)$.

Remark 6. We guess that the sufficient condition $\mu_k \leq 1$ in Theorem 2 should be $\mu_k \leq \mu(k)$ with $1 < \mu(k) < 2$.

Remark 7. This paper is a revised version of the electronic preprint [**6**] and the third one in a series of articles including [**7, 9, 10, 11, 12, 13, 14, 15, 18**].

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