Applicable Analysis and Discrete Mathematics
available online at http://pefmath.etf.rs

Appl. Anal. Discrete Math. **15** (2021), 486–509.
https://doi.org/10.2298/AADM210511032M

A MINIMAX APPROXIMANT IN THE THEORY OF ANALYTIC INEQUALITIES

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The aim of this paper is to examine the families of monotonically stratified functions with respect to one parameter and the connections of such families of functions with certain results stemming from the Theory of Analytic Inequalities. The obtained results are applied to the Cusa-Huygens inequality and some related inequalities.

1. INTRODUCTION

In the Theory of Analytic Inequalities, numerous analytic inequalities with one parameter on a given interval are considered. Let

$$\varphi_p : (a,b) \longrightarrow \mathbb{R},$$

be a family of functions defined on the interval (a,b), for a parameter $p \in \mathbb{D}$ ($\mathbb{D} \subseteq \mathbb{R}$). These functions may also be considered as the functions with two variables: x and p. Let us consider the problem of determining the maximal subset $I \subseteq \mathbb{D}$ such that for $p \in I$

$$\varphi_p(x) > 0$$

holds for all $x \in (a,b)$ (if such subset exists). Similarly, we consider the problem of determining the maximal subset $J \subseteq \mathbb{D}$ such that for $p \in J$

$$\varphi_p(x) < 0$$

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2020 Mathematics Subject Classification. Primary 33B10; Secondary 26D05
Keywords and Phrases: Exponential Inequalities, Sinc Function

holds for all $x \in (a,b)$ (if such subset exists). In this paper, assume that $\mathbb{D} = \mathbb{R}^+$, the subsets I and J are non-empty and

$$I \cup J \subsetneqq \mathbb{R}^+.$$

In the examples we have considered, we examine the sign of the function $\varphi_p(x)$, for all $x \in (a,b)$ and all $p \in \mathbb{R}^+ \backslash (I \cup J)$. Additionally, we prove that there exists the unique value p_0 of the parameter $p \in \mathbb{R}^+$, for which the infimum of the error

$$d^{(p)} = \sup_{x \in (a,b)} |\varphi_p(x)| \tag{1}$$

is attained. For such a value p_0, the function $\varphi_{p_0}(x)$ is called *the minimax approximant* on (a,b).

The importance of introducing the minimax approximants in specific cases is to extend the existing inequalities to a wider range of parameters. The obtained results may be of importance for certain applications of the Theory of Analytic Inequalities [**16**], [**28**], [**29**]. Let us notice that the minimax approximant can be defined for some other examples of metrics, too.

2. MAIN RESULTS

In this section, we provide assertions by means of which we prove the existence of minimax approximants for one particular class of the families of functions. Also, we provide auxiliary theorems on the basis of which we prove, in the next section, that three specific families of functions belong to this class.

For functions $\varphi_p(x)$, where $x \in (a,b)$, $p \in \mathbb{R}^+$ we say that they are *increasingly stratified functions* if $(\forall p_1, p_2 \in \mathbb{R}^+)\, p_1 < p_2 \iff \varphi_{p_1}(x) < \varphi_{p_2}(x)$ holds for all $x \in (a,b)$, and, conversely *decreasingly stratified functions* if $(\forall p_1, p_2 \in \mathbb{R}^+)\, p_1 < p_2 \iff \varphi_{p_1}(x) > \varphi_{p_2}(x)$ holds for all $x \in (a,b)$.

Theorem 1. *Let $\varphi_p(x)$ be a family of functions that are continuous with respect to $x \in (a,b)$ for each $p \in \mathbb{R}^+$ and increasingly stratified for $p \in \mathbb{R}^+$, and let c, d be in $\mathbb{R}^+$ such that $c < d$. If*:

(a) *$\varphi_c(x) < 0$ and $\varphi_d(x) > 0$ for all $x \in (a,b)$, and at the endpoints $\varphi_c(a+) = \varphi_c(b-) = \varphi_d(a+) = 0$ and $\varphi_d(b-) \in \mathbb{R}^+$ hold;*

(b) *the functions $\varphi_p(x)$ are continuous with respect to $p \in (c,d)$ for each $x \in (a,b)$ and $\varphi_p(b-)$ is continuous with respect to $p \in (c,d)$ too;*

(c) *for all $p \in (c,d)$, there exists a right neighbourhood of the point a in which $\varphi_p(x) < 0$ holds;*

(d) *for all $p \in (c,d)$ the function $\varphi_p(x)$ has exactly one extremum $t^{(p)}$ on (a,b), which is minimum;*

then there exists exactly one solution p_0, for $p\in\mathbb{R}^+$, of the following equation

$$|\varphi_p(t^{(p)})| = \varphi_p(b-)$$

and for $d_0 = |\varphi_{p_0}(t^{(p_0)})| = \varphi_{p_0}(b-)$ we have

$$d_0 = \inf_{p\in\mathbb{R}^+} \sup_{x\in(a,\,b)} |\varphi_p(x)|\,.$$

Proof. First we prove the existence and the uniqueness of the solution p_0 of $|\varphi_p(t^{(p)})| = \varphi_p(b-)$. Both functions $\varphi_p(b-)$ and $\varphi_p(t^{(p)})$ are continuous with respect to $p\in(c,d)$. From (b) this holds for $\varphi_p(b-)$, and the other function is continuous because of the increasingly stratification of the functions $\varphi_p(x)$: for $p_1,p_2\in(c,d)$ and $p_1>p_2$ it follows that $\varphi_{p_1}(t^{(p_1)})>\varphi_{p_2}(t^{(p_2)})$ and $\varphi_{p_1}(t^{(p_1)})-\varphi_{p_2}(t^{(p_2)}) < \varphi_{p_1}(t^{(p_2)})-\varphi_{p_2}(t^{(p_2)})$ (minimum of φ_{p_1} is attained at $t^{(p_1)}$), therefore

$$|\varphi_{p_1}(t^{(p_1)})-\varphi_{p_2}(t^{(p_2)})| < |\varphi_{p_1}(t^{(p_2)})-\varphi_{p_2}(t^{(p_2)})|$$

and from (b) $\varphi_p(t^{(p_2)})$ is continious with respect to $p\in(c,d)$. Hence, the function $|\varphi_p(t^{(p)})|-\varphi_p(b-)$ is continuous with respect to $p\in(c,d)$, with a positive limit when p approaches $c+$ and a negative one when p approaches $d-$, so there exists a solution $p_0\in(c,d)$.

Let us assume that for $p\in(c,d)$ there exist two different solutions $p'<p''$ of the given equation, that is, $|\varphi_{p'}(t^{(p')})| = \varphi_{p'}(b-)$ and $|\varphi_{p''}(t^{(p'')})| = \varphi_{p''}(b-)$. As $p'<p''$, we have $\varphi_{p'}(b-) < \varphi_{p''}(b-)$, i.e. $|\varphi_{p'}(t^{(p')})| < |\varphi_{p''}(t^{(p'')})|$. By the negativity of the minimum of the functions $\varphi_{p'}$ and $\varphi_{p''}$ it follows that $\varphi_{p'}(t^{(p')}) > \varphi_{p''}(t^{(p'')})$. Further, due to increasing stratification of the functions φ_p, $p'<p''$ implies $\varphi_{p'}(t^{(p'')})<\varphi_{p''}(t^{(p'')})$, while $\varphi_{p'}(t^{(p')})<\varphi_{p'}(t^{(p'')})$ holds because $t^{(p')}$ is the minimum of the function $\varphi_{p'}$. Therefore $\varphi_{p'}(t^{(p')})<\varphi_{p''}(t^{(p'')})$, a contradiction.

Let p_0 be the unique solution of the equation $|\varphi_p(t^{(p)})| = \varphi_p(b-)$ and

$$d_0 = |\varphi_{p_0}(t^{(p_0)})| = \varphi_{p_0}(b-).$$

Let $d^{(p)} = \sup_{x\in(a,\,b)}|\varphi_p(x)|$. For $p\le c$ we have $\varphi_p(x)<0$, so due to the shape and stratification of φ_p it follows that $d^{(p)}\ge|\varphi_p(t^{(c)})|\ge|\varphi_c(t^{(c)})|\ge|\varphi_{p_0}(t^{(p_0)})| = d_0$. For $p\ge d$ the functions $\varphi_p(x)$ are positive and $d^{(p)}\ge|\varphi_p(b-)|\ge\varphi_d(b-)\ge d_0$. If $c\le p\le d$, then let us notice $d^{(p)}=\max\{|\varphi_p(t^{(p)})|,\varphi_p(b-)\}$ and $d^{(p)}\ge d_0$. Namely, if $c\le p<p_0$ then $\varphi_p(t^{(p)})\le\varphi_{p_0}(t^{(p_0)}) \Leftrightarrow |\varphi_p(t^{(p)})|\ge d_0$ and $\varphi_p(b-)\le\varphi_{p_0}(b-)=d_0$, while if $p_0\le p\le d$ then $\varphi_p(t^{(p)})\ge\varphi_{p_0}(t^{(p_0)}) \Leftrightarrow |\varphi_p(t^{(p)})|\le d_0$ and $\varphi_p(b-)\ge\varphi_{p_0}(b-)=d_0$. It follows from the previous discussion that $d^{(p)}\ge d_0$ for all $p>0$, so

$$d_0 = \inf_{p\in\mathbb{R}^+} \sup_{x\in(a,\,b)} |\varphi_p(x)|.$$

□

In order to determine the sign of the function $\varphi_p(x)$ for $x \in (a,b)$ and for all values of the parameter $p \in R^+$, we formulate the following theorem, which is a consequence of the previous one.

Theorem 2. *For functions $\varphi_p(x)$ that satisfy the assumptions of Theorem 1, the following assertions hold true:*

1) *If $p\in(0,c]$ then*
$$x\in(a,b) \implies \varphi_p(x) \le \varphi_c(x) < 0.$$

2) *If $p\in(c,d)$ and the function $\varphi_p(x)$ has exactly one zero $x_0^{(p)}\in(a,b)$, then the following implications are true*
$$x\in(a,x_0^{(p)}) \implies \varphi_p(x) < 0,$$
and
$$x\in(x_0^{(p)},b) \implies \varphi_p(x) > 0.$$

3) *If $p\in[d,\infty)$ then*
$$x\in(a,b) \implies \varphi_p(x) \ge \varphi_d(x) > 0.$$

Analogous theorems can be stated for decreasingly stratified functions.

Theorem 1'. *Let $\varphi_p(x)$ be a family of functions that are continuous with respect to $x\in(a,b)$ for each $p\in\mathbb{R}^+$ and decreasingly stratified for $p\in\mathbb{R}^+$, and let c,d be in $\mathbb{R}^+$ such that $c<d$. If:*

(a) *$\varphi_c(x) > 0$ and $\varphi_d(x) < 0$ for all $x \in (a,b)$, and at the endpoints $\varphi_c(a+) = \varphi_d(b-) = \varphi_d(a+) = 0$ and $\varphi_c(b-)\in\mathbb{R}^+$ hold;*

(b) *the functions $\varphi_p(x)$ are continuous with respect to $p\in(c,d)$ for each $x\in(a,b)$ and $\varphi_p(b-)$ is continuous with respect to $p\in(c,d)$ too;*

(c) *for all $p\in(c,d)$, there exists a right neighbourhood of the point a in which $\varphi_p(x)<0$ holds;*

(d) *for all $p\in(c,d)$ the function $\varphi_p(x)$ has exactly one extremum $t^{(p)}$ on (a,b), which is minimum;*

then there exists exactly one solution p_0, for $p\in\mathbb{R}^+$, of the following equation
$$|\varphi_p(t^{(p)})| = \varphi_p(b-)$$
and for $d_0 = |\varphi_{p_0}(t^{(p_0)})| = \varphi_{p_0}(b-)$ we have
$$d_0 = \inf_{p\in\mathbb{R}^+} \sup_{x\in(a,b)} |\varphi_p(x)|.$$

Theorem 2'. *For functions $\varphi_p(x)$ that satisfy the assumptions of Theorem 1', the following assertions hold true:*

1) *If* $p\in(0,c]$ *then*
$$x\in(a,b) \implies \varphi_p(x) \geq \varphi_c(x) > 0.$$

2) *If* $p\in(c,d)$ *and the function* $\varphi_p(x)$ *has exactly one zero* $x_0^{(p)}\in(a,b)$*, then the following implications are true*
$$x\in(a,x_0^{(p)}) \implies \varphi_p(x) < 0,$$
and
$$x\in(x_0^{(p)},b) \implies \varphi_p(x) > 0.$$

3) *If* $p\in[d,\infty)$ *then*
$$x\in(a,b) \implies \varphi_p(x) \leq \varphi_d(x) < 0.$$

In particular examples, it is proved that the observed classes of functions satisfy the conditions of Theorem 1 or Theorem 1'. The proofs are based on Theorem 2.1 [**27**] which is in this paper called Nike theorem, as well as on one modification of Nike theorem, which we state and prove in the extension.

Theorem 3 (Nike theorem)**.** *Let* $f:(0,c)\longrightarrow\mathbb{R}$ *be* m *times differentiable function (for some* $m\geq 2$, $m\in\mathbb{N}$) *satisfying the following conditions*:

(a) $f^{(m)}(x)>0$ *for* $x\in(0,c)$*;*

(b) *there is a right neighbourhood of zero in which the following inequalities are true:*
$$f<0,\ f'<0,\ \ldots,\ f^{(m-1)}<0;$$

(c) *there is a left neighbourhood of c in which the following inequalities are true:*
$$f>0,\ f'>0,\ \ldots,\ f^{(m-1)}>0.$$

Then the function f *has exactly one zero* $x_0\in(0,c)$*, and* $f(x)<0$ *for* $x\in(0,x_0)$ *and* $f(x)>0$ *for* $x\in(x_0,c)$*. Also, the function* f *has exactly one local minimum on the interval* $(0,c)$*. More precisely, there is exactly one point* $t\in(0,c)$ $\big($*in fact* $t\in(0,x_0)\big)$ *such that* $f(t)<0$ *is the smallest value of the function* f *on the interval* $(0,c)$ *and particularly on* $(0,x_0)$*.*

Remark 1. *For some applications of the Nike theorem in the Analytic Inequality Theory, see* [**27**] *and* [**3**].

In the case when it is not possible to apply the Nike theorem, the following theorem is applied, which gives sufficient conditions that the function on the interval has exactly one zero and exactly one minimum (see section 3.3).

Theorem 4. *Let* $f : (0,c) \longrightarrow \mathbb{R}$ *be* m *times differentiable function* $\big($*for some* $m \geq 2,\ m \in \mathbb{N}$$\big)$ *satisfying the following conditions*:

(a) $f^{(m)}$ *has exactly one zero* x_m *on* $(0,c)$ *such that* $f^{(m)} > 0$ *on* $(0,x_m)$ *and* $f^{(m)} < 0$ *on* (x_m,c)*;*

(b) *there is a right neighbourhood of zero in which the following inequalities are true*

$$f < 0, f' < 0, ..., f^{(m-1)} < 0;$$

(c) *there is a left neighbourhood of* c *in which the following inequalities are true*

$$f > 0, f' > 0, ..., f^{(m-1)} > 0.$$

Then the function f *has exactly one zero* $x_0 \in (0,c)$ *and* $f(x) < 0$ *for* $x \in (0,x_0)$ *and* $f(x) > 0$ *for* $x \in (x_0,c)$*. The function* f *has exactly one minimum on the interval* $(0,c)$*, i.e. there is exactly one point* $t \in (0,c)$ $\big($*in fact* $t \in (0,x_0)$$\big)$ *such that* $f(t) < 0$ *is the smallest value of the function* f *on the interval* $(0,c)$ *and particularly on* $(0,x_0)$*.*

Proof. From (a) it follows that the function $f^{(m-1)}$ on $(0,c)$ has exactly one local maximum at x_m in which $f^{(m-1)}(x_m) > 0$, because $f^{(m-1)}$ decreases over (x_m,c), and due to (c) there is a left neighbourhood of the point c in which $f^{(m-1)} > 0$. Due to (b) there exists a right neighbourhood of zero in which $f^{(m-1)} < 0$ holds on $(0,c)$ and since $f^{(m-1)}$ increases on $(0,x_m)$, it implies that $f^{(m-1)}$ has exactly one zero x_{m-1} on $(0,x_m)$ $\big($as well as on $(0,c)\big)$ and that $f^{(m-1)} < 0$ on $(0,x_{m-1})$ and $f^{(m-1)} > 0$ on (x_{m-1},c).
From the above, on $(0,c)$ the function $f^{(m-2)}$ has exactly one local minimum at the point $x_{m-1} \in (0,c)$. Since $f^{(m-2)}$ decreases on $(0,x_{m-1})$, from (b) it follows that $f^{(m-2)}(x_{m-1}) < 0$, and since $f^{(m-2)}$ increases on (x_{m-1},c), from (c) it follows that $f^{(m-2)}$ has exactly one zero x_{m-2} on (x_{m-1},c) (and thereby on $(0,c)$) and that $f^{(m-2)} < 0$ on $(0,x_{m-2})$ and $f^{(m-2)}(x) > 0$ on (x_{m-2},c).
For $m \geq 3$, analogous to the proof for $f^{(m-2)}$, it is concluded that on $(0,c)$ function $f^{(m-3)}$ has exactly one local minimum at x_{m-2} in which $f^{(m-3)}(x_{m-2}) < 0$ and exactly one zero x_{m-3} on (x_{m-2},c) (as well as on $(0,c)$) and that $f^{(m-3)} < 0$ on $(0,x_{m-3})$ and $f^{(m-3)} > 0$ on (x_{m-3},c), etc. Therefore, the function f has exactly one local minimum at $t = x_1$ on the interval $(0,c)$ in which $f(t) < 0$ and exactly one zero x_0 on (t,c) (as well as on $(0,c)$) and that $f < 0$ on $(0,x_0)$ and $f > 0$ on (x_0,c). □

Remark 2. *Let us notice that Theorem* 3*, as well as Theorem* 4*, claim that the local minimum at* t *is the only extremum of the function* f *on* $(0,c)$*, which is shown in their proofs.*

3. EXAMPLES OF DETERMINING THE MINIMAX APPROXIMANT

3.1. The case of Cusa-Huygens inequality

The subject of this subsection is the family of functions:

$$\varphi_p(x) = x - \frac{(p+1)\sin x}{p+\cos x}$$

which are continuous on $\left(0, \frac{\pi}{2}\right)$ for each $p \in \mathbb{R}^+$ and increasingly stratified for $p \in \mathbb{R}^+$ [**8**], too. Furthermore, let

$$A = 2/(\pi-2) = 1.75193\ldots \quad \text{and} \quad B = 2.$$

Obviously, $\varphi_A(0+) = \varphi_A(\pi/2-) = \varphi_B(0+) = 0$ and $\varphi_B(\pi/2-) \in \mathbb{R}^+$. The functions $\varphi_p(x)$ are continuous with respect to $p \in (A, B)$ for each $x \in \left(0, \frac{\pi}{2}\right)$ and $\varphi_p(\pi/2-)$ is continuous with respect to $p \in (A, B)$. Based on the order in the family of functions $\varphi_p(x)$ with respect to p, the following statements hold.

Statement 1. *If* $p \in (0, A] = (0, 1.75193\ldots]$ *then* :

$$\varphi_p(x) \leq \varphi_A(x) = x - \frac{\frac{\pi}{\pi-2}\sin x}{\frac{2}{\pi-2} + \cos x} < 0,$$

for $x \in \left(0, \frac{\pi}{2}\right)$.

Remark 3. *The inequality* $\varphi_A(x) < 0$, $x \in (0, \pi/2)$, *can be written as*:

$$x < \frac{\frac{\pi}{\pi-2}\sin x}{\frac{2}{\pi-2} + \cos x} \qquad \big(\text{for } x \in (0, \pi/2)\big),$$

which has been proved in [**25**].

Statement 2. *If* $p \in [B, \infty)$ *then* :

$$\varphi_p(x) \geq \varphi_B(x) = x - \frac{3\sin x}{2+\cos x} > 0,$$

for $x \in \left(0, \frac{\pi}{2}\right)$.

Remark 4. *The inequality* $\varphi_B(x) > 0$, $x \in (0, \pi/2)$, *is in fact the well-known Cusa-Huygens inequality*:

$$x > \frac{3\sin x}{2+\cos x} \qquad \big(\text{for } x \in (0, \pi/2)\big).$$

Let us emphasize that some of the analytic inequalities such as the Cusa-Huygens type inequalities for the arcsine function

$$\frac{3\,t}{2+\sqrt{1-t^2}} \le \arcsin t \le \frac{\frac{\pi}{\pi-2}\,t}{\frac{2}{\pi-2}+\sqrt{1-t^2}},$$

for $t\in[0,1]$ are used in specific engineering research [**1**].

Next, we explore the functions $\varphi_p(x)$ for any fixed value of the parameter

$$p \in (A,B)$$

and we determine the sign of the function $\varphi_p(x)$ on $(0,\pi/2)$. For this purpose, we consider the derivatives of $\varphi_p(x)$ with respect to x:

$$\begin{aligned}
\varphi_p'(x) &= \frac{d\varphi_p}{dx}(x) \\
&= \frac{(1-\cos x)\left(p^2-p-1-\cos x\right)}{(p+\cos x)^2} \\
\varphi_p''(x) &= \frac{d^2\varphi_p}{dx^2}(x) \\
&= \frac{(p+1)\sin x\left(p^2-2-p\cos x\right)}{(p+\cos x)^3} \\
\varphi_p'''(x) &= \frac{d^3\varphi_p}{dx^3}(x) \\
&= \frac{(p+1)}{(p+\cos x)^4}\Bigg(\cos x\,p^3+4\left(1-\cos^2 x\right)p^2+\left(\cos^3 x-4\cos x\right)p+ \\
&\quad +4\cos^2 x-6\Bigg) \\
\varphi_p^{(iv)}(x) &= \frac{d^4\varphi_p}{dx^4}(x) \\
&= -\frac{(p+1)\sin x}{(p+\cos x)^5}\Bigg(p^4-11\cos x\,p^3+\left(11\cos^2 x-20\right)p^2+ \\
&\quad +\left(-\cos^3 x+20\cos x\right)p-8\cos^2 x+24\Bigg)
\end{aligned}$$

Let us denote the following factor of the fourth derivative $\varphi_p^{(iv)}(x)$ by

$$Q_p(x) = -p^4+11\cos x\,p^3-\left(11\cos^2 x-20\right)p^2+\left(\cos^3 x-20\cos x\right)p+8\cos x^2-24$$

for $x\in(0,\pi/2)$. We have:

$$\begin{aligned}
Q_p(x) &= p\cos^3 x+(8-11p^2)\cos^2 x+(11p^3-20p)\cos x-p^4+20p^2-24 \\
&> 0+(8-11p^2)+0+(-p^4+20p^2-24) = -p^4+9p^2-16>0\,,\ \text{for } p\in(A,B)\,.
\end{aligned}$$

Consequently

$$\varphi_p^{(iv)}(x) > 0$$

for $x \in (0, \pi/2)$. For the function $\varphi_p(x)$, the following Taylor's expansion about zero holds

$$\varphi_p(x) = \frac{p-2}{6(p+1)}x^3 + o\,(x^4).$$

The above expansion enables us to conclude that there exists a right neighbourhood of zero in which

$$\varphi_p(x) < 0,\ \varphi_p'(x) < 0,\ \varphi_p''(x) < 0,\ \varphi_p'''(x) < 0,$$

hold. On the other hand, for $\varphi_p(x)$ the following Taylor's expansion holds about $\pi/2$:

$$\varphi_p(x) = \frac{(\pi-2)p-2}{2p} + \frac{p^2-p-1}{p^2}\left(x-\frac{\pi}{2}\right) + \frac{(p+1)(p^2-2)}{2p^3}\left(x-\frac{\pi}{2}\right)^2 + \frac{(p+1)(2p^2-3)}{3p^4}\left(x-\frac{\pi}{2}\right)^3 + o\left(\left(x-\frac{\pi}{2}\right)^3\right).$$

Therefore, there exists a left neighbourhood of $\pi/2$ in which

$$\varphi_p(x) > 0,\ \varphi_p'(x) > 0,\ \varphi_p''(x) > 0,\ \varphi_p'''(x) > 0,$$

hold. By applying the Nike theorem, it follows that the function φ_p has exactly one zero $x_0^{(p)} \in (0, \pi/2)$ and that $\varphi_p(x) < 0$ for $x \in (0, x_0^{(p)})$ while $\varphi_p(x) > 0$, for $x \in (x_0^{(p)}, \pi/2)$. Also, the function φ_p has exactly one local minimum on the interval $(0, \pi/2)$, i.e. there is exactly one point $t^{(p)} \in (0, x_0) \subset (0, \pi/2)$ such that $\varphi_p\big(t^{(p)}\big) < 0$ is the minimal value of the function $\varphi_p(x)$ on the interval $(0, x_0)$, i.e. $(0, \pi/2)$.

Based on the previous consideration and Theorem 2, we can establish the following

Theorem 5. 1) *If* $p \in (0, A]$ *then*

$$x \in \left(0, \frac{\pi}{2}\right) \implies x < \frac{(A+1)\sin x}{A+\cos x} \le \frac{(p+1)\sin x}{p+\cos x}.$$

2) *If* $p \in (A, B)$ *then*

$$x \in \left(0, x_0^{(p)}\right) \implies x < \frac{(p+1)\sin x}{p+\cos x},$$

and

$$x \in \left(x_0^{(p)}, \pi/2\right) \implies x > \frac{(p+1)\sin x}{p+\cos x}.$$

3) *If* $p \in [B, \infty)$ *then*

$$x \in \left(0, \frac{\pi}{2}\right) \implies x > \frac{(B+1)\sin x}{B+\cos x} \ge \frac{(p+1)\sin x}{p+\cos x}.$$

Remark 5. *The solution of the equation* $\varphi_p(x) = 0$ *with respect to* p *is* $p = g(x) = \dfrac{x\cos x - \sin x}{\sin x - x}$, *defined on* $(0, \pi/2)$. *The assertion of the previous theorem also follows on the basis of the monotonic stratification of the family* $\varphi_p(x)$ *and the inequality*

$$A < g(x) < B,$$

as well as the fact that $g(x)$ *is a monotonically decreasing function on* $(0, \pi/2)$ *such that*

$$\lim_{x\to 0+} g(x) = B \qquad \textit{and} \qquad \lim_{x\to \pi/2-} g(x) = A.$$

The proof of the inequality $g' < 0$ *on* $(0, \pi/2)$ *can be obtained, for example, by using the mixed trigonometric polynomial functions and the method presented in* [**26**], [**24**] *and* [**7**]. *For* $p \in (A, B)$, *the value of* $x_0^{(p)}$ *can also be determined as a unique solution of the equation* $g(x) = p$.

At the end of this section, we single out the minimax approximation of the Cusa-Huygens inequality, which minimizes the error (1) for the considered family $\varphi_p(x)$. Based on Theorem 1, we can give the following

Theorem 6. *There is exactly one solution of the following equation wih respect to parameter* $p \in (A, B)$

$$\left|\varphi_p\left(t^{(p)}\right)\right| = \varphi_p(\pi/2-).$$

The solution is numerically determined as

$$p_0 = 1.78114\ldots.$$

For value

$$d_0 = \left|\varphi_{p_0}\left(t^{(p_0)}\right)\right| = \varphi_{p_0}(\pi/2-) = 0.0093601\ldots,$$

the following holds

$$d_0 \;=\; \inf_{p\in\mathbb{R}^+} \sup_{x\in(0,\frac{\pi}{2})} |\varphi_p(x)|.$$

Corollary 1. *For value* $p_0 = 1.78114\ldots$, *the minimax approximant is determined as*

$$\varphi_{p_0}(x) = x - \frac{2.78114\ldots \sin x}{1.78114\ldots + \cos x}$$

and it determines the Cusa minimax approximation:

$$x \approx \frac{2.78114 \sin x}{1.78114 + \cos x} \qquad \left(\textit{for } x \in \left(0, \frac{\pi}{2}\right)\right).$$

The following consequence of previous theorems gives applicable inequalities.

Theorem 7. *For* $p_0 = 1.78114\ldots$ *the function* φ_{p_0} *has exactly one zero* $x_0^{(p_0)} = 1.47584\ldots \in (0, \pi/2)$. *The following inequalities hold true*:

$$x < \frac{(p_0+1)\sin x}{p_0 + \cos x}, \qquad \left(\textit{for } \; x \in (0, x_0)\right)$$

and

$$\frac{(p_0+1)\sin x}{p_0+\cos x} < x, \qquad \left(\mathit{for}\ x\in\left(x_0,\frac{\pi}{2}\right)\right).$$

Note that some of these claims have been proven in [**8**] by methods developed only for this family of functions. In this section, as well as in the next two sections, the proposed statements are based on the results from section 2.

3.2. The case of Sandor's inequality (1)

We give another example, similar to the previous one, established on the basis of Theorem 2.1 of [**37**]:

Theorem 8. (J. Sándor) *The constants*

$$C = 1 \quad \mathit{and} \quad D = \frac{\ln(\pi/2)}{\ln(3/2)} = 1.11373\ldots$$

are the best possible, for which the following inequalities hold true:

$$\left(\frac{\cos x+2}{3}\right)^D < \frac{\sin x}{x} < \left(\frac{\cos x+2}{3}\right)^C \qquad \left(\mathit{for}\ \ x\in(0,\pi/2)\right).$$

We observe the family of functions

$$\varphi_p(x) = \left(\frac{\cos x+2}{3}\right)^p - \frac{\sin x}{x}.$$

Obviously, they are decreasingly stratified for $p\in\mathbb{R}^+$ and continuous with respect to $x \in \left(0,\frac{\pi}{2}\right)$ for each $p\in\mathbb{R}^+$ and with respect to $p\in(C,D)$ for each $x\in\left(0,\frac{\pi}{2}\right)$. Also, the function $\varphi_p(\pi/2-)$ is continuous with respect to $p\in(C,D)$ and $\varphi_C(0+) = \varphi_D(0+) = \varphi_D(\pi/2-) = 0$ and $\varphi_C(\pi/2-)\in\mathbb{R}^+$ hold.

The following two introductory statements are true.

Statement 3. *If* $p\in(0,C]$, *then*

$$\varphi_p(x) \geq \varphi_C(x) = \frac{\cos x+2}{3} - \frac{\sin x}{x} > 0$$

for all $x\in\left(0,\frac{\pi}{2}\right)$.

Statement 4. *If $p \in [D, \infty)$, then*

$$\varphi_p(x) \le \varphi_D(x) = \left(\frac{\cos x + 2}{3}\right)^D - \frac{\sin x}{x} < 0$$

for all $x \in \left(0, \frac{\pi}{2}\right)$.

In order to find the sign of the function $\varphi_p(x)$ over $\left(0, \frac{\pi}{2}\right)$ for

$$p \in (C, D)$$

we apply the Nike theorem again.

The derivatives of the function $\varphi_p(x)$ with respect to x are:

$$\begin{aligned}
\varphi_p'(x) &= \frac{d\varphi_p}{dx}(x) \\
&= -\frac{\cos x}{x} + \frac{\sin x}{x^2} - \frac{p \sin x}{3}\left(\frac{\cos x + 2}{3}\right)^{p-1}, \\
\varphi_p''(x) &= \frac{d^2\varphi_p}{dx^2}(x) \\
&= \frac{-px^3(p\cos^2 x + 2\cos x - p + 1)\left(\frac{\cos x + 2}{3}\right)^{p-1} + 3(\cos x + 2)(2x\cos x + (x^2 - 2)\sin x)}{3x^3(\cos x + 2)}, \\
\varphi_p'''(x) &= \frac{d^3\varphi_p}{dx^3}(x) \\
&= \frac{Q_p(x)}{3x^4(\cos x + 2)^2},
\end{aligned}$$

where

$$Q_p(x) = px^4 \sin x(p^2\cos^2 x + (6p-2)\cos x - p^2 + 3p + 2)\left(\frac{\cos x + 2}{3}\right)^{p-1} +$$
$$+ 3(\cos x + 2)^2((x^3 - 6x)\cos x + (-3x^2 + 6)\sin x).$$

With the notations $r_p(x) = p^2\cos^2 x + (6p-2)\cos x - p^2 + 3p + 2$ and $s(x) = (x^3 - 6x)\cos x + (-3x^2 + 6)\sin x$ we have

$$Q_p(x) = p\,x^4 \sin x\, r_p(x)\left(\frac{\cos x + 2}{3}\right)^{p-1} + 3\,(\cos x + 2)^2\, s(x).$$

It will be shown that the conditions of the Nike theorem are true in this case.

We prove that $\varphi_p''' > 0$, which is equivalent to $Q_p(x) > 0$, for all $x \in \left(0, \frac{\pi}{2}\right)$ and all $p \in (C, D)$. Under the given conditions, the first term in $Q_p(x)$ is positive and the second term is negative. Namely, the functions $p, p^2, 6p-2$ and $-p^2 + 3p + 2$ are positive (and increasing) on the given interval, so

$$r_p(x) > \cos^2 x + 4\cos x + 4 = (\cos x + 2)^2$$

while

$$s(x) < 0,$$

because $s(0) = 0$ and $s'(x) = -x^3 \sin x < 0$.

Since $\left(\dfrac{\cos x + 2}{3}\right)^{p-1}$ is a decreasing function with respect both to x and p, we obtain:

$$\left(\frac{\cos x + 2}{3}\right)^{p-1} > \left(\frac{2}{3}\right)^{D-1} > 0.95.$$

Therefore, the first term of $Q_p(x)$ is greater than $x^4 \sin x(\cos x + 2)^2 \cdot 0.95$, so:

$$Q_p(x) > (\cos x + 2)^2(0.95\, x^4 \sin x + 3\, s(x)).$$

Let $m(x) = 0.95\, x^4 \sin x + 3\, s(x)$. Now, it is sufficient to prove that $m(x)$ is positive, which follows from $m(0) = 0$ and $m'(x) = x^3(0.8 \sin x + 0.95\, x \cos x) > 0$.

We check the other conditions of the theorem, by using the Taylor's expansion of the function φ_p about zero:

$$\varphi_p(x) = \frac{1-p}{6} x^2 + o(x^3).$$

It follows that there exists a right neighbourhood of zero in which $\varphi_p < 0$, $\varphi_p' < 0$ and $\varphi_p'' < 0$ hold.

In $x = \pi/2$ the following holds

$$\varphi_p\left(\frac{\pi}{2}-\right) = \left(\frac{2}{3}\right)^p - \frac{2}{\pi} > 0 \quad \left(\Longleftrightarrow p < \frac{\ln \frac{\pi}{2}}{\ln \frac{3}{2}}\right),$$

and

$$\varphi_p'\left(\frac{\pi}{2}-\right) = \frac{4}{\pi^2} - \frac{p}{3}\left(\frac{2}{3}\right)^{p-1} > 0$$

because $\varphi_p'\left(\frac{\pi}{2}-\right)$ is an increasing function with respect to p on the given interval. Thus $\varphi_p'\left(\frac{\pi}{2}-\right) > \varphi_1'\left(\frac{\pi}{2}-\right) = 0.071954... > 0$, and

$$\varphi_p''\left(\frac{\pi}{2}-\right) = \frac{\pi^3\, p\, (p-1)\left(\frac{2}{3}\right)^{p-1} + 12\,(\pi^2 - 8)}{6\,\pi^3} > 0.$$

Therefore, there exists a left neighbourhood of the point $x = \dfrac{\pi}{2}$ in which $\varphi_p > 0$, $\varphi_p' > 0$ and $\varphi_p'' > 0$ hold.

As in the previous section, by applying the Nike theorem, it follows that the function φ_p has exactly one zero $x_0^{(p)} \in \left(0, \frac{\pi}{2}\right)$, (wherein $\varphi_p(x) < 0$ for all $x \in (0, x_0^{(p)})$ and

$\varphi_p(x)>0$ for all $x\in\big(x_0^{(p)},\pi/2)\big)$, as well as exactly one local minimum $t^{(p)}\in\Big(0,\dfrac{\pi}{2}\Big)$, i.e. that there exists exactly one point $t^{(p)}\in\Big(0,\dfrac{\pi}{2}\Big)$ such that $\varphi_p(t^{(p)})\ \big(<0\big)$ is the smallest value of the function $\varphi_p(x)$ under the given conditions.

Based on the previous consideration and Theorem 2', we can give the following

Theorem 9. 1) *If* $p\in(0,C]$, *then*

$$x\in\Big(0,\frac{\pi}{2}\Big)\implies\frac{\sin x}{x}<\frac{\cos x+2}{3}\le\Big(\frac{\cos x+2}{3}\Big)^p.$$

2) *If* $p\in(C,D)$, *then*

$$x\in(0,x_0^{(p)})\implies\frac{\sin x}{x}>\Big(\frac{\cos x+2}{3}\Big)^p,$$

and

$$x\in(x_0^{(p)},\pi/2)\implies\frac{\sin x}{x}<\Big(\frac{\cos x+2}{3}\Big)^p.$$

3) *If* $p\in[D,\infty)$, *then*

$$x\in\Big(0,\frac{\pi}{2}\Big)\implies\frac{\sin x}{x}>\Big(\frac{\cos x+2}{3}\Big)^D\ge\Big(\frac{\cos x+2}{3}\Big)^p.$$

Remark 6. *The equality* $\varphi_p(x)=0$ *is equivalent to* $p=g(x)=\ln\dfrac{x}{\sin x}\Big/\ln\dfrac{3}{2+\cos x}$, *for* $x\in 0,\pi/2)$. *The previous theorem can be proved on the basis of the monotonic stratification of the family* $\varphi_p(x)$ *and the inequality*

$$C<g(x)<D,$$

as well as the fact that $g(x)$ *is a monotonically increasing function on* $(0,\pi/2)$ *such that*

$$\lim_{x\to0+}g(x)=C\qquad\text{and}\qquad\lim_{x\to\pi/2-}g(x)=D.$$

Hence, Statements 3 *and* 4 *follow. For* $p\in(C,D)$ *the value* $x_0^{(p)}$ *can also be determined as a unique solution of the equation* $g(x)=p$.

Also, based on Theorem 1', a particular statement for the observed family of functions is valid.

Theorem 10. *There is exactly one solution of the following equation with respect to the parameter* $p\in(C,D)$

$$\Big|\varphi_p\Big(t^{(p)}\Big)\Big|=\varphi_p(\pi/2-)$$

and the solution is numerically determined as

$$p_0=1.08716\ldots.$$

For value
$$d_0 = \left|\varphi_{p_0}\left(t^{(p_0)}\right)\right| = \varphi_{p_0}(\pi/2-) = 0.0068956\ldots\,,$$
we have
$$d_0 = \inf_{p\in\mathbb{R}^+} \sup_{x\in(0,\frac{\pi}{2})} |\varphi_p(x)|\,.$$

Corollary 2. *For value* $p_0 = 1.08716\ldots$ *the minimax approximant is*
$$\varphi_{p_0}(x) = \left(\frac{\cos x + 2}{3}\right)^{1.08716\ldots} - \frac{\sin x}{x}$$
which determines the corresponding minimax approximation:
$$\frac{\sin x}{x} \approx \left(\frac{\cos x + 2}{3}\right)^{1.08716} \qquad \left(\textit{for } x\in\left(0,\frac{\pi}{2}\right)\right).$$

Based on all previous considerations, we can establish the following

Theorem 11. *For the value of the parameter* $p_0 = 1.08716\ldots$, *the function* φ_{p_0} *has exactly one zero* $x_0^{(p_0)} = 0.98856\ldots \in (0,\pi/2)$ *and the following inequalities are true*:
$$\frac{\sin x}{x} > \left(\frac{\cos x + 2}{3}\right)^{p_0}, \qquad \left(\textit{for } x\in(0,x_0)\right)$$
and
$$\frac{\sin x}{x} < \left(\frac{\cos x + 2}{3}\right)^{p_0}, \qquad \left(\textit{for } x\in\left(x_0,\frac{\pi}{2}\right)\right).$$

3.3. The case of Sandor's inequality (2)

In the next example, we can see that the observed family of functions does not satisfy all of the conditions of the Nike theorem. In consequence we use Theorem 4 in order to determine the minimax approximation. This example is obtained on the basis of Theorem 2.2 of [**37**]:

Theorem 12. (J. Sándor) *The constants*
$$E = \frac{\ln(\pi/2)}{\ln 2} = 0.65149\ldots \quad \textit{and} \quad F = \frac{2}{3} = 0.66666\ldots$$
are the best possible, for which the following inequalities hold true:
$$\left(\frac{\cos x + 1}{2}\right)^F < \frac{\sin x}{x} < \left(\frac{\cos x + 1}{2}\right)^E \qquad \left(\textit{for } x\in(0,\pi/2)\right).$$

Let us consider the family of functions

$$\varphi_p(x) = \frac{\sin x}{x} - \left(\frac{\cos x + 1}{2}\right)^p$$

which are continuous with respect to x on $\left(0, \frac{\pi}{2}\right)$ for each $p \in (0, \infty)$ and increasingly stratified for $p \in (0, \infty)$, too. They are continuous with respect to $p \in (E, F)$ for each x on $\left(0, \frac{\pi}{2}\right)$ and $\varphi_p(\pi/2-)$ is continuous with respect to p on (E, F). It can be easily checked that $\varphi_E(0+) = \varphi_E(\pi/2-) = \varphi_F(0+) = 0$ and $\varphi_F(\pi/2-) \in \mathbb{R}^+$.

The following two introductory statements are true.

Statement 5. *For $p \in (0, E]$, we have*

$$\varphi_p(x) \le \varphi_E(x) = \frac{\sin x}{x} - \left(\frac{\cos x + 1}{2}\right)^E < 0$$

for all $x \in \left(0, \frac{\pi}{2}\right)$.

Statement 6. *For $p \in [F, \infty)$, we have*

$$\varphi_p(x) \ge \varphi_F(x) = \frac{\sin x}{x} - \left(\frac{\cos x + 1}{2}\right)^F > 0$$

for all $x \in \left(0, \frac{\pi}{2}\right)$.

It remains to examine the sign of the functions $\varphi_p(x)$ on $\left(0, \frac{\pi}{2}\right)$ for

$$p \in (E, F).$$

In this sense, we compute the derivatives of functions $\varphi_p(x)$ with respect to x:

$$\begin{aligned}
\varphi'_p(x) &= \frac{d\varphi_p}{dx}(x) \\
&= \frac{\cos x}{x} - \frac{\sin x}{x^2} + \frac{p \sin x}{2}\left(\frac{\cos x + 1}{2}\right)^{p-1}, \\
\varphi''_p(x) &= \frac{d^2\varphi_p}{dx^2}(x) \\
&= \frac{px^3(p\cos x - p + 1)\left(\frac{\cos x + 1}{2}\right)^{p-1} - (2x^2 - 4)\sin x - 4x\cos x}{2x^3},
\end{aligned}$$

$$\begin{aligned}\varphi_p'''(x) &= \frac{d^3\varphi_p}{dx^3}(x)\\ &= \frac{1}{2x^4(\cos x+1)}\Bigg(px^4\sin x\cdot\Big(-p^2\cos x+p^2-3p+1\Big)\Big(\frac{\cos x+1}{2}\Big)^{p-1}-\\ &\quad -2(\cos x+1)\Big((x^3-6x)\cos x+(-3x^2+6)\sin x\Big)\Bigg),\end{aligned}$$

$$\begin{aligned}\varphi_p^{(iv)}(x) &= \frac{d^4\varphi_p}{dx^4}(x)\\ &= \frac{1}{2x^5(\cos x+1)}\Bigg(px^5\sin x\cdot\Big(-p^3\cos^2 x-(-2p^3+6p^2-4p+1)\cos x-\\ &\quad -p^3+6p^2-7p+2\Big)\Big(\frac{\cos x+1}{2}\Big)^{p-1}+\\ &\quad +2\,(\cos x+1)\Big((4x^3-24x)\cos x+(x^4-12x^2+24)\sin x\Big)\Bigg),\end{aligned}$$

$$\varphi_p^{(v)}(x) = \frac{d^5\varphi_p}{dx^5}(x) = \frac{Q_p(x)}{2x^6(\cos x+1)^2},$$

wherein

$$\begin{aligned}Q_p(x) &= p\,x^6\sin x\cdot r_p(x)\left(\frac{\cos x+1}{2}\right)^{p-1}+2\,(\cos x+1)^2\,s(x),\\ r_p\,(x) &= p^4\cos^2 x+\big(-2p^4+10p^3-10p^2+5p-1\big)\cos x+\\ &\quad +p^4-10p^3+25p^2-20p+5,\\ s\,(x) &= \big(x^5-20x^3+120x\big)\cos x-5\,\big(x^4-12x^2+24\big)\sin x.\end{aligned}$$

We will prove that

$$\varphi_p^{(v)}(x)<0,$$

i.e. that $Q_p(x)<0$ for $x\in\left(0,\frac{\pi}{2}\right)$ and $p\in(E,F)$. As $s(x)<0$ due to $s(0)=0$ and $s'(x)=-x^5\sin x<0$, the second term in $Q_p(x)$ is negative. On the other hand, in order to bound the first term, the polynomials $g_1(p)=-2p^4+10p^3-10p^2+5p-1$ and $g_2(p)=p^4-10p^3+25p^2-20p+5$ are observed, which are increasing on (E,F). On this interval we have $g_1(p)<g_1\left(\frac{2}{3}\right)=\frac{37}{81}$ and $g_2(p)<g_2\left(\frac{2}{3}\right)=\frac{1}{81}$, so $r_p(x)<r(x)=\frac{16\cos^2 x+37\cos x+1}{81}$. Then

$$Q_p(x)<\frac{2}{3}\,x^6\sin x\cdot r(x)\left(\frac{\cos x+1}{2}\right)^{p-1}+2\,(\cos x+1)^2\,s(x).$$

Let $q_p(x)=\frac{2}{3}x^6\sin x\cdot r(x)\left(\frac{\cos x+1}{2}\right)^{p-1}+2\,(\cos x+1)^2\,s(x)$. As $\cos x<1$, $(\cos x+1)^2>\frac{9}{4}$ and $\left(\frac{\cos x+1}{2}\right)^{p-1}<\left(\frac{3}{4}\right)^{E-1}<1.1055$, for every $x\in\left(0,\frac{\pi}{3}\right)$ and $p\in(E,F)$, we have $r(x)<\frac{2}{3}$ and

$$Q_p(x)<q_p(x)<0.5\,x^6\sin x+4.5\,s(x).$$

Let $m(x)=0.5\,x^6\sin x+4.5\,s(x)$. Now, it is sufficient to prove that $m(x)<0$. Since $m(0)=0$ and $m'(x)=0.5\,x^5\,(x\cos x-3\sin x)<0$ $\big($which follows from $m_1(x)=x\cos x-3\sin x$ and $m_1(0)=0$, $m_1'(x)=-x\sin x-2\cos x<0\big)$ it also holds that $m(x)<0$, and therefore also $\varphi_p^{(v)}(x)<0$ for $\left(0,\frac{\pi}{3}\right)$.

For $x\in\left[\frac{\pi}{3},\frac{\pi}{2}\right)$ and $p\in(E,F)$ the inequalities $\cos x\le\frac{1}{2}$, $(\cos x+1)^2>1$ and $\left(\frac{\cos x+1}{2}\right)^{p-1}\le\left(\frac{1}{2}\right)^{F-1}<1.281$ hold, therefore $r(x)<\frac{47}{162}$, so

$$Q_p(x)<q_p(x)<0.25\,x^6\sin x+2\,s(x)$$

holds. For $n(x)=0.25\,x^6\sin x+2\,s(x)$ and $x\in\left(\frac{\pi}{3},\frac{\pi}{2}\right)$ it follows $Q_p(x)<q_p(x)<n(x)$, so it is sufficient to prove that $n(x)<0$. Since $n(0)=0$ and $n'(x)=0.25x^5(x\cos x-2\sin x)<0$ $\big($which follows from $n_1(x)=x\cos x-2\sin x$, $n_1(0)=0$ and $n_1'(x)=-x\sin x-\cos x<0\big)$, then $n(x)<0$ holds and thus $\varphi_p^{(v)}(x)<0$ on $\left(\frac{\pi}{3},\frac{\pi}{2}\right)$.

The Taylor expansion for the function φ_p at the point zero is

$$\varphi_p(x)=\left(-\frac{1}{6}+\frac{\pi}{4}\right)x^2+\left(\frac{1}{120}+\frac{1}{96}p-\frac{1}{32}p^2\right)x^4+o(x^5)$$

from which it is easy to conclude that for $p\in(E,F)$ there is a right neighbourhood of zero in which $\varphi(x)<0$, $\varphi'(x)<0$, $\varphi''(x)<0$, $\varphi'''(x)<0$, while $\varphi^{(iv)}(x)>0$.

Now we prove that there is a left neighborhood of $\frac{\pi}{2}$ where $\varphi>0$, $\varphi'>0$ and $\varphi''>0$, and $\varphi^{(iv)}<0$, while φ''' has not the same sign for all $p\in(E,F)$. Namely,

$$\varphi\left(\frac{\pi}{2}-\right)>0\Longleftrightarrow p>E,$$

$$\varphi'\left(\frac{\pi}{2}-\right)=-\frac{4}{\pi^2}+p\,2^{-p}>0.65\cdot 2^{-\frac{2}{3}}-\frac{4}{\pi^2}>0.004>0,$$

$$\varphi''\left(\frac{\pi}{2}-\right)>0\Longleftrightarrow p\,(1-p)\,\pi^3 2^{-p}-2\,\pi^2+16>0$$

which follows from $p\,(1-p)\,\pi^3\,2^{-p}-2\,\pi^2+16>0.65\cdot\frac{1}{3}\cdot\pi^3\cdot 2^{-\frac{2}{3}}>0.4928>0$.

Then

$$\varphi^{(iv)}(x)<0\Longleftrightarrow(1-p)(-p^2+5p-2)\,\pi^5 p\,2^{-p}-2\,(\pi^4-48\pi^2+384)>0$$

which follows from $(1-p)(-p^2+5p-2)\pi^5 p\,2^{-p}-2(\pi^4-48\pi^2+384)>\frac{1}{3}\cdot(-0.65^2+5\cdot 0.65-2)\pi^5\cdot 0.65\cdot 2^{-\frac{2}{3}}-16>18>0$. It is interesting that φ''' has not a constant sign at $\frac{\pi}{2}$. It is positive for $p\in(E,p_1)$ and negative for $p\in(p_1,F)$, where $p_1=0.66329\ldots$ is determined numerically.

Since $\varphi_p^{(v)} < 0$ on $\left(0, \dfrac{\pi}{2}\right)$, it follows that $\varphi_p^{(iv)}$ decreases, and from $\varphi^{(iv)} > 0$ on a right neighborhood of zero and $\varphi^{(iv)} > 0$ on a left neighborhood of $\dfrac{\pi}{2}$, it also follows that on this interval $\varphi^{(iv)}$ has exactly one zero x_4 and that $\varphi^{(iv)} > 0$ on $(0, x_4)$ and $\varphi^{(iv)} < 0$ on $\left(x_4, \dfrac{\pi}{2}\right)$. Therefore, φ''' has exactly one local maximum x_4 on $\left(0, \dfrac{\pi}{2}\right)$, and because $\varphi''' > 0$ holds on a right neighborhood of zero, it follows that on $\left(0, \dfrac{\pi}{2}\right)$ two cases occur: **1)** $\varphi''' > 0$ (for $p \leq p_1$) or **2)** φ''' has exactly one zero $x_3 \in \left(x_4, \dfrac{\pi}{2}\right)$ (as well as on $(0, \pi/2)$) and $\varphi''' > 0$ on $(0, x_3)$ and $\varphi''' < 0$ on $\left(x_3, \dfrac{\pi}{2}\right)$.

In the first case by applying the Nike theorem, and in the second, by applying Theorem 4, it follows that function φ_p has exactly one zero $x_0^{(p)} \in (0, \pi/2)$ and $\varphi_p < 0$ for $x \in (0, x_0^{(p)})$ and $\varphi_p > 0$ for $x \in (x_0^{(p)}, \pi/2)$. Also, the function φ_p has exactly one minimum on $(0, \pi/2)$, i.e. there is exactly one point $t^{(p)} \in (0, x_0) \subset (0, \pi/2)$ such that $\varphi_p\big(t^{(p)}\big)(<0)$ is the smallest value of the function φ_p on $(0, x_0)$.

Based on above proofs and Theorems 1 and 2, we can give the following

Theorem 13. 1) *If* $p \in (0, E]$, *then*

$$x \in \left(0, \frac{\pi}{2}\right) \implies \frac{\sin x}{x} < \left(\frac{\cos x + 1}{2}\right)^E \leq \left(\frac{\cos x + 1}{2}\right)^p.$$

2) *If* $p \in (E, F)$, *then*

$$x \in (0, x_0^{(p)}) \implies \frac{\sin x}{x} < \left(\frac{\cos x + 1}{2}\right)^p,$$

and

$$x \in (x_0^{(p)}, \pi/2) \implies \frac{\sin x}{x} > \left(\frac{\cos x + 1}{2}\right)^p.$$

3) *If* $p \in [F, \infty)$, *then*

$$x \in \left(0, \frac{\pi}{2}\right) \implies \frac{\sin x}{x} > \left(\frac{\cos x + 1}{2}\right)^F \geq \left(\frac{\cos x + 1}{2}\right)^p.$$

Remark 7. *The equality* $\varphi_p = 0$ *is equivalent to* $p = g(x) = \ln \dfrac{x}{\sin x} \Big/ \ln \dfrac{2}{1 + \cos x}$, *for* $x \in (0, \pi/2)$. *The statement of the previous theorem also follows on the basis of stratification of the family* $\varphi_p(x)$ *and the inequality*

$$E < g(x) < F,$$

as well as the fact that $g(x)$ *is a monotonically decreasing function on* $(0, \pi/2)$ *such that*

$$\lim_{x \to 0+} g(x) = F \qquad \textit{and} \qquad \lim_{x \to \pi/2-} g(x) = E.$$

From there, Statements 5 *and* 6 *follow. For* $p \in (E, F)$ *the value* $x_0^{(p)}$ *can also be determined as the unique solution of the equation* $g(x) = p$.

Theorem 14. *There is exactly one solution of the following equation with respect to* $p \in (E, F)$

$$\left|\varphi_p\left(t^{(p)}\right)\right| = \varphi_p(\pi/2-).$$

The solution is numerically determined as

$$p_0 = 0.65462\ldots.$$

For value

$$d_0 = \left|\varphi_{p_0}\left(t^{(p_0)}\right)\right| = \varphi_{p_0}(\pi/2-) = 0.00138038\ldots,$$

we have

$$d_0 = \inf_{p\in\mathbb{R}^+} \sup_{x\in(0,\frac{\pi}{2})} |\varphi_p(x)|.$$

Corollary 3. *For the value* $p_0 = 0.65462\ldots$ *the minimax approximant is*

$$\varphi_{p_0}(x) = \frac{\sin x}{x} - \left(\frac{\cos x + 1}{2}\right)^{0.65462\ldots}$$

and the corresponding minimax approximation is:

$$\frac{\sin x}{x} \approx \left(\frac{\cos x + 1}{2}\right)^{0.65462} \qquad \left(\textit{for } x \in \left(0, \frac{\pi}{2}\right)\right).$$

Now we are in a position to establish the following

Theorem 15. *For the value of the parameter* $p_0 = 0.65462\ldots$, *the function* φ_{p_0} *has exactly one zero* $x_0^{(p_0)} = 0.97676\ldots \in (0, \pi/2)$ *and the following inequalities are true*:

$$\frac{\sin x}{x} < \left(\frac{\cos x + 1}{2}\right)^{p_0}, \qquad \left(\textit{for } x \in (0, x_0)\right)$$

and

$$\frac{\sin x}{x} > \left(\frac{\cos x + 1}{2}\right)^{p_0}, \qquad \left(\textit{for } x \in \left(x_0, \frac{\pi}{2}\right)\right).$$

4. CONCLUSION

One of the aims of this paper was to explore the validity of some analytic inequalities with a parameter on wider intervals and thus provide a complete answer in which form the corresponding inequalities can be applied to all the values of the parameter $p \in \mathbb{R}^+$, as discussed above. In the previous section, the examples of minimax approximations are presented, which, from the approximation theory point of view, minimize the error in the considered sense. Let us notice that a procedure analogous to the procedure developed in this paper can be applied for a large class of inequalites [**16**], [**28**], [**29**], [**36**], [**47**] and [**2**]–[**7**], [**9**],[**10**], [**12**]–[**15**], [**17**], [**19**]–[**23**], [**30**]–[**33**], [**38**], [**40**]–[**46**], [**48**]–[**51**] .

The proves considered in this paper are based on the Nike theorem and its extension. The families of functions being considered are of the same type geometrically. This research shows that it is possible to consider analytic inequalities with a parameter of a similar type, for instance, see [**34**], [**35**], [**18**] (according [**36**]) and also [**39**], [**11**], whereby their geometry would require the application of some other new statements, similar to the Nike theorem. Such families of analytic inequalities with one parameter will be the subject of our future research.

Acknowledgments. The authors are grateful to professors Ivana Jovović, Cristinel Mortici and anonymous referees for their careful corrections to and valuable comments on the original version of this paper. Research of the first author was supported in part by the Serbian Ministry of Education, Science and Technological Development, under Projects ON 174032 & III 44006 which are realized at the School of Electrical Engineering, University of Belgrade.

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(Received 11. 05. 2021.)
(Revised 04. 10. 2021.)

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