Applicable Analysis and Discrete Mathematics
available online at http://pefmath.etf.rs

Appl. Anal. Discrete Math. **15** (2021), 467–485.
https://doi.org/10.2298/AADM201227028Y

A NEW CHAIN OF INEQUALITIES INVOLVING THE TOADER-QI, LOGARITHMIC AND EXPONENTIAL MEANS

*Zhen-Hang Yang and Jing-Feng Tian**

In this paper, we establish an interesting chain of sharp inequalities involving Toader-Qi mean, exponential mean, logarithmic mean, arithmetic mean and geometric mean. This greatly improves some existing results.

1. INTRODUCTION

Throughout the paper, we assume that $a, b > 0$ with $a \neq b$. The classical logarithmic mean, exponential mean and power mean of order p are defined by

$$L \equiv L(a,b) = \frac{a-b}{\ln a - \ln b},\ E \equiv E(a,b) = e^{-1}\left(\frac{b^b}{a^a}\right)^{1/(b-a)},$$

$$A_q \equiv A_q(a,b) = \left(\frac{a^q + b^q}{2}\right)^{1/q} \text{ if } q \neq 0 \text{ and } A_0 \equiv A_0(a,b) = \sqrt{ab},$$

respectively. As usual, $A \equiv A(a,b) = A_1(a,b)$ and $G \equiv G(a,b) = A_0(a,b)$ denote the arithmetic and geometric means, respectively (see [**5**]). A new mean, Toader-Qi mean $TQ \equiv TQ(a,b)$, given by

$$TQ(a,b) = \frac{2}{\pi}\int_0^{\pi/2} a^{\cos^2\theta} b^{\sin^2\theta} d\theta = \sqrt{ab}\, I_0\left(\ln\sqrt{\frac{a}{b}}\right)$$

*Corresponding author. Jing-Feng Tian
2020 Mathematics Subject Classification. Primary 39B62; Secondary 26E60, 33B10.
Keywords and Phrases. Toader-Qi Mean, Modified Bessel Function, Exponential Mean, Logarithmic Mean, Hyperbolic Function.

appeared in [**12, 18**], where

$$I_v(x) = \sum_{n=0}^{\infty} \frac{(x/2)^{2n+v}}{n!\Gamma(v+n+1)}, \quad x \in \mathbb{R}, \quad v \in \mathbb{R} \backslash \{-1, -2, ...\}, \tag{1}$$

is the modified Bessel function of the first kind of order v ([**4, 9, 19, 44**]), $\Gamma(x)$ is the gamma function (see [**17, 40**]).

There are many interesting inequalities for these means, see for example, [**3, 6, 8, 10, 13, 14, 15, 20, 22, 25, 26, 29**].

Here we mention several chains of inequalities for these means. The first of which is:

$$G < L < A_{1/3} < \frac{A+2G}{3} < A_{1/2} < \frac{2A+G}{3} < A_{2/3} < E < A_{\ln 2} \tag{2}$$

(see [**7, 11, 16, 23, 24**]), where every inequality is sharp. The second is:

$$\sqrt{AG} < \sqrt{LE} < \frac{L+E}{2} < \frac{A+G}{2}, \tag{3}$$

which appeared in [**2**]. The third was proven in [**12**]:

$$L < TQ < \frac{A+G}{2} < \frac{2A+G}{3} < E. \tag{4}$$

The fourth was established in [**30**, Theorem 3.4], [**42**, Equation (20)],

$$L^{3/4}A^{1/4} < TQ < \sqrt{LA_{2/3}} < \frac{3}{4}L + \frac{1}{4}A. \tag{5}$$

The fifth states that

$$\sqrt{\frac{e}{\pi}}\sqrt{LE} < \frac{2^{4/3}}{\sqrt{\pi}}\sqrt{LA_{2/3}} < TQ < \sqrt{LA_{2/3}} < \sqrt{LE}, \tag{6}$$

which was shown in [**32**], [**42**, Remark 4], where every inequality is sharp.

Inspired these chains of inequalities, the purpose of this paper is to further establish a new chain of inequalities involving those means mentioned above, that is,

$$\theta_2 A_{2/3} < \theta_1 \Phi_{r_0} < \frac{TQ^2}{L} < \Phi_{r_0} < A_{2/3} < \Psi_{p_0} < E < \Psi_{p_0^*} < \lambda_{p_0}\Psi_{p_0}, \tag{7}$$

with the best constants

$$r_0 = \frac{1198}{901} = 1.329..., \quad p_0 = \frac{22}{63} = 0.349..., \quad p_0^* = \frac{1}{90e^{-1} - 30} = 0.321...,$$
$$\theta_2 = \frac{\sqrt{8}}{\pi} = 0.900..., \quad \theta_1 = \frac{2640}{901\pi} = 0.932..., \quad \lambda_{p_0} = \frac{660}{241e} = 1.007..,$$

where

$$\Phi_r \equiv \Phi_r(a,b) = \frac{1}{3}\frac{157A^2 + 157rAG + (83r-37)G^2}{(78-r)A + (81r-38)G}, \; r \in \left(-\frac{1}{2}, 78\right), \tag{8}$$

$$\Psi_p \equiv \Psi_p(a,b) = \frac{(30p+1)A^2 + (28-15p)AG + (16-15p)G^2}{45(pA + (1-p)G)}, \quad p \in (0,\infty). \tag{9}$$

Those new inequalities in the chain of inequalities (7) are contained in the following three theorems.

Theorem 1. *Let $\Phi_r(a,b)$ be defined by (8). The double inequality*

$$\beta_1 \Phi_{r_0}(a,b) < \frac{TQ^2(a,b)}{L(a,b)} < \Phi_{r_0}(a,b) \tag{10}$$

holds with the best constants $r_0 = 1198/901$, $\beta_1 = 2640/(901\pi) = 0.932...$.

Theorem 2. *Let $\Phi_r(a,b)$ be defined by (8). The double inequality*

$$\beta_2 A_{2/3}(a,b) < \Phi_{r_0}(a,b) < A_{2/3}(a,b) \tag{11}$$

holds with the best constants $2/3$ and $\beta_2 = 901\sqrt{2}/1320 = 0.965...$.

Theorem 3. *Let $\Psi_p(a,b)$ be defined by (9). The inequalities*

$$A_{2/3}(a,b) < \Psi_{p_0}(a,b) < E(a,b) < \Psi_{p_0^*}(a,b) < \lambda_{p_0}\Psi_{p_0}(a,b) \tag{12}$$

holds with the best constants $2/3$, $p_0 = 22/63$, $p_0^* = 1/\left(90e^{-1} - 30\right)$ and $\lambda_{p_0} = 660/(241e)$.

To prove the above three theorems, we need hyperbolic functions representations of those means (see [**27**, Lemma 3]). Without loss of generality, we assume that $b > a > 0$ and let $t = \ln\sqrt{b/a}$. Then

$$\frac{L(a,b)}{\sqrt{ab}} = \frac{\sinh t}{t}, \quad \frac{E(a,b)}{\sqrt{ab}} = \exp\left(\frac{t}{\tanh t} - 1\right), \tag{13}$$

$$\frac{TQ(a,b)}{\sqrt{ab}} = I_0(t), \quad \frac{A_q(a,b)}{\sqrt{ab}} = \cosh^{1/q}(qt) \quad \text{for } q \neq 0, \tag{14}$$

$$\frac{\Phi_r(a,b)}{\sqrt{ab}} = \frac{1}{3}\frac{157\cosh^2 t + 157r\cosh t + 83r - 37}{(78-r)\cosh t + 81r - 38} = \Phi_r(t), \tag{15}$$

$$\frac{\Psi_p(a,b)}{\sqrt{ab}} = \frac{1}{45}\frac{(30p+1)\cosh^2 t + (28-15p)\cosh t + (16-15p)}{p\cosh t + (1-p)} = \Psi_p(t), \tag{16}$$

where $r \in (-1/2, 78)$ and $p \in (0,\infty)$.

The rest of this paper is organized as follows. In the next section, some useful lemmas are collected. The proofs of Theorems 1–3 are given in Sections 3–5. In the last section, several remarks are presented.

2. LEMMAS AND A TOOL

2.1 Four lemmas

The following four lemmas are needed to prove Theorem 1.

Lemma 1 ([**30**, Lemma 2.8])**.** *The formula*

$$I_0^2(t) = \sum_{n=0}^{\infty} \frac{(2n)!}{2^{2n} n!^4} t^{2n} \tag{17}$$

holds for $t > 0$*.*

Lemma 2. *The sequence*

$$s_n = \frac{(2n)!\,(2n+1)!}{2^{4n} n!^4} \tag{18}$$

is strictly decreasing for $n \geq 0$ *with* $\lim_{n\to\infty} s_n = 2/\pi$*. Moreover, the double inequality*

$$\frac{15}{\pi}\frac{2n+1}{15n+4} < s_n < \frac{1225}{256}\frac{2n+1}{15n+4} \tag{19}$$

holds for $n \geq 3$*.*

Proof. The monotonicity of s_n was proven in [**30**, Lemma 2.5]. To prove (19), we set

$$s_n^* = \frac{15n+4}{2n+1} s_n = (15n+4)\frac{(2n)!^2}{2^{4n} n!^4}.$$

Since

$$\frac{s_{n+1}^*}{s_n^*} - 1 = -\frac{1}{4}\frac{n-3}{(15n+4)(n+1)^2} \leq 0$$

for $n \geq 3$, the sequence $\{s_n^*\}_{n\geq 3}$ is decreasing, and therefore,

$$\frac{1225}{256} = s_3^* > s_n^* > \lim_{n\to\infty} s_n^* = \lim_{n\to\infty}\left(\frac{15n+4}{2n+1}s_n\right) = \frac{15}{2}\frac{2}{\pi} = \frac{15}{\pi},$$

which implies (19). □

Lemma 3. *We have*

$$\sum_{k=0}^{n} \frac{2^{2k}}{(2k+1)!\,(2n-2k)!} = \frac{3^{2n+1}+1}{4\,(2n+1)!}. \tag{20}$$

Proof. By binomial theorem we have

$$\begin{aligned}(x+1)^{2n+1} &= \sum_{\ell=0}^{2n+1}\binom{2n+1}{\ell}x^{\ell} = \sum_{k=0}^{n}\binom{2n+1}{2k}x^{2k} + \sum_{k=0}^{n}\binom{2n+1}{2k+1}x^{2k+1},\\ (x-1)^{2n+1} &= \sum_{\ell=0}^{2n+1}\binom{2n+1}{\ell}(-1)^{2n+1-\ell}\,x^{\ell}\\ &= -\sum_{k=0}^{n}\binom{2n+1}{2k}x^{2k} + \sum_{k=0}^{n}\binom{2n+1}{2k+1}x^{2k+1}.\end{aligned}$$

It follows that

$$\frac{(x+1)^{2n+1}+(x-1)^{2n+1}}{2} = \sum_{k=0}^{n}\binom{2n+1}{2k+1}x^{2k+1} = \sum_{k=0}^{n}\frac{(2n+1)!x^{2k+1}}{(2k+1)!\,(2n-2k)!},$$

which, by putting $x=2$, proves the desired identity. □

Lemma 4. *Let*

$$a_n^* = \sum_{k=0}^{n}\frac{(2k)!}{2^{2k}k!^4\,(2n-2k)!}. \tag{21}$$

Then the following inequalities hold:

$$a_n^* < \frac{1}{6}\frac{3^{2n+1}+1}{(2n+1)!} + \frac{643}{16\,588\,800}\frac{1}{(2n-8)!} \quad \textit{for } n\geq 5, \tag{22}$$

$$a_n^* > \frac{15}{4\pi}\frac{3^{2n+1}+1}{(15n+4)\,(2n)!} \quad \textit{for } n\geq 3. \tag{23}$$

Proof. (i) By Lemma 2 we see that, for $5\leq k\leq n$,

$$s_k = \frac{(2k)!\,(2k+1)!}{2^{4k}k!^4} \leq s_5 = \frac{43\,659}{65\,536} = 0.666\,18... < \frac{2}{3}. \tag{24}$$

Due to

$$\frac{(2k)!}{2^{2k}k!^4\,(2n-2k)!} = \frac{s_k\times 2^{2k}}{(2k+1)!\,(2n-2k)!} < \frac{2}{3}\frac{2^{2k}}{(2k+1)!\,(2n-2k)!}$$

and identity (20), we obtain

$$\begin{aligned}
a_n^* &= \sum_{k=5}^{n}(\cdot)+\sum_{k=0}^{4}(\cdot) \\
&< \frac{2}{3}\sum_{k=5}^{n}\frac{2^{2k}}{(2k+1)!\,(2n-2k)!}+\sum_{k=0}^{4}\frac{s_k\times 2^{2k}}{(2k+1)!\,(2n-2k)!} \\
&= \frac{2}{3}\sum_{k=0}^{n}\frac{2^{2k}}{(2k+1)!\,(2n-2k)!}-\frac{2}{3}\sum_{k=0}^{4}\frac{2^{2k}}{(2k+1)!\,(2n-2k)!} \\
&\quad+\sum_{k=0}^{4}\frac{s_k\times 2^{2k}}{(2k+1)!\,(2n-2k)!}=\frac{1}{6}\frac{3^{2n+1}+1}{(2n+1)!}+\sum_{k=0}^{4}\frac{(s_k-2/3)\,2^{2k}}{(2k+1)!\,(2n-2k)!}.
\end{aligned}$$

If we prove

$$\tau:=\sum_{k=0}^{4}\frac{(s_k-2/3)\,2^{2k}}{(2k+1)!\,(2n-2k)!}<\frac{643}{16\,588\,800}\frac{1}{(2n-8)!},$$

then the first inequality (22) follows. In fact,

$$\begin{aligned}
\tau &= \frac{1}{(2n-8)!}\sum_{k=0}^{4}\frac{(s_k-2/3)\,2^{2k}}{(2k+1)!}\frac{(2n-8)!}{(2n-2k)!} \\
&< \frac{1}{(2n-8)!}\sum_{k=0}^{4}\frac{(s_k-2/3)\,2^{2k}}{(2k+1)!}\frac{(2\times 5-8)!}{(2\times 5-2k)!}=\frac{643}{16\,588\,800}\frac{1}{(2n-8)!},
\end{aligned}$$

which proves (22).

(ii) Using the monotonicity of s_k and inequality (19) gives

$$\begin{aligned}
a_n^* &= \sum_{k=0}^{n}\frac{s_k\times 2^{2k}}{(2k+1)!\,(2n-2k)!}>\sum_{k=0}^{n}\frac{s_n\times 2^{2k}}{(2k+1)!\,(2n-2k)!} \\
&> \frac{2}{\pi}\frac{n+1/2}{n+4/15}\sum_{k=0}^{n}\frac{2^{2k}}{(2k+1)!\,(2n-2k)!}=\frac{2}{\pi}\frac{n+1/2}{n+4/15}\frac{3^{2n+1}+1}{4\,(2n+1)!},
\end{aligned}$$

where the last equality holds due to (20). This proves (23), and the proof of this lemma is done. □

2.2 NP and PN-type polynomials

To prove Theorem 3, we recall the NP or PN-type polynomials (see [**41**]). A polynomial

$$P_n(x)=-\sum_{k=0}^{m}a_kx^k+\sum_{k=m+1}^{n}a_kx^k$$

is called an NP-type polynomial if $a_k \geq 0$ for $0 \leq k \leq n$ and $a_m, a_n > 0$. Correspondingly, $-P_n(x)$ is called a PN-type polynomial. For such polynomials, we have a simple but efficient signs rule (see [**33, 35, 38**]).

Proposition 1. *Let $P_n(x)$ be an NP-type polynomial. There is an $x_0 > 0$ such that $P_n(x) < 0$ for all $x \in (0, x_0)$ and $P(x_0) > 0$ for $x \in (x_0, \infty)$.*

Remark 1. For an NP-type polynomial, it follows from Proposition 1 that if there is an $x_1 > 0$ such that $P_n(x_1) < 0$, then $P_n(x) < 0$ for all $x \in (0, x_1)$; if $P_n(x_1) > 0$, then $P_n(x) > 0$ for all $x \in (x_1, \infty)$.

Remark 2. More general, $P_\infty(x)$ and $-P_\infty(x)$ are called NP and PN-type power series, respectively. Such type power series appear frequently in certain special functions. A similar signs rule was proven in [**36, 39**] and plays a key role in the study of means and special functions, see for example, [**21, 28, 31, 34, 37, 43**].

3. PROOF OF THEOREM 1

Proof of Theorem 1. Substituting $r_0 = 1198/901$ into (15) we have

$$\Phi_{r_0}(t) = \frac{1}{120} \frac{901 \cosh^2 t + 1198 \cosh t + 421}{11 \cosh t + 10}. \tag{25}$$

Using hyperbolic functions representations of $L(a, b)$, $TQ(a, b)$ and $\Phi_r(a, b)$ given in (13), (14) and (25), the double inequality (10) is equivalent to

$$\beta_1 \Phi_{r_0}(t) < \frac{t \times I_0(t)^2}{\sinh t} < \Phi_{r_0}(t) \tag{26}$$

holds for $t > 0$, which is in turn equivalent to

$$\beta_1 f_2(t) < f_1(t) < f_2(t),$$

where

$$\begin{aligned} f_1(t) &= \left(\frac{11}{21} \cosh t + \frac{10}{21}\right) I_0(t)^2, \\ f_2(t) &= \frac{\sinh t}{t} \left(\frac{901}{2520} \cosh^2 t + \frac{1198}{2520} \cosh t + \frac{421}{2520}\right). \end{aligned}$$

Using the representation (17), Cauchy product formula and "product into sum" formulas for the hyperbolic functions, $f_1(t)$ and $f_2(t)$ can be expressed in power series:

$$\begin{aligned} f_1(t) &= \frac{11}{21} \sum_{n=0}^{\infty} \frac{(2n)!}{2^{2n} n!^4} t^{2n} \sum_{n=0}^{\infty} \frac{t^{2n}}{(2n)!} + \frac{10}{21} \sum_{n=0}^{\infty} \frac{(2n)!}{2^{2n} n!^4} t^{2n} \\ &= \sum_{n=0}^{\infty} \left(\frac{11}{21} \sum_{k=0}^{n} \frac{(2k)!}{2^{2k} k!^4 (2n-2k)!} + \frac{10}{21} \frac{(2n)!}{2^{2n} n!^4} \right) t^{2n} := \sum_{n=0}^{\infty} a_n t^{2n}, \end{aligned}$$

where

$$a_n = \frac{11}{21}a_n^* + \frac{10}{21}\frac{(2n)!}{2^{2n}n!^4} \quad \text{with} \quad a_n^* = \sum_{k=0}^{n}\frac{(2k)!}{2^{2k}k!^4\,(2n-2k)!}; \tag{27}$$

$$\begin{aligned} f_2(t) &= \frac{901\sinh 3t + 2396\sinh 2t + 2585\sinh t}{10\,080t} \\ &= \sum_{n=0}^{\infty}\frac{901\times 3^{2n+1} + 599\times 2^{2n+3} + 2585}{10080\,(2n+1)!}t^{2n} := \sum_{n=0}^{\infty} b_n t^{2n}, \end{aligned}$$

where

$$b_n = \frac{1}{10080}\frac{901\times 3^{2n+1} + 599\times 2^{2n+3} + 2585}{(2n+1)!}. \tag{28}$$

If we prove $c_n = a_n - b_n \le 0$ and $d_n = a_n - \beta_1 b_n > 0$ for all $n \ge 0$, then the desired inequalities follow.

(i) We now prove $c_n = a_n - b_n \le 0$ for all $n \ge 0$. A direct verification gives $c_n = 0$ for $0 \le n \le 3$ and

$$c_4 = -\frac{241}{18\,063\,360}, \qquad c_5 = -\frac{14\,137}{4470\,681\,600},$$

$$c_6 = -\frac{3569\,057}{11\,158\,821\,273\,600}, \quad c_7 = -\frac{22\,652\,689}{1171\,676\,233\,728\,000}.$$

For $n \ge 8$, since $s_n < 2/3$, we have

$$\frac{(2n)!}{2^{2n}n!^4} = s_n\frac{2^{2n}}{(2n+1)!} < \frac{2}{3}\frac{2^{2n}}{(2n+1)!}.$$

This in combination with the inequality (22) yields

$$\begin{aligned} a_n &= \frac{11}{21}a_n^* + \frac{10}{21}\frac{(2n)!}{2^{2n}n!^4} \\ &< \frac{11}{21}\frac{1}{6}\frac{3^{2n+1}+1}{(2n+1)!} + \frac{11}{21}\frac{643}{16\,588\,800}\frac{1}{(2n-8)!} + \frac{10}{21}\frac{2}{3}\frac{2^{2n}}{(2n+1)!} \\ &= \frac{1}{126}\frac{40\times 2^{2n} + 33\times 3^{2n} + 11}{(2n+1)!} + \frac{7073}{348\,364\,800}\frac{1}{(2n-8)!}. \end{aligned}$$

Then

$$\begin{aligned} c_n &= a_n - b_n < \frac{1}{126}\frac{40\times 2^{2n} + 33\times 3^{2n} + 11}{(2n+1)!} + \frac{7073}{348\,364\,800}\frac{1}{(2n-8)!} \\ &\quad -\frac{1}{10080}\frac{901\times 3^{2n+1} + 599\times 2^{2n+3} + 2585}{(2n+1)!} = -\frac{c_n'}{10\,080\,(2n+1)!}, \end{aligned}$$

where

$$c'_n = 1592 \times 2^{2n} + 63 \times 3^{2n} + 1705 - \frac{7073}{34\,560} \frac{(2n+1)!}{(2n-8)!}.$$

It is easy to check that c'_n satisfies the recurrence relation

$$c'_{n+1} - \frac{(2n+3)(2n+2)}{(2n-6)(2n-7)} c'_n = \frac{6c''_n}{(n-1)(n-3)(2n-7)}, \tag{29}$$

where

$$c''_n = 21 \frac{8n^2 - 61n + 93}{n-1} \times 3^{2n} + 796 \frac{2n^2 - 19n + 27}{n-1} \times 2^{2n} - 5115.$$

Since the coefficients of 3^{2n} and 2^{2n} are both increasing in n for $n \geq 8$, so is c''_n, and therefore, we obtain

$$c''_n \geq c''_8 = \frac{105\,922\,257\,660}{7} > 0 \text{ for } n \geq 8.$$

This together with the relation (29) and $c'_8 = 1010\,859\,825 > 0$ yields $c'_n > 0$ for $n \geq 8$, which implies $c_n < 0$ for $n \geq 8$. This proves $c_n = a_n - b_n \leq 0$ for all $n \geq 0$.

(ii) Second, we show $d_n = a_n - \beta_1 b_n > 0$ for all $n \geq 0$. An easy check gives

$$d_0 = 1 - \frac{2640}{901\pi} > 0, \quad d_1 = \frac{16}{21} - \frac{14\,080}{6307\pi} > 0, \quad d_2 = \frac{71}{288} - \frac{3905}{5406\pi} > 0.$$

For $n \geq 3$, using the inequality (19) we have

$$\frac{(2n)!}{2^{2n} n!^4} = s_n \frac{2^{2n}}{(2n+1)!} > \frac{15}{\pi} \frac{2n+1}{15n+4} \frac{2^{2n}}{(2n+1)!} = \frac{15}{\pi} \frac{2^{2n}}{(15n+4)(2n)!}.$$

This fact together with the inequality (23) gives

$$\begin{aligned} a_n &= \frac{11}{21} a^*_n + \frac{10}{21} \frac{(2n)!}{2^{2n} n!^4} \\ &> \frac{11}{21} \frac{15}{4\pi} \frac{3^{2n+1}+1}{(15n+4)(2n)!} + \frac{10}{21} \frac{15}{\pi} \frac{2^{2n}}{(15n+4)(2n)!} \\ &= \frac{5}{28\pi} \frac{33 \times 3^{2n} + 40 \times 2^{2n} + 11}{(2n)!\,(15n+4)}. \end{aligned}$$

Then

$$\begin{aligned} d_n &= a_n - \beta_1 b_n > \frac{5}{28\pi} \frac{33 \times 3^{2n} + 40 \times 2^{2n} + 11}{(2n)!\,(15n+4)} \\ &\quad - \frac{2640}{901\pi} \frac{901 \times 3^{2n+1} + 599 \times 2^{2n+3} + 2585}{10080\,(2n+1)!} \\ &= \frac{1}{75\,684\pi} \frac{d'_n}{(15n+4)(2n+1)!}, \end{aligned}$$

where

$$d_n' = 69\,377 \times 3^{2n+1} - (62\,520n - 14\,863) \times 2^{2n+3} - 55\,(10\,104n + 1433)\,.$$

It is readily checked that d_n' has the recurrence relation

$$d_{n+1}' - 9d_n' = 10\,(62\,520n - 64\,879) \times 2^{2n+2} + 880\,(5052n + 85) > 0$$

for $n \geq 3$. This in conjunction with $d_3' = 61\,560\,660 > 0$ yields $d_n' > 0$ for $n \geq 3$, and therefore, so is d_n. Then $d_n = a_n - \beta_1 b_n > 0$ for all $n \geq 0$.

Since

$$\begin{aligned}\lim_{t\to 0} \frac{f_1(t)}{f_2(t)} &= \lim_{t\to 0} \frac{t \times I_0(t)^2}{\Phi_{r_0}(t)\sinh t} = 1,\\ \lim_{t\to \infty} \frac{f_1(t)}{f_2(t)} &= \lim_{t\to \infty} \frac{t \times I_0(t)^2}{\Phi_{r_0}(t)\sinh t} = \frac{2640}{901\pi},\end{aligned}$$

where the second limit is valid due to the asymptotic formula

$$I_0(t) \sim \frac{e^t}{\sqrt{2\pi t}} \quad \text{as } t \to \infty$$

(see [**1**]). Consequently, the inequalities proven are sharp, thereby completing the proof. □

4. PROOF OF THEOREM 2

Proof of Theorem 2. Using hyperbolic functions representations of $E(a,b)$ and $\Phi_r(a,b)$ given in (13) and (25), the double inequality (11) is equivalent to

$$\beta_2 \cosh^{3/2}\left(\frac{2t}{3}\right) < \Phi_{r_0}(t) < \cosh^{3/2}\left(\frac{2t}{3}\right)$$

holds for $t > 0$. Let

$$\begin{aligned} g(t) &= \ln \Phi_{r_0}(t) - \frac{3}{2}\ln\left(\cosh\frac{2t}{3}\right) = \ln \frac{901\cosh^2 t + 1198\cosh t + 421}{120}\\ &\quad - \ln(11\cosh t + 10) - \frac{3}{2}\ln\left(\cosh\frac{2t}{3}\right).\end{aligned}$$

Differentiation yields

$$g'(t) = \frac{2\,(901\cosh t + 599)\sinh t}{901\cosh^2 t + 1198\cosh t + 421} - \frac{11\sinh t}{11\cosh t + 10} - \frac{\sinh(2t/3)}{\cosh(2t/3)},$$

which, by factoring, can be written as

$$\begin{aligned} & (11\cosh t+10)\left(901\cosh^2 t+1198\cosh t+421\right)\frac{\cosh(2t/3)}{\sinh(t/3)}\times g'(t) \\ = & \left(9911\cosh^2 t+18\,020\cosh t+7349\right)\frac{\sinh t}{\sinh(t/3)}\cosh\frac{2t}{3} \\ & -(11\cosh t+10)\left(901\cosh^2 t+1198\cosh t+421\right)\frac{\sinh(2t/3)}{\sinh(t/3)} := g_1(t/3). \end{aligned}$$

Making changes of variables $t=3s$, $\cosh s=x$ and using the following formulas

$$\begin{aligned} \frac{\sinh(2t/3)}{\sinh(t/3)} &= \frac{\sinh(2s)}{\sinh s}=2\cosh s=2x, \\ \frac{\sinh t}{\sinh(t/3)} &= \frac{\sinh(3s)}{\sinh s}=4\cosh^2 s-1=4x^2-1, \\ \cosh\frac{2t}{3} &= \cosh(2s)=2\cosh^2 s-1=2x^2-1, \\ \cosh t &= \cosh(3s)=4\cosh^3 s-3\cosh s=4x^3-3x, \end{aligned}$$

then factoring, we obtain

$$\begin{aligned} g_1(s) = & -(x-1)\left(133\,376x^6-25\,200x^5-225\,264x^4\right. \\ & \left.+86\,696x^3+89\,640x^2-55\,131x+7349\right), \end{aligned}$$

which can be written as

$$\begin{aligned} g_1(s) = & -(x-1)\left[133\,376(x-1)^6+775\,056(x-1)^5+1649\,376(x-1)^4\right. \\ & \left.+1601\,160(x-1)^3+746\,784(x-1)^2+157\,437(x-1)+11\,466\right]<0 \end{aligned}$$

for $x=\cosh s>1$ with $s=t/3>0$. Therefore, $g'(t)<0$ for $t>0$.

A simple computation yields

$$\begin{aligned} \lim_{t\to 0}\exp g(t) &= \lim_{t\to 0}\frac{\Phi_{r_0}(t)}{\cosh^{3/2}(2t/3)}=1, \\ \lim_{t\to\infty}\exp g(t) &= \lim_{t\to\infty}\frac{\Phi_{r_0}(t)}{\cosh^{3/2}(2t/3)}=\frac{901}{330\sqrt{8}}. \end{aligned}$$

By the monotonicity of $g(t)$, the desired double inequality follows.

Final, we show that the constant $2/3$ is the best. Since

$$\lim_{t\to 0}\frac{\Phi_{r_0}(t)-\cosh^{1/q}(qt)}{t^2}=-\frac{1}{2}\left(q-\frac{2}{3}\right),$$

the necessary condition for the inequality $\Phi_{r_0}(a,b)<A_q(a,b)$ to hold is $q\geq 2/3$. Taking into account the increasing property of $q\mapsto\cosh^{1/q}(qt)$, we see that $q=2/3$ is the best constant. This completes the proof. $\square$

5. PROOF OF THEOREM 3

Before proving Theorem 3, we observe the monotonicity of the ratio

$$H_p(t) = \frac{\exp(t\coth t - 1)}{\Psi_p(t)} = 45\frac{[p\cosh t + (1-p)]\exp(t\coth t - 1)}{(30p+1)\cosh^2 t + (28-15p)\cosh t + (16-15p)}.$$

Theorem 4. *For $p > 0$, let $\Psi_p(t)$ be defined by (16). (i) If $p \geq 22/63$, then the function $H_p(t)$ is strictly increasing from on $(0,\infty)$ onto $(1,\lambda_p)$, where $\lambda_p = 90pe^{-1}/(30p+1)$. Therefore, the double inequality*

$$\Psi_p(t) < \exp\left(\frac{t}{\tanh t} - 1\right) < \lambda_p\Psi_p(t) \tag{30}$$

holds for $t > 0$.

(ii) If $0 < p \leq \left(\sqrt{61}+1\right)/30$, then $H_p(t)$ is strictly decreasing from on $(0,\infty)$ onto $(\lambda_p, 1)$. Thus the double inequality

$$\lambda_p\Psi_p(t) < \exp\left(\frac{t}{\tanh t} - 1\right) < \Psi_p(t) \tag{31}$$

holds for $t > 0$.

(iii) If $\left(\sqrt{61}+1\right)/30 < p < 22/63$, then there is a $t_0 > 0$ such that the $H_p(t)$ is strictly decreasing on $(0,t_0)$ and increasing on (t_0,∞). Consequently, the inequality

$$\exp\left(\frac{t}{\tanh t} - 1\right) < \max\{1,\lambda_p\}\Psi_p(t) \tag{32}$$

holds for $t > 0$. In particular, if $\lambda_p \leq 1$, that is, $\left(\sqrt{61}+1\right)/30 < p \leq 1/\left(90e^{-1}-30\right) = p_0^*$, then we have

$$\exp\left(\frac{t}{\tanh t} - 1\right) < \Psi_p(t) \tag{33}$$

for $t > 0$.

Proof. Let

$$\begin{aligned} h_1(t) &= \ln H_p(t) = \frac{t\cosh t}{\sinh t} - 1 + \ln 45 + \ln(p\cosh t + 1 - p) \\ &\quad - \ln\left[(30p+1)\cosh^2 t + (28-15p)\cosh t + 16 - 15p\right]. \end{aligned}$$

Then

$$\lim_{t\to 0} h_1(t) = 0 \quad \text{and} \quad \lim_{t\to\infty} h_1(t) = \ln\frac{90p}{30p+1} - 1 = \ln\lambda_p.$$

Differentiation yields

$$\begin{aligned} h_1'(t) &= -\frac{t}{\sinh^2 t} + \frac{\cosh t}{\sinh t} + p\frac{\sinh t}{p\cosh t - p + 1} \\ &\quad -\frac{[2(30p+1)\cosh t + (28-15p)]\sinh t}{(30p+1)\cosh^2 t + (28-15p)\cosh t + 16 - 15p} \\ &= \frac{h_3(\cosh t)\sinh t - t}{\sinh^2 t} := \frac{h_2(t)}{\sinh^2 t}, \qquad (34) \end{aligned}$$

where

$$h_3(x) = x + \frac{p\left(x^2-1\right)}{px-p+1} - \frac{[2(30p+1)x + (28-15p)]\left(x^2-1\right)}{(30p+1)x^2 + (28-15p)x + 16 - 15p};$$

$$h_2'(t) = h_3'(\cosh t)\sinh^2 t + h_3(\cosh t)\cosh t - 1 = h_4(\cosh t), \qquad (35)$$

where

$$h_4(x) = \left(x^2-1\right)h_3'(x) + xh_3(x) - 1.$$

Direct computation and simplification give

$$h_4(x) = \frac{(x-1)^3(x+1)\times h_5(x)}{(px-p+1)^2\left[(30p+1)x^2 + (28-15p)x + 16 - 15p\right]^2}, \qquad (36)$$

where

$$\begin{aligned} h_5(x) &= p(30p+1)\left(15p^2-p-1\right)x^3 \\ &\quad -\left(1350p^4 - 1710p^3 + 57p^2 + 118p + 2\right)x^2 \\ &\quad +\left(1350p^4 - 3375p^3 + 2367p^2 - 226p - 88\right)x \\ &\quad -\left(450p^4 - 1680p^3 + 2279p^2 - 1290p + 240\right). \end{aligned}$$

To treat easily the sign of $h_5(x)$ on $(1,\infty)$, we make a change of variable $x = y+1$, where $y > 0$, to obtain

$$\begin{aligned} h_5(y+1) &= \left(450p^4 - 15p^3 - 31p^2 - p\right)y^3 + \left(1665p^3 - 150p^2 - 121p - 2\right)y^2 \\ &\quad +\left(2160p^2 - 465p - 92\right)y + 15(63p - 22) \\ &: = c_3(p)y^3 + c_2(p)y^2 + c_1(p)y + c_0(p). \end{aligned}$$

Clearly, every coefficient $c_i(p)$ $(i = 3,2,1,0)$ is an NP-type polynomial. By Proposition 1 there is a unique $p_i > 0$ such that $c_i(p) < 0$ for $p \in (0, p_i)$ and $c_i(p) > 0$ for $p \in (p_i, \infty)$, $i = 3,2,1,0$. Since

$$c_3(p) = p(30p+1)\left(15p^2 - p - 1\right),$$

we see that

$$p_3 = \frac{1+\sqrt{61}}{30} = 0.293...;$$

due to

$$c_2\left(\frac{8}{25}\right)=-\frac{4754}{3125}<0 \text{ and } c_2\left(\frac{1}{3}\right)=\frac{8}{3}>0,$$

we see that $p_2\in(0.325,0.333)$. Also, we have

$$p_1=\frac{1+\sqrt{112\,345}}{1440}=0.340... \quad\text{ and }\quad p_0=\frac{22}{63}=0.349....$$

Obviously, $p_3<p_2<p_1<p_0$. We have thus the changes pattern of signs of coefficients $c_i(p)$:

p	$c_0(p)$	$c_1(p)$	$c_2(p)$	$c_0(p)$
$(0,p_3)$	$-$	$-$	$-$	$-$
(p_3,p_2)	$-$	$-$	$-$	$+$
(p_2,p_1)	$-$	$-$	$+$	$+$
(p_1,p_0)	$-$	$+$	$+$	$+$
(p_0,∞)	$+$	$+$	$+$	$+$

From the above table, we find that: (i) $h_5(y+1)>0$ if $p\in[p_0,\infty)$; (ii) $h_5(y+1)<0$ if $p\in(0,p_3]$; (iii) if $p\in(p_3,p_0)$ then $h_5(y+1)$ is an NP-type polynomial of y, and by Proposition 1 there is a $y_0>0$ such that $h_5(y+1)<0$ for $y\in(0,y_0)$ and $h_5(y+1)>0$ for $y\in(y_0,\infty)$. Next we prove the desired result by distinguishing three cases.

Case 1: $p\in[p_0,\infty)$. Since $h_5(y+1)>0$ for $y>0$, so is $h_4(x)$ for $x>1$, and then $h_2'(t)>0$ for $t>0$ by (35), which implies that $h_2(t)>h_2(0)=0$. This, due to (34), indicates that $h_1'(t)>0$ for $t>0$, that is, $H_p(t)$ is strictly increasing on $(0,\infty)$. It follows that

$$0=\lim_{t\to0}\ln H_p(t)<\ln H_p(t)<\lim_{t\to\infty}\ln H_p(t)=\ln\lambda_p,$$

which implies the double inequality (30).

Case 2: $p\in(0,p_3]$. In the same way, $H_p(t)$ is strictly decreasing on $(0,\infty)$ and the double inequality (31) follows.

Case 3: $p\in(p_3,p_0)$. We see that there is an $x_0=y_0+1>1$ such that $h_5(x)<0$ for $x\in(1,x_0)$ and $h_5(x)>0$ for $x\in(x_0,\infty)$, so is $h_4(x)$ by the relation (36). It follows from (35) that $h_2(t)$ is decreasing on $(0,t_0)$ and increasing on (t_0,∞), where $t_0=\operatorname{arcosh}x_0$. Since $h_2(0)=0$,

$$\begin{aligned}\lim_{x\to\infty}h_3(x) &= \lim_{x\to\infty}\left[x+\frac{p\left(x^2-1\right)}{px+1-p}\right.\\ &\qquad\left.-\frac{\left[2(30p+1)x+(28-15p)\right]\left(x^2-1\right)}{(30p+1)x^2+(28-15p)x+16-15p}\right]\\ &= \frac{15p^2-p-1}{p(30p+1)},\end{aligned}$$

$$\begin{aligned}\lim_{t\to\infty} h_2(t) &= \lim_{t\to\infty}\left[h_3(\cosh t)-\frac{t}{\sinh t}\right]\lim_{t\to\infty}\sinh t\\ &= \operatorname{sgn}\left[\frac{15p^2-p-1}{p(30p+1)}\right]\infty=\infty,\end{aligned}$$

therefore, there is a $t_1 \in (t_0,\infty)$ such that $h_2(t)<0$ for $t\in(0,t_1)$ and $h_2(t)>0$ for $t\in(t_1,\infty)$. This, by (34), implies that $h_1(t)$ is decreasing on $(0,t_1)$ and increasing on (t_1,∞), then so is $H_p(t)$. It then follows that

$$H_p(t)<\max\{1,\lambda_p\},$$

which implies (32). Evidently, the inequality (33) holds for $t>0$ when $\max\{1,\lambda_p\}=1$, that is, $p_3<p\le 1/\left(90e^{-1}-30\right)=p_0^*$.

This completes the proof. □

We are now in a position to prove Theorem 3.

Prove Theorem 3. (i) We first prove the second and third inequalities in (12) hold with the best constants p_0 and p_0^*. They are clearly equivalent to

$$\Psi_{p_0}(t)<\exp\left(\frac{t}{\tanh t}-1\right)<\Psi_{p_0^*}(t) \tag{37}$$

for $t>0$ with the best p_0 and p_0^*. The inequality (37) follows from the inequalities (30) and (33). The optimality of p_0 and p_0^* is due to the two facts. The first of which is the monotonicity of $\Psi_p(t)$ with respect to p. In fact, we have

$$\frac{\partial\Psi_p(t)}{\partial p}=-\frac{1}{45}\frac{(\cosh t-1)^3}{(p\cosh t+1-p)^2}<0.$$

The second is the necessary conditions for (37) to hold for all $t>0$, that is,

$$\begin{aligned}\lim_{t\to0}\frac{\exp(t\coth t-1)-\Psi_p(t)}{t^6} &= \frac{1}{360}\left(p-\frac{22}{63}\right)\ge0,\\ \lim_{t\to\infty}\frac{\exp(t\coth t-1)-\Psi_p(t)}{\cosh t} &= \frac{2}{e}-\frac{2}{3}-\frac{1}{45p}\le0,\end{aligned}$$

which give $p\ge p_0$ and $p\le p_0^*$, respectively.

(ii) A direct computation yields

$$\begin{aligned}&\Psi_{p_0^*}(t)-\lambda_{p_0}\Psi_{p_0}(t)=-\left(\frac{660}{241e}-1\right)\\ &\times\frac{(1446-418e)x^2+(2868-1021e)x+1356-451e}{(22x+41)(ex+90-31e)}<0,\end{aligned}$$

where $x=\cosh t>1$, which proves the fourth inequality of (12). Since

$$\lim_{t\to\infty}\frac{\Psi_{p_0^*}(t)}{\Psi_{p_0}(t)}=\frac{660}{241e}=\lambda_{p_0},$$

the constant λ_{p_0} is the best possible.

(iii) Final, we prove the first inequality of (12). Assume that $b > a > 0$ and let $a/b = x^6 \in (0,1)$. Then this inequality is equivalent to

$$j(x) = \left[\frac{1}{15} \frac{241 \left(1+x^6\right)^2 /4 + 478 \left(1+x^6\right) x^3/2 + 226x^6}{22\left(1+x^6\right)/2 + 41x^3} \right]^2 - \left(\frac{1+x^4}{2} \right)^3 > 0$$

for $x \in (0,1)$. Factoring yields

$$j(x) = \frac{1}{3600} \frac{(x-1)^4 \times j_{20}(x)}{\left(11x^6 + 41x^3 + 11\right)^2} > 0,$$

where

$$\begin{aligned} j_{20}(x) \quad = \quad & 3631x^{20} + 14\,524x^{19} + 36\,310x^{18} + 127\,512x^{17} + 183\,303x^{16} \\ & +98\,856x^{15} + 485\,982x^{14} + 738\,792x^{13} + 88\,047x^{12} + 469\,432x^{11} \\ & +1222\,582x^{10} + 469\,432x^{9} + 88\,047x^{8} + 738\,792x^{7} + 485\,982x^{6} \\ & +98\,856x^{5} + 183\,303x^{4} + 127\,512x^{3} + 36\,310x^{2} + 14\,524x + 3631, \end{aligned}$$

which is clearly positive for $x \in (0,1)$.

It remains to show that the constant $2/3$ is the best. Since

$$\lim_{t \to 0} \frac{\cosh^{1/q}(qt) - \Psi_{p_0}(t)}{t^2} = \frac{1}{2}\left(q - \frac{2}{3}\right),$$

the necessary condition for the inequality $A_q(a,b) < \Psi_{p_0}(a,b)$ to hold is $q \le 2/3$. This in conjunction with the increasing property of $q \mapsto \cosh^{1/q}(qt)$ reveals that $q = 2/3$ is the best constant.

We thus complete the proof. □

6. CONCLUSION

In this paper, we established a nice chain of sharp inequalities for some means (7), where the inequalities from the first to fourth are given in Theorems 1 and 2, while ones from the fifth to eighth are given by Theorem 3. Moreover, this chain can be written as

$$\sqrt{\theta_2 L A_{2/3}} \quad < \quad \sqrt{\theta_1 L \Phi_{r_0}} < TQ < \sqrt{L\Phi_{r_0}} < \sqrt{LA_{2/3}}, \tag{38}$$

$$A_{2/3} \quad < \quad \Psi_{p_0} < E < \Psi_{p_0^*} < \lambda_{p_0}\Psi_{p_0}. \tag{39}$$

Clearly, as an upper bound for the Toader-Qi mean, $\sqrt{L\Phi_{r_0}}$ is better than $\sqrt{LA_{2/3}}$, which improves a known result given in [**42**]. As a lower bound for the exponential mean E, Ψ_{p_0} is superior to $A_{2/3}$, which is also a refinement of Stalarsky's inequality in [**16**].

REFERENCES

1. M. Abramowitz, I. A. Stegun: *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, National Bureau of Standards, Applied Mathematics Series 55.* Dover Publications, New York and Washington, 1972.
2. H. Alzer: *Ungleichungen für Mittelwerte.* Arch. Math., **47** (1986), 422–426.
3. H. Alzer: *Two inequalities for means.* C. R. Math. Pep. Acad. Sci. Canada, **9** (1987), 11–16.
4. Á. Baricz: *Bounds for Turánians of modified Bessel functions.* Expo. Math., **33** (2015), 223–251.
5. P. S. Bullen, D. S. Mitrinović, P. M. Vasić: *Means and Their Inequalities.* Dordrecht, 1988.
6. B. C. Carlson: *The logarithmic mean.* Amer. Math. Monthly, **79** (1972), 615–618.
7. T.-P. Lin: *The power mean and the logarithmic mean.* Amer. Math. Monthly, **81**(1974), 879–883.
8. D. S. Mitrinović: *Analytic Inequalities.* Springer, New York, 1970.
9. E. Neuman: *Inequalities involving modified Bessel functions of the first kind.* J. Math. Anal. Appl., **171** (1992), 532–536.
10. E. Neuman, J. Sándor: *On certain means of two arguments and their extensions.* Int. J. Math. Math. Sci., **2003** (2003), 981–993.
11. Zs. Páles: *Inequalities for differences of powers.* J. Math. Anal. Appl., **131** (1988), 271–281.
12. F. Qi, X.-T. Shi, F.-F. Liu, Z.-H. Yang: *A double inequality for an integral mean in terms of the exponential and logarithmic means.* Period Math. Hung., **75** (2017), 180–189.
13. Y.-F. Qiu, M.-K. Wang, Y.-M. Chu, G.-D. Wang: *Two sharp inequalities for Lehmer mean, identric mean and logarithmic mean.* J. Math. Inequal., **5** (2011), 301–306.
14. J. Sándor: *On the identric and logarithmic means.* Aequat. Math., **40** (1990), 261–270.
15. J. Sándor: *On certain identities for means.* Studia Univ. Babes-Bolyai Math., **38** (1993), 7–14.
16. K. B. Stolarsky: *Generalizations of the Logarithmic Mean.* Math. Mag., **48** (1975), 87–92.
17. J.-F. Tian, Z. Yang: *Asymptotic expansions of Gurland's ratio and sharp bounds for their remainders.* J. Math. Anal. Appl., **493** (2021), 124545.
18. Gh. Toader: *Some mean values related to the arithmetic-geometric mean.* J. Math. Anal. Appl., **218**(1998), 358–368.

19. G. N. Watson: *A Treatise on the Theory of Bessel Functions.* Cambridge University Press, Cambridge, 1922.

20. W.-F. Xia, Y.-M. Chu, G.-D. Wang: *The optimal upper and lower power mean bounds for a convex combination of the arithmetic and logarithmic means.* Abstr. Appl. Anal., **2010** (2010), Art.ID 604804.

21. Z. Yang, J.-F. Tian: *Monotonicity rules for the ratio of two Laplace transforms with applications.* J. Math. Anal. Appl., **470** (2019), 821–845.

22. Z.-H. Yang: *On the homogeneous functions with two parameters and its monotonicity.* J. Inequal. Pure Appl. Math., **6** (2005), Art. 101.

23. Z.-H. Yang: *On the log-convexity of two-parameter homogeneous functions.* Math. Inequal. Appl., **10** (2007), 499–516.

24. Z.-H. Yang: *On the monotonicity and log-convexity of a four-parameter homogeneous mean.* J. Inequal. Appl., **2008** (2008), Art. ID 149286.

25. Z.-H. Yang: *New sharp bounds for logarithmic mean and identric mean.* J. Inequal. Appl., **2013** (2013), 116.

26. Z.-H. Yang: *On converses of some comparison inequalities for homogeneous means.* Hacet. J. Math. Stat., **46** (2017), 629–644.

27. Z.-H. Yang: *New sharp bounds for identric mean in terms of logarithmic mean and arithmetic mean.* J. Math. Inequal., **6** (2012), 533–543.

28. Z.-H. Yang: *Sharp approximations for the complete elliptic integrals of the second kind by one-parameter means.* J. Math. Anal. Appl., **467** (2018), 446–461.

29. Z.-H. Yang, Y.-M. Chu: *An optimal inequalities chain for bivariate means.* J. Math. Inequal., **9** (2015), 331–343.

30. Z.-H. Yang, Y.-M. Chu: *On approximating the modified Bessel function of the first kind and Toader-Qi mean.* J.Inequal. Appl., **2016** (2016), 40.

31. Z.-H. Yang, Y.-M. Chu: *Inequalities for certain means in two arguments.* J. Inequal. Appl., **2015** (2015), 299.

32. Z.-H. Yang, Y.-M. Chu, Y.-Q. Song: *Sharp bounds for Toader-Qi mean in terms of logarithmic and identric means.* Math. Inequal. Appl., **19** (2016), 721–730.

33. Z.-H. Yang, Y.-M. Chu, X.-J, Tao: *A double inequality for the trigamma function and its applications.* Abstr. Appl. Anal., **2014** (2014), Art. ID 702718.

34. Z.-H. Yang, Y.-M. Chu, X.-H. Zhang: *Necessary and sufficient conditions for functions involving the psi function to be completely monotonic.* J. Inequal. Appl., **2015** (2015), 157.

35. Z.-H. Yang, W.-M. Qian, Y.-M. Chu, W. Zhang: *On rational bounds for the gamma function.* J. Inequal. Appl., **2017** (2017), 210.

36. Z.-H. Yang, W.-M. Qian, Y.-M. Chu, W. Zhang: *On approximating the arithmetic-geometric mean and complete elliptic integral of the first kind.* J. Math. Anal. Appl., **462** (2018), 1714–1726.

37. Z.-H. Yang, W.-M. Qian, W. Zhang, Y.-M. Chu: *Notes on the complete elliptic integral of the first kind.* Math. Inequal. Appl., **23** (2020), 77–93.

38. Z.-H. Yang, J. Tian: *Monotonicity and sharp inequalities related to gamma function.* J. Math. Inequal., **12** (2018), 1–22.

39. Z.-H. Yang, J. Tian: *Convesity and monotonicity for elliptic integrals of the first kind and applications.* Appl. Anal. Discrete Math., **13** (2019), 240–260.

40. Z.-H. Yang, J.-F. Tian: *Sharp bounds for the ratio of two zeta functions.* J. Comput. Appl. Math., **364** (2020), 112359.(

41. Z.-H. Yang, J.-F. Tian, M.-K. Wang: *A positive answer to Bhatia–Li conjecture on the monotonicity for a new mean in its parameter.* Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM, **114** (2020), 126.

42. Z.-H. Yang, J.-F. Tian, Y.-R. Zhu: *New sharp bounds for the modified Bessel function of the first kind and Toader-Qi mean.* Math., **8** (2020), 901.

43. Z.-H. Yang, J.-F. Tian, Y.-R. Zhu: *A rational approximation for the complete elliptic integral of the first kind.* Math., **8** (2020), 635.

44. Z.-H. Yang, S.-Z. Zheng: *Monotonicity of the ratio of modified Bessel functions of the first kind with applications.* J. Inequal. Appl., **2018** (2018), 57.

Zhen-Hang Yang
Engineering Research Center of Intelligent Computing
for Complex Energy Systems of Ministry of Education,
North China Electric Power University,
Baoding, Hebei, 071003, P.R. China,
Department of Science and Technology Information,
Zhejiang Electric Power Company Research Institute,
Hangzhou, Zhejiang, 310014, P.R. China
E-mail: *yzhkm@163.com*

(Received 27. 12. 2020.)
(Revised 16. 09. 2021.)

Jing-Feng Tian
Department of Mathematics and Physics,
North China Electric Power University,
Baoding, Hebei, 071003, P.R. China
E-mail: *tianjf@ncepu.edu.cn*