Applicable Analysis and Discrete Mathematics
available online at http://pefmath.etf.rs

Appl. Anal. Discrete Math. **18** (2024), 116–147.
https://doi.org/10.2298/AADM230310001M

ASYMPTOTIC EXPANSIONS OF STABLE, STABILIZABLE AND STABILIZED MEANS WITH APPLICATIONS

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In this paper we present a complete asymptotic expansion of a symmetric homogeneous stable (balanced), stabilizable and stabilized mean. By including known asymptotic expansions of parametric means it is shown how the obtained coefficients are used to solve the problem of identifying stable means within classes of parametric means under consideration, how to disprove some mean is stabilizable or stabilized and how to obtain best possible parameters such that given mean is sub-stabilizable with a pair of parametric means.

1. INTRODUCTION

Consider *bivariate mean* M, i.e. function $M\colon \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$\min(s,t) \leq M(s,t) \leq \max(s,t), \tag{1}$$

where $(s,t) \mapsto \min(s,t)$ and $(s,t) \mapsto \max(s,t)$ are considered as trivial means. With symmetry and homogeneity defined in usual way, let M, N, K, be three homogeneous symmetric means and let

$$\mathcal{R}(K,N,M)(s,t) = K\Big(N\big(s, M(s,t)\big), N\big(M(s,t),t\big)\Big).$$

$\mathcal{R}$ is also called the resultant mean–map of K, N and M ([**30**]). Observe the following functional equation

$$M(s,t) = \mathcal{R}(M,M,M)(s,t) = M\Big(M\big(s, M(s,t)\big), M\big(M(s,t),t\big)\Big), \tag{2}$$

2020 Mathematics Subject Classification. 26E60; 41A60; 39B22.
Keywords and Phrases. Complete asymptotic expansion, Composite means, Recursive relation, Power means, Asymptotic inequalities.

which has been examined by many authors in various settings.

G. Aumann ([**2**]) studied constructions of means of several arguments and corresponding iterative algorithms, more precisely, augmentation of mean to $n+1$ arguments with mean of n arguments given. He called such mean the upper mean. While studying the opposite procedure, i.e. the reduction process ([**3**]), he introduced the lower mean. While trying to determine when these two processes are inverse to each other, for $n = 2$ *composite functional equation* (2) appeared. G. Aumann proved that this functional equation in class of analytic means on $\mathbb{C}^2$ is characteristic to the analytic quasi-arithmetic means. L. R. Berrone ([**4**]) presented key results from Aumann's two papers pointing out non-equivalence of complex methods within class of real variable means and also analyzed generalizations of Aumann functional equation which involves general weighting operators.

T. Kiss ([**19**]), calling the equation (2) *balancing property*, solved it without differentiability assumptions in the class of two-variable means, which contains the class of Matkowski means.

In this paper we follow definitions introduced by M. Raïssouli who, relying on equation (2), also introduced a notion of stabilizable and stabilized mean.

Definition 1 ([**30**])**.** A symmetric mean M is said to be:

1. *Stable* (*balanced*), if $\mathcal{R}(M, M, M) = M$, that is, (2) holds.
2. (K, N)-*stabilizable*, if for two nontrivial stable means K and N the following relation is satisfied:

$$M(s,t) = \mathcal{R}(K, M, N)(s,t) = K\Big(M\big(s, N(s,t)\big), M\big(N(s,t), t\big)\Big). \tag{3}$$

3. (K, N)-*stabilized*, if for two nontrivial stable means K and N the following relation is satisfied:

$$M(s,t) = \mathcal{R}(K, N, M)(s,t) = K\Big(N\big(s, M(s,t)\big), N\big(M(s,t), t\big)\Big). \tag{4}$$

It can easily be seen that the arithmetic A mean is stable, the (binomial) power mean B_p is also stable for all real numbers p and more general, every cross mean is stable ([**30**]). The logarithmic and identric means are known not to be stable. Stabilizable or stabilized mean does not need to be stable by itself. Furthermore, geometric mean G is simultaneously (A, H)-stabilized and (H, A)-stabilized, while the Heron mean *He* is (A, G)-stabilized. The power logarithmic mean L_p, also known as generalized logarithmic mean, is (B_p, A)-stabilizable, the power difference mean D_p, i.e. Stolarsky mean $E_{p,p+1}$, is (A, B_p)-stabilizable, the power exponential mean I_p, i.e. Stolarsky mean $E_{p,p}$, is (G, B_p)-stabilizable and the second power logarithmic mean l_p, i.e. Stolarsky mean $E_{p,0}$, is (B_p, G)-stabilizable ([**30**]). For all real numbers p and q, Stolarsky mean $E_{p,q}$ is (B_{q-p}, B_p)-stabilizable ([**32**]). Precise definitions of those means will be given in Section 5.

A given mean can be stabilizable with respect to two distinct couples of means. For instance, logarithmic mean L is simultaneously (A, G)-stabilizable and

(H, A)-stabilizable. On the other side, for two given stable means M_1 and M_2, such that $M_1 \leq M_2$ and M_1 is strict and cross mean, there exists one and only one (M_1, M_2)-stabilizable mean M such that $M_1 \leq M \leq M_2$ ([**33**]).

There are various applications of the stability and stabilizability. As an extension of the stabilizability concept, A. Gasmi and M. Raïssouli ([**15**]) introduced the generalized stabilizability for means. Regarding mean inequalities which have been studied extensively, M. Raïssouli ([**31**]) presented an approach for obtaining refinements in a convenient manner. He has shown how to obtain in a recursive way an infinite number of lower and upper bounds starting from an arbitrary lower and upper bounds of a stabilizable mean.

For two nontrivial stable comparable means the (strict) sub-stabilizability and super-stabilizability concept can be introduced with the appropriate inequality sign in (3).

Definition 2 ([**34**])**.** Let K, N be two nontrivial stable comparable means. Mean M is called

1. (K, N)-*sub-stabilizable*, if $\mathcal{R}(K, M, N) \leq M$ and M is between K and N,
2. (K, N)-*super-stabilizable*, if $M \leq \mathcal{R}(K, M, N)$ and M is between K and N.

For example, geometric mean G is (G, A)-super-stabilizable (but not strictly), arithmetic mean A is (G, A)-sub-stabilizable, logarithmic mean L is strictly (G, A) - super-stabilizable and strictly (A, H)-sub-stabilizable, identric mean I is strictly (A, G)-sub-stabilizable ([**34**]). The first Seiffert mean is strictly (G, A) - super-stabilizable ([**1**]).

In the above mentioned papers some open problems appeared from which we shall single out the following.

a. Find all pairs (p, q) such that Gini means $G_{p,q}$ and Stolarsky means $E_{p,q}$ are stable ([**30**]).

b. Prove or disprove that the first Seiffert mean P is not stabilizable ([**30**]).

c. Find the best real numbers $p > 0$ and $q > 0$ for which the first Seiffert means P is strictly (B_p, B_q)-sub-stabilizable ([**34**]).

d. Ascertain if the second Seiffert mean T and the Neuman-Sándor mean *NS* are strictly (B_p, B_q)-sub-stabilizable for some real numbers $p > 0$, $q > 0$ ([**34**]).

Throughout this paper, whenever we consider (K, N)-sub/super-stabilizable or stabilized mean it will be assumed that K and N are nontrivial stable means.

The aim of this paper is to apply the previously developed techniques of asymptotic expansions on the equations (2), (3) and (4) in order to obtain the asymptotic expansion of stable, stabilizable and stabilized mean.

The (formal) series $\sum_{n=0}^{\infty} a_n \varphi_n(x)$ is said to be an *asymptotic expansion* of a function $f(x)$ as $x \to x_0$, with respect to asymptotic sequence $(\varphi_n(x))_{n \in \mathbb{N}_0}$, if for each $N \in \mathbb{N}_0$

$$f(x) \sim \sum_{n=0}^{N} a_n \varphi_n(x) + o(\varphi_N(x)).$$

Approximation of the function to a given accuracy is achieved by approaching the variable to a certain fixed point or a point at infinity. A small number of members of this series ensures a good approximation. Taylor series can also be seen as an asymptotic as $x \to 0$. Asymptotic series may be convergent or divergent. For a given asymptotic sequence, asymptotic representation is unique. Conversely, asymptotic series represents a class of asymptotically equal functions. Theoretical background from theory of asymptotic expansions can be found in [**14**].

For a symmetric homogeneous stable mean we find coefficients a_n in the asymptotic power series expansion of the form

$$M(x-t, x+t) \sim \sum_{n=0}^{\infty} a_n t^{2n} x^{-2n+1}, \text{ as } x \to \infty, \tag{5}$$

which will be given by a recursive relation. It will be shown that the asymptotic representation (5) is sufficient to obtain the general form

$$M(x+s, x+t) \sim \sum_{n=0}^{\infty} a_n(s,t) x^{-n+1}, \text{ as } x \to \infty, \tag{6}$$

where $a_n \equiv a_{2n}(-t, t)$.

Based on the asymptotic expansions, more precisely, on positivity of the first non-zero coefficient, a notion of asymptotic inequality can be introduced. Recall its definition.

Definition 3 ([**38**])**.** Let $F(s,t)$ be any homogeneous function such that

$$F(x+s, x+t) = c_k(t,s) x^{-k+1} + \mathcal{O}(x^{-k}).$$

If $c_k(s,t) > 0$ for all s and t, then we say F is *asymptotically* greater than zero, and write

$$F \succ 0.$$

Asymptotic inequality is considered as necessary relation between comparable means. Namely, if $F \geq 0$, then $F \succ 0$, which has been proved in the same paper. Furthermore, for the asymptotic inequalities it is sufficient to observe the case $s = -t$ as explained in [**8**]. Asymptotic inequalities were used to obtain the best possible parameters in convex combinations of means which include Seiffert ([**38**]) and Neuman-Sándor ([**13**]) means and to obtain the best possible parameters such that inequality between some parametric means holds ([**11**]). In this paper asymptotic inequalities will be used to treat the case of sub-stabilizability with power means.

This paper is organized as follows. In Section 2 we state some fundamental results regarding operations with asymptotic expansions and show the auxiliary result which will be used afterwards. In Section 3 we obtain asymptotic expansion of the resultant mean-map provided that all three means involved possess asymptotic expansion. Using this result in order, we obtained the asymptotic expansion of stable, stabilizable and stabilized mean. In Section 4 we obtain necessary conditions for mean N to be simultaneously (K, M) and (M, K)-stabilizable, for mean M to be simultaneously (K, N) and (N, K)-stabilized and for mean M to be simultaneously (K, N)-stabilizable and (K, N)-stabilized. With respect to known asymptotic expansions of parametric means derived in [**11**] and [**38**], in Section 5 it will be shown how the obtained coefficients are used to solve the problem of identifying stable means within classes of parametric means under consideration, how to disprove a mean is stabilizable or stabilized and how to obtain best possible parameters such that given mean is sub-stabilizable with a pair of parametric means. Recursive formulas were evaluated using computer algebra system *Mathematica*. In Section 6 we sum up all the results, emphasize our contribution to the open questions from cited papers and state new conjectures which arose from this paper.

2. PRELIMINARIES

Suppose that all means involved here have the asymptotic expansions as $x \to \infty$ of the following type

$$M(x-t, x+t) \sim \sum_{n=0}^{\infty} a_n^M t^{2n} x^{-2n+1}, \tag{7}$$

$$N(x-t, x+t) \sim \sum_{n=0}^{\infty} a_n^N t^{2n} x^{-2n+1}, \tag{8}$$

$$K(x-t, x+t) \sim \sum_{n=0}^{\infty} a_n^K t^{2n} x^{-2n+1}. \tag{9}$$

Operations with asymptotic power series are conducted in very intuitive manner. Asymptotic expansion of a linear combination corresponds to expansion with the same linear combination done term-wise. Coefficients in product of two asymptotic power series are defined by convolution. Also, two asymptotic power series can be divided with the result given in a form of asymptotic series as described in [**11**, Lemma 1.1.]. The composition has asymptotic expansion whose coefficients can be obtained by formal substitution and rearrangement of terms ([**14**, p. 20]). Under some reasonable assumptions, asymptotic power series can be differentiated and integrated term by term ([**14**, p. 21]). In the sequel we state the fundamental result on transformations which is about power of an asymptotic series. Coefficients of the new series, which depend on the power r and initial sequence $\mathbf{a} = (a_n)_{n\in\mathbb{N}_0} = (a_0, a_1, a_2, \ldots)$, will be denoted here as $P[n, r, \mathbf{a}]$. We assume all sequences are enumerated from 0.

Lemma 4 ([**7, 17**])**.** *Let*

$$g(x) \sim \sum_{n=0}^{\infty} a_n x^{-n}$$

be a given asymptotic expansion (for $x \to \infty$) of $g(x)$ with $a_0 \neq 0$. Then for all real r it holds

$$[g(x)]^r \sim \sum_{n=0}^{\infty} P[n, r, \mathbf{a}] x^{-n},$$

where $P[0, r, \mathbf{a}] = a_0^r$ and

$$P[n, r, \mathbf{a}] = \frac{1}{n a_0} \sum_{k=1}^{n} [k(1+r) - n] a_k P[n-k, r, \mathbf{a}], \quad n \in \mathbb{N}.$$

Remark 5. It may be useful to consider $P[n, r, \mathbf{a}]$ as the coefficient by the x^{-n} in the r-th power of series assigned to sequence $\mathbf{a}$, especially when r is a nonnegative integer, wherefrom following useful relations follow easily.

1. $P[n, 0, \mathbf{a}] = \delta_n$, $n \in \mathbb{N}_0$, where δ_n stands for a single-argument Kronecker delta function.

2. $P[n, 1, \mathbf{a}] = a_n$, $n \in \mathbb{N}_0$.

3. $P[0, r, \mathbf{a}] = a_0^r$, $r \in \mathbb{R}$.

The following auxiliary sequences will be used to express main results. Let

$$\begin{aligned} \mathbf{g} &\coloneqq (1, a_1, 0, a_2, 0, a_3, \ldots) \\ \mathbf{h} &\coloneqq (2, -1, a_1, 0, a_2, 0, a_3, \ldots). \end{aligned} \tag{10}$$

and also

$$\begin{aligned} \tilde{\mathbf{g}} &\coloneqq (1, -a_1, 0, -a_2, 0, -a_3, \ldots) \\ \tilde{\mathbf{h}} &\coloneqq (2, 1, a_1, 0, a_2, 0, a_3, \ldots). \end{aligned}$$

Let us denote by $D(m, n, k)$ and $S(m, n, k)$ terms which will appear within the inner sums later in proof of Theorem 7:

$$\begin{aligned} D(m, n, k) = {} & P[k, 2n, \tilde{\mathbf{g}}] P[m - 2n - k, -2n + 1, \tilde{\mathbf{h}}] \\ & - P[k, 2n, \mathbf{g}] P[m - 2n - k, -2n + 1, \mathbf{h}], \end{aligned} \tag{11}$$

$$\begin{aligned} S(m, n, k) = {} & P[k, 2n, \tilde{\mathbf{g}}] P[m - 2n - k, -2n + 1, \tilde{\mathbf{h}}] \\ & + P[k, 2n, \mathbf{g}] P[m - 2n - k, -2n + 1, \mathbf{h}]. \end{aligned} \tag{12}$$

Some of the coefficients $D(m, n, k)$ and $S(m, n, k)$ are equal to zero because of the relations between sequences $\mathbf{g}$ and $\tilde{\mathbf{g}}$ and also $\mathbf{h}$ and $\tilde{\mathbf{h}}$.

Lemma 6. *For $m \in \mathbb{N}_0$, $n \in \{0, 1, \ldots, \lfloor \frac{m}{2} \rfloor\}$, $k \in \{0, 1, \ldots, m-2n\}$, it holds*

$$D(m,n,k) = \begin{cases} 0, & m \text{ even}, \\ -2P[k, 2n, \mathbf{g}]P[m-2n-k, -2n+1, \mathbf{h}], & m \text{ odd}, \end{cases} \tag{13}$$

and

$$S(m,n,k) = \begin{cases} 2P[k, 2n, \mathbf{g}]P[m-2n-k, -2n+1, \mathbf{h}], & m \text{ even}, \\ 0, & m \text{ odd}. \end{cases} \tag{14}$$

Proof. Let us define (generating) functions

$$G(x) = \sum_{k=0}^{\infty} g_k x^{-k}, \qquad \tilde{G}(x) = \sum_{k=0}^{\infty} \tilde{g}_k x^{-k},$$

whose r-th power can be expressed as

$$[G(x)]^r = \sum_{j=0}^{\infty} P[j, r, \mathbf{g}]x^{-j}, \qquad [\tilde{G}(x)]^r = \sum_{j=0}^{\infty} P[j, r, \tilde{\mathbf{g}}]x^{-j}.$$

Connection between coefficients in the expansion of the power of functions G and $\tilde{G}$ can be established using underlying series A_1, the generating function of a sequence $\overline{\mathbf{a}} = (a_1, 0, a_2, 0, \ldots)$:

$$A_1(x) = \sum_{k=0}^{\infty} a_{k+1} x^{-2k}.$$

It holds

$$\begin{aligned}[G(x)]^r &= (1 + x^{-1}A_1(x))^r = \sum_{k=0}^{\infty} \binom{r}{k} x^{-k} A_1(x)^k \\ &= \sum_{k=0}^{\infty} \binom{r}{k} x^{-k} \sum_{l=0}^{\infty} P[l, k, \overline{\mathbf{a}}]x^{-2l} = \sum_{j=0}^{\infty} \sum_{l=0}^{\lfloor \frac{j}{2} \rfloor} \binom{r}{j-2l} P[l, j-2l, \overline{\mathbf{a}}]x^{-j}\end{aligned}$$

and

$$\begin{aligned}[\tilde{G}(x)]^r &= (1 - x^{-1}A_1(x))^r = \sum_{k=0}^{\infty} \binom{r}{k} (-1)^k x^{-k} A_1(x)^k \\ &= \sum_{k=0}^{\infty} \binom{r}{k} (-1)^k x^{-k} \sum_{l=0}^{\infty} P[l, k, \overline{\mathbf{a}}]x^{-2l} = \sum_{j=0}^{\infty} (-1)^j \sum_{l=0}^{\lfloor \frac{j}{2} \rfloor} \binom{r}{j-2l} P[l, j-2l, \overline{\mathbf{a}}]x^{-j}\end{aligned}$$

wherefrom it follows that

$$P[j, r, \mathbf{g}] = (-1)^j P[j, r, \tilde{\mathbf{g}}]. \tag{15}$$

Furthermore, let

$$H(x) = \sum_{k=0}^{\infty} h_k x^{-k}, \qquad \tilde{H}(x) = \sum_{k=0}^{\infty} \tilde{h}_k x^{-k},$$

and also

$$[H(x)]^r = \sum_{j=0}^{\infty} P[j, r, \mathbf{h}] x^{-j}, \qquad [\tilde{H}(x)]^r = \sum_{j=0}^{\infty} P[j, r, \tilde{\mathbf{h}}] x^{-j}.$$

If A_2 denotes the generating function of a sequence $\tilde{\mathbf{a}} = (2, 0, a_1, 0, a_2, \ldots)$:

$$A_2(x) = 2 + \sum_{k=1}^{\infty} a_k x^{-2k},$$

then the r-th power of functions H and $\tilde{H}$ can be written as

$$\begin{aligned}
&[H(x)]^r = (A_2(x) - x^{-1})^r = A_2(x)^r (1 - x^{-1} A_2(x)^{-1})^r \\
&= \sum_{k=0}^{\infty} \binom{r}{k} (-1)^k x^{-k} A_2(x)^{r-k} = \sum_{k=0}^{\infty} \binom{r}{k} (-1)^k x^{-k} \sum_{l=0}^{\infty} P[l, r-k, \tilde{\mathbf{a}}] x^{-2l} \\
&= \sum_{j=0}^{\infty} (-1)^j \sum_{l=0}^{\lfloor \frac{j}{2} \rfloor} \binom{r}{j-2l} P[l, r+2l-j, \tilde{\mathbf{a}}] x^{-j}
\end{aligned}$$

and similarly

$$\begin{aligned}
[\tilde{H}(x)]^r &= A_2(x)^r (1 + x^{-1} A_2(x)^{-1})^r = \sum_{k=0}^{\infty} \binom{r}{k} x^{-k} A_2(x)^{r-k} \\
&= \sum_{k=0}^{\infty} \binom{r}{k} x^{-k} \sum_{l=0}^{\infty} P[l, r-k, \tilde{\mathbf{a}}] x^{-2l} = \sum_{j=0}^{\infty} \sum_{l=0}^{\lfloor \frac{j}{2} \rfloor} \binom{r}{j-2l} P[l, r+2l-j, \tilde{\mathbf{a}}] x^{-j}
\end{aligned}$$

wherefrom it follows that

$$P[j, r, \mathbf{h}] = (-1)^j P[j, r, \tilde{\mathbf{h}}]. \tag{16}$$

Combining relations (15) and (16) with the definitions of D and S ((11) and (12)) gives the relations (13) and (14). □

3. MAIN RESULTS

In all three notions, stable, stabilizable and stabilized, the similar composition appears. The following theorem establishes the asymptotic expansion of the resultant mean–map of K, N and M:

$$R(x-t,x+t) \coloneqq \mathcal{R}(K,N,M)(x-t,x+t) \sim \sum_{m=0}^{\infty} a_m^R t^{2m} x^{-2m+1}. \tag{17}$$

Afterwards, this composition will be used with $R = K = N = M$ to obtain the asymptotic expansion of stable mean M, with $R = N$ to obtain the asymptotic expansion of (K,M)-stabilizable mean N, and with $R = M$ to obtain the asymptotic expansion of (K,N)-stabilized mean M.

Theorem 7. *Let homogeneous symmetric means M, N and K have the asymptotic expansions* (7)*,* (8) *and* (9)*. Then the coefficients $(a_m^R)_{m\in\mathbb{N}_0}$ in the asymptotic expansion* (17) *are given by the formula:*

$$a_m^R = \sum_{n=0}^{m} a_n^K \sum_{k=0}^{m-n} P[k,2n,\mathbf{d}]P[m-n-k,-2n+1,\mathbf{s}], \quad m\in\mathbb{N}_0, \tag{18}$$

where $\mathbf{d} = (d_m)_{m\in\mathbb{N}_0}$, $\mathbf{s} = (s_m)_{m\in\mathbb{N}_0}$, with

(19)
$$d_m = -\frac{1}{2}\sum_{n=0}^{m} a_n^N \sum_{k=0}^{2m+1-2n} P[k,2n,\mathbf{g}^M]P[2m+1-2n-k,-2n+1,\mathbf{h}^M], \quad m\in\mathbb{N}_0,$$

(20)
$$s_m = \frac{1}{2}\sum_{n=0}^{m} a_n^N \sum_{k=0}^{2m-2n} P[k,2n,\mathbf{g}^M]P[2m-2n-k,-2n+1,\mathbf{h}^M], \quad m\in\mathbb{N}_0,$$

and $\mathbf{g}^M$ and $\mathbf{h}^M$ are defined as in (10) *with $a_m = a_m^M$.*

Proof. First, we shall start from the composition $N\big(x-t, M(x-t,x+t)\big)$. We write arguments $x-t$ and $M = M(x-t,x+t)$ in form of difference and sum of terms $\frac{1}{2}(M-x+t)$ and $\frac{1}{2}(M+x-t)$. Then we apply expansion (8), substitute M with its asymptotic expansion (7), use Lemma 4 and rearrange sums to obtain the following:

(21)

$$\begin{aligned} &N\big(x-t, M(x-t,x+t)\big)\\ &= N\Big(\tfrac{1}{2}(M+x-t) - \tfrac{1}{2}(M-x+t), \tfrac{1}{2}(M+x-t) + \tfrac{1}{2}(M-x+t)\Big) \end{aligned}$$

$$\sim \sum_{n=0}^{\infty} a_n^N \Big(\tfrac{1}{2}(M-x+t)\Big)^{2n} \Big(\tfrac{1}{2}(M+x-t)\Big)^{-2n+1}$$
$$\sim \frac{1}{2}\sum_{n=0}^{\infty} a_n^N \Big(1+\sum_{k=1}^{\infty} a_k^M t^{2k-1}x^{-2k+1}\Big)^{2n} \Big(2-\frac{t}{x}+\sum_{j=1}^{\infty} a_j^M t^{2j}x^{-2j}\Big)^{-2n+1} t^{2n}x^{-2n+1}$$
$$\sim \frac{1}{2}\sum_{n=0}^{\infty} a_n^N \sum_{k=0}^{\infty} P[k,2n,\mathbf{g}^M]t^k x^{-k} \sum_{j=0}^{\infty} P[j,-2n+1,\mathbf{h}^M]t^j x^{-j} t^{2n}x^{-2n+1}$$
$$\sim \frac{1}{2}\sum_{m=0}^{\infty}\sum_{n=0}^{\lfloor \frac{m}{2}\rfloor} a_n^N \sum_{k=0}^{m-2n} P[k,2n,\mathbf{g}^M]P[m-2n-k,-2n+1,\mathbf{h}^M]t^m x^{-m+1}.$$

With similar procedure, we have

(22)

$$N\big(M(x-t,x+t),x+t\big) \sim \sum_{n=0}^{\infty} a_n^N \Big(\tfrac{1}{2}(x+t-M)\Big)^{2n} \Big(\tfrac{1}{2}(x+t+M)\Big)^{-2n+1}$$
$$\sim \frac{1}{2}\sum_{n=0}^{\infty} a_n^N \Big(1-\sum_{k=1}^{\infty} a_k^M t^{2k-1}x^{-2k+1}\Big)^{2n} \Big(2+\frac{t}{x}+\sum_{j=1}^{\infty} a_j^M t^{2j}x^{-2j}\Big)^{-2n+1} t^{2n}x^{-2n+1}$$
$$\sim \frac{1}{2}\sum_{n=0}^{\infty} a_n^N \sum_{k=0}^{\infty} P[k,2n,\tilde{\mathbf{g}}^M]t^k x^{-k} \sum_{j=0}^{\infty} P[j,-2n+1,\tilde{\mathbf{h}}^M]t^j x^{-j} t^{2n}x^{-2n+1}$$
$$\sim \frac{1}{2}\sum_{m=0}^{\infty}\sum_{n=0}^{\lfloor \frac{m}{2}\rfloor} a_n^N \sum_{k=0}^{m-2n} P[k,2n,\tilde{\mathbf{g}}^M]P[m-2n-k,-2n+1,\tilde{\mathbf{h}}^M]t^m x^{-m+1}.$$

Now the left hand side in (17) can be written as

$$K(X-T,X+T)=\sum_{n=0}^{\infty} a_n^K T^{2n}X^{-2n+1}, \tag{23}$$

where X and T are such that their difference equals $N(x-t,M)$ and their sum equals $N(M,x+t)$, with $M=M(x-t,x+t)$. We may further analyze T and X. With use of (21), (22), (11), (13) and (19) we obtain the following

(24)

$$T=\frac{1}{2}\big(N(M(x-t,x+t),x+t)-N(x-t,M(x-t,x+t))\big)$$
$$\sim \frac{1}{4}\sum_{m=0}^{\infty}\sum_{n=0}^{\lfloor \frac{m}{2}\rfloor} a_n^N \sum_{k=0}^{m-2n} \big(P[k,2n,\tilde{\mathbf{g}}^M]P[m-2n-k,-2n+1,\tilde{\mathbf{h}}^M]$$
$$- P[k,2n,\mathbf{g}^M]P[m-2n-k,-2n+1,\mathbf{h}^M]\big)t^m x^{-m+1}$$
$$\sim \frac{1}{4}\sum_{m=0}^{\infty}\sum_{n=0}^{\lfloor \frac{m}{2}\rfloor} a_n^N \sum_{k=0}^{m-2n} D(m,n,k)t^m x^{-m+1}$$

$$
\begin{aligned}
&\sim \frac{1}{4}\sum_{m=0}^{\infty}\sum_{n=0}^{m} a_n^N \sum_{k=0}^{2m+1-2n} D(2m+1,n,k)t^{2m+1}x^{-2m}\\
&\sim -\frac{1}{2}\sum_{m=0}^{\infty}\sum_{n=0}^{m} a_n^N \sum_{k=0}^{2m+1-2n} P[k,2n,\mathbf{g}^M]P[2m+1-2n-k,-2n+1,\mathbf{h}^M]t^{2m+1}x^{-2m}\\
&\sim \sum_{m=0}^{\infty} d_m t^{2m+1}x^{-2m}
\end{aligned}
$$

and similarly, with use of (21), (22), (12), (14) and (20) we obtain the following

(25)

$$
\begin{aligned}
X &= \frac{1}{2}\Big(N\big(M(x-t,x+t),x+t\big)+N\big(x-t,M(x-t,x+t)\big)\Big)\\
&\sim \frac{1}{4}\sum_{m=0}^{\infty}\sum_{n=0}^{\lfloor\frac{m}{2}\rfloor} a_n^N \sum_{k=0}^{m-2n} \big(P[k,2n,\tilde{\mathbf{g}}^M]P[m-2n-k,-2n+1,\tilde{\mathbf{h}}^M]\\
&\qquad\qquad + P[k,2n,\mathbf{g}^M]P[m-2n-k,-2n+1,\mathbf{h}^M]\big)t^m x^{-m+1}\\
&\sim \frac{1}{4}\sum_{m=0}^{\infty}\sum_{n=0}^{\lfloor\frac{m}{2}\rfloor} a_n^N \sum_{k=0}^{m-2n} S(m,n,k)t^m x^{-m+1}\\
&\sim \frac{1}{4}\sum_{m=0}^{\infty}\sum_{n=0}^{m} a_n^N \sum_{k=0}^{2m-2n} S(2m,n,k)t^{2m}x^{-2m+1}\\
&\sim \frac{1}{2}\sum_{m=0}^{\infty}\sum_{n=0}^{m} a_n^N \sum_{k=0}^{2m-2n} P[k,2n,\mathbf{g}^M]P[2m-2n-k,-2n+1,\mathbf{h}^M]t^{2m}x^{-2m+1}\\
&\sim \sum_{m=0}^{\infty} s_m t^{2m}x^{-2m+1}.
\end{aligned}
$$

Finally, from (17), by including expansions of K (9), T (24) and X (25) in (23), then using Lemma 4 and rearranging sums we obtain:

$$
\begin{aligned}
R &= K(X-T,X+T)\\
&\sim \sum_{n=0}^{\infty} a_n^K \Big(\sum_{k=0}^{\infty} d_k t^{2k+1}x^{-2k}\Big)^{2n}\Big(\sum_{j=0}^{\infty} s_j t^{2j}x^{-2j+1}\Big)^{-2n+1}\\
&\sim \sum_{n=0}^{\infty} a_n^K \Big(\sum_{k=0}^{\infty} d_k t^{2k}x^{-2k}\Big)^{2n}\Big(\sum_{j=0}^{\infty} s_j t^{2j}x^{-2j}\Big)^{-2n+1} t^{2n}x^{-2n+1}\\
&\sim \sum_{n=0}^{\infty} a_n^K \sum_{k=0}^{\infty} P[k,2n,\mathbf{d}]t^{2k}x^{-2k}\sum_{j=0}^{\infty} P[j,-2n+1,\mathbf{s}]t^{2j}x^{-2j}t^{2n}x^{-2n+1}
\end{aligned}
$$

$$\sim \sum_{m=0}^{\infty} \sum_{n=0}^{m} a_n^K \sum_{k=0}^{m-n} P[k, 2n, \mathbf{d}] P[m-n-k, -2n+1, \mathbf{s}] t^{2m} x^{-2m+1}.$$

$\square$

Remark 8. Asymptotic expansion of the composite mean $K(N_1, N_2)$ has been derived by Burić and Elezović ([**6**]) for two types of asymptotic power series, general ($\sum_{n=0}^{\infty} \gamma_n t^n x^{-n+1}$) in Theorem 2.1. and symmetric ($\sum_{n=0}^{\infty} \gamma_n t^{2n} x^{-2n+1}$) in Theorem 2.2. In our case means $N_1(s,t) = N(s, M(s,t))$ and $N_2(s,t) = N(M(s,t), t)$ are not symmetric and would require non-symmetric treatment. But due to the specificity of the means N_1 and N_2, which is that while their difference is antisymmetric, their sum is symmetric, in the end the composition $K(N_1, N_2)$ is symmetric for symmetric means K, N and M. Symmetric form of the asymptotic expansion of $K(N_1, N_2)$ would be difficult to deduce just by applying Theorem 2.1. from the above mentioned paper so we needed to conduct the similar procedure starting from the beginning in order to obtain the desired result.

According to Theorem 7, first few coefficients a_m^R are as follows:

(26)
$$\begin{aligned}
a_0^R &= 1,\\
a_1^R &= \tfrac{1}{4}\big(a_1^K + 2a_1^M + a_1^N\big),\\
a_2^R &= \tfrac{1}{16}\Big(a_2^K + 8a_2^M + a_1^N + 2a_1^M(1+2a_1^M)a_1^N - a_1^K\big(3a_1^N + a_1^M(2+8a_1^N)\big) + a_2^N\Big),\\
a_3^R &= \tfrac{1}{64}\Big(a_3^K + 32a_3^M + \big(1 - 2a_1^M(1+2a_1^M)^2 + 8a_2^M + 32a_1^M a_2^M\big)a_1^N\\
&\quad - a_2^K\big(7a_1^N + 2a_1^M(3+8a_1^N)\big) + a_1^K\big(a_1^N(-3+4a_1^N) - 8a_2^M(1+4a_1^N)\\
&\quad + 4(a_1^M)^2(1+a_1^N)(1+4a_1^N) + 2a_1^M\big(a_1^N(3+8a_1^N) - 8a_2^N\big) - 7a_2^N\big)\\
&\quad + 6a_2^N + 6a_1^M(3+4a_1^M)a_2^N + a_3^N\Big).
\end{aligned}$$

3.1 Stable means

In order to obtain asymptotic expansion of stable mean M we need to use Theorem 7 with $R = N = K = M$. The idea is to express coefficient $a_m = a_m^M$ using lower terms, i.e. in form of recursive relation.

Theorem 9. *Let homogeneous symmetric stable mean M have the asymptotic expansion* (5) *with $a_m = a_m^M$. Then $a_0 = 1$, $a_1 \in \mathbb{R}$ and for $m \geq 2$ coefficients a_m are given by the recursive formula:*

(27)
$$\begin{aligned}
a_m = \frac{2^{2m-1}}{2^{2m-2}-1}\Bigg(&\frac{1}{2}\sum_{n=1}^{m-1} a_n \sum_{k=0}^{2m-2n} P[k, 2n, \mathbf{g}] P[2m-2n-k, -2n+1, \mathbf{h}]\\
&+ \sum_{n=1}^{m-1} a_n \sum_{k=0}^{m-n} P[k, 2n, \mathbf{d}] P[m-n-k, -2n+1, \mathbf{s}]\Bigg), \quad m \geq 2,
\end{aligned}$$

where $\mathbf{g}$ *and* $\mathbf{h}$ *are defined in* (10) *and* $\mathbf{d}$ *and* $\mathbf{s}$ *are defined by* (19) *and* (20).

Proof. Proof is based on definition formula (2) with variables $x-t$ and $x+t$:

$$M(x-t,x+t) = M\Big(M\big(x-t, M(x-t,x+t)\big), M\big(M(x-t,x+t), x+t\big)\Big). \tag{28}$$

Proof is divided into three parts. The asymptotic expansion of the left hand side has the form (5) while the coefficients in the asymptotic expansion of the right hand side will be obtained as a consequence of Theorem 7. Then term a_m with the highest index will be identified. The corresponding coefficients will be equated wherefrom the recursive formula (27) will be deduced.

I. Asymptotic expansion of the right-hand side of (28). From Theorem 7, with $a_m = a_m^M = a_m^N = a_m^K$, we have

$$a_m^R = \sum_{n=0}^{m} a_n \sum_{k=0}^{m-n} P[k, 2n, \mathbf{d}] P[m-n-k, -2n+1, \mathbf{s}], \quad m \in \mathbb{N}_0, \tag{29}$$

where

$$d_m = -\frac{1}{2}\sum_{n=0}^{m} a_n \sum_{k=0}^{2m+1-2n} P[k, 2n, \mathbf{g}] P[2m+1-2n-k, -2n+1, \mathbf{h}], \quad m \in \mathbb{N}_0, \tag{30}$$

and

$$s_m = \frac{1}{2}\sum_{n=0}^{m} a_n \sum_{k=0}^{2m-2n} P[k, 2n, \mathbf{g}] P[2m-2n-k, -2n+1, \mathbf{h}], \quad m \in \mathbb{N}_0. \tag{31}$$

II. Extracting higher indexed term a_m. In this step of the proof we shall detect the higher indexed term of the sequence $\mathbf{a}$ contained in a_m^R. Simple computations reveal that $a_0^R = 1$ and $a_1^R = a_1$ as can also be seen from the list of coefficients (26). For $m \geq 2$ we may divide the sum on the right hand side of (29) into three parts, $n = 0$, $n \in \{1, \dots, m-1\}$ and $n = m$:

$$\begin{aligned} a_m^R = a_0 &\sum_{k=0}^{m} P[k, 0, \mathbf{d}] P[m-k, 1, \mathbf{s}] \\ &+ \sum_{n=1}^{m-1} a_n \sum_{k=0}^{m-n} P[k, 2n, \mathbf{d}] P[m-n-k, -2n+1, \mathbf{s}] \\ &+ a_m P[0, 2m, \mathbf{d}] P[0, -2m+1, \mathbf{s}]. \end{aligned}$$

According to Remark 5, $P[k, 0, \mathbf{d}] = \delta_k$, $P[m, 1, \mathbf{s}] = s_m$, $P[0, 2m, \mathbf{d}] = d_0^{2m}$ and $P[0, -2m+1, \mathbf{s}] = s_0^{-2m+1}$ and hence

$$a_m^R = a_0 s_m + \sum_{n=1}^{m-1} a_n \sum_{k=0}^{m-n} P[k, 2n, \mathbf{d}] P[m-n-k, -2n+1, \mathbf{s}] + a_m d_0^{2m} s_0^{-2m+1}.$$

Using formula (31) for s_m and substituting a_0, d_0 and s_0 with 1, $\frac{1}{2}$ and 1 respectively, we obtain

$$\begin{aligned} a_m^R = \frac{1}{2}\sum_{n=0}^{m} a_n \sum_{k=0}^{2m-2n} P[k,2n,\mathbf{g}]P[2m-2n-k,-2n+1,\mathbf{h}] \\ + \sum_{n=1}^{m-1} a_n \sum_{k=0}^{m-n} P[k,2n,\mathbf{d}]P[m-n-k,-2n+1,\mathbf{s}] + 2^{-2m}a_m. \end{aligned}$$

We will divide the first sum into three parts, $n=0$, $n\in\{1,\dots,m-1\}$ and $n=m$, where for $n=0$ we have

$$\frac{1}{2}a_0\sum_{k=0}^{2m} P[k,0,\mathbf{g}]P[2m-k,1,\mathbf{h}] = \frac{1}{2}a_0\sum_{k=0}^{2m} h_{2m}\delta_k = \frac{1}{2}a_0\sum_{k=0}^{2m} a_m\delta_k = \frac{1}{2}a_m,$$

and for $n=m$ we have

$$\frac{1}{2}a_m P[0,2m,\mathbf{g}]P[0,-2m+1,\mathbf{h}] = \frac{1}{2}a_m g_0^{2m} h_0^{-2m+1} = a_m 2^{-2m}.$$

Now we continue to analyze a_m^R with that information included and terms with a_m grouped together:

$$\begin{aligned} a_m^R &= (2^{-1}+2^{-2m+1})a_m \\ &+ \frac{1}{2}\sum_{n=1}^{m-1} a_n \sum_{k=0}^{2m-2n} P[k,2n,\mathbf{g}]P[2m-2n-k,-2n+1,\mathbf{h}] \\ &+ \sum_{n=1}^{m-1} a_n \sum_{k=0}^{m-n} P[k,2n,\mathbf{d}]P[m-n-k,-2n+1,\mathbf{s}]. \end{aligned} \tag{32}$$

Let $i_{max}(\cdot)$ denote the highest index i such that a_i appears within term inside the parenthesis. That is,

$$i_{max}(g_k) = \lfloor\frac{k+1}{2}\rfloor, \qquad i_{max}(h_k) = \lfloor\frac{k}{2}\rfloor.$$

In (32), $P[k,2n,\mathbf{g}]$ according to Lemma 4 depends only on finite sequence $(g_0,\dots,g_k)$ and hence

$$i_{max}(P[k,2n,\mathbf{g}]) = \max_{j\in\{0,\dots,k\}}(i_{max}(g_j)) = \lfloor\frac{k+1}{2}\rfloor.$$

Also $P[2m-2n-k,-2n+1,\mathbf{h}]$ from the same formula depends only on finite sequence $(h_0,\dots,h_{2m-2n-k})$ and hence

$$i_{max}(P[2m-2n-k,-2n+1,\mathbf{h}]) = \max_{j\in\{0,\dots,2m-2n-k\}}(i_{max}(h_j)) = \lfloor m-n-\frac{k}{2}\rfloor.$$

The highest index that appears in sum in the second row of (32) is

$$\begin{aligned}
&i_{max}\Big(\sum_{n=1}^{m-1} a_n \sum_{k=0}^{2m-2n} P[k,2n,\mathbf{g}]P[2m-2n-k,-2n+1,\mathbf{h}]\Big)\\
&\leq \max_{\substack{n\in\{1,\dots,m-1\}\\ k\in\{0,\dots,2m-2n\}}} (m-1, i_{max}(P[k,2n,\mathbf{g}]), i_{max}(P[2m-2n-k,-2n+1,\mathbf{h}])\\
&= \max_{\substack{n\in\{1,\dots,m-1\}\\ k\in\{0,\dots,2m-2n\}}} \Big(m-1, \lfloor\frac{k+1}{2}\rfloor, \lfloor m-n-\frac{k}{2}\rfloor\Big) = m-1.
\end{aligned}$$

Regarding the third row of (32) first we observe d_m. From formula (30) we have

$$\begin{aligned}
-2d_m &= a_0 \sum_{k=0}^{2m+1} P[k,0,\mathbf{g}]P[2m+1-k,1,\mathbf{h}]\\
&\quad + \sum_{n=1}^{m} a_n \sum_{k=0}^{2m+1-2n} P[k,2n,\mathbf{g}]P[2m+1-2n-k,-2n+1,\mathbf{h}]\\
&= h_{2m+1} + \sum_{n=1}^{m} a_n \sum_{k=0}^{2m+1-2n} P[k,2n,\mathbf{g}]P[2m+1-2n-k,-2n+1,\mathbf{h}],
\end{aligned}$$

and hence

$$i_{max}(d_m) \leq \max(i_{max}(h_{2m+1}), m, i_{max}(g_{2m-1}), i_{max}(h_{2m-1})) = m.$$

From the discussion before we may also see that

$$i_{max}(s_m) = m.$$

Combining derived relations finally gives

$$\begin{aligned}
&i_{max}\Big(\sum_{n=1}^{m-1} a_n \sum_{k=0}^{m-n} P[k,2n,\mathbf{d}]P[m-n-k,-2n+1,\mathbf{s}]\Big)\\
&\leq \max_{\substack{n\in\{1,\dots,m-1\}\\ k\in\{0,\dots,m-n\}}} (m-1, i_{max}(d_k), i_{max}(s_{m-n-k}))\\
&\leq \max_{\substack{n\in\{1,\dots,m-1\}\\ k\in\{0,\dots,m-n\}}} (m-1, k, m-n-k)\\
&= m-1.
\end{aligned}$$

III. Equating coefficients from the left-hand and the right-hand side of (28). For a stable mean M, the expansions of the left and right side in (28) must be equal, that is, $a_m = a_m^R$ for $m \in \mathbb{N}_0$. Coefficient $a_0^R = 1$ which is in agreement with property (1). Next, we have free coefficient a_1. Furthermore, the connection between a_m^R and a_m for $m \geq 2$ in (32) is also linear so equating those coefficients (32) finally gives the relation (27) which completes the proof. □

For convenience, we give here the first few coefficients a_m^R:

$$
\begin{aligned}
a_0^R &= 1,\\
a_1^R &= a_1,\\
a_2^R &= \tfrac{1}{16}a_1(1+a_1)(1-4a_1) + \tfrac{5}{8}a_2,\\
a_3^R &= \tfrac{1}{64}(1+a_1)\Big(a_1\big(1+2a_1(-3+6a_1+8a_1^2)-8a_2\big)+6a_2\Big) + \tfrac{17}{32}a_3,\\
a_4^R &= \tfrac{1}{256}\Big(-56a_1^5 - 48a_1^6 + 33a_2^2 + 24a_1^4(1+10a_2)\\
&\qquad + a_1^3(22+300a_2) + 15(a_2+a_3) + 3a_1^2(-3+8a_2+4a_3)\\
&\qquad + a_1\big(1+3a_2(-7+32a_2)+18a_3\big)\Big) + \tfrac{65}{128}a_4.
\end{aligned} \tag{33}
$$

For a stable mean coefficients a_m^R must be equal to a_m. Using (27) we obtain asymptotic expansion of a stable mean. With successive substitutions done, all the subsequent coefficients can be seen as polynomials in variable a_1. Asymptotic expansion up to five terms of a symmetric, homogeneous stable mean in variables $(x-t, x+t)$ has the form:

$$
\begin{aligned}
&M(x-t,x+t) = x + a_1t^2x^{-1} + \tfrac{1}{6}a_1(1+a_1)(1-4a_1)t^4x^{-3}\\
&\quad + \tfrac{1}{90}a_1(1+a_1)\left(6-31a_1+36a_1^2+64a_1^3\right)t^6x^{-5}\\
&\quad + \tfrac{1}{2520}a_1(1+a_1)\big(90-531a_1+937a_1^2+568a_1^3-3088a_1^4-2176a_1^5\big)t^8x^{-7}\\
&\quad + \mathcal{O}(x^{-9}).
\end{aligned} \tag{34}
$$

Proposition 10. *Bi-variate homogeneous symmetric mean M with asymptotic expansion* (5) *has the asymptotic expansion* (6) *where for $s \neq \pm t$*

$$
a_m(s,t) = 2^{-m}\sum_{n=0}^{\lfloor \frac{m}{2} \rfloor} a_n \binom{1-2n}{m-2n} (t-s)^{2n}(t+s)^{m-2n}, \quad m \in \mathbb{N}_0.
$$

Proof. Let $\alpha = \frac{t+s}{2}$ and $\beta = \frac{t-s}{2}$. Then

$$
\begin{aligned}
M(x+s,x+t) &= M(x+\alpha-\beta, x+\alpha+\beta)\\
&= \sum_{n=0}^{\infty} a_n\beta^{2n}(x+\alpha)^{-2n+1}\\
&= \sum_{n=0}^{\infty} a_n\beta^{2n}x^{-2n+1}\sum_{k=0}^{\infty}\binom{-2n+1}{k}\alpha^k x^{-k}\\
&= \sum_{m=0}^{\infty}\sum_{n=0}^{\lfloor \frac{m}{2} \rfloor} a_n\binom{1-2n}{m-2n}\beta^{2n}\alpha^{m-2n}x^{-m+1}
\end{aligned}
$$

and the proof is complete. $\square$

Combining Proposition 10 with asymptotic expansion (34) we get the following result.

Corollary 11. *For a homogeneous symmetric stable mean M with asymptotic expansion* (5) *holds*

$$\begin{aligned}M(x+s,x+t) &= x+\tfrac{1}{2}(s+t)+\tfrac{1}{4}(s-t)^2a_1x^{-1}-\tfrac{1}{8}(s-t)^2(s+t)a_1x^{-2}\\&+\tfrac{1}{16}\left(a_1\left(s^2-t^2\right)^2-\tfrac{1}{6}a_1(a_1+1)(4a_1-1)(s-t)^4\right)x^{-3}\\&+\tfrac{1}{64}a_1(s-t)^2(s+t)\left((a_1+1)(4a_1-1)(s-t)^2-2(s+t)^2\right)x^{-4}\\&+\tfrac{1}{64}\Big((s-t)^2(s+t)^4a_1-(s-t)^4(s+t)^2a_1(1+a_1)(-1+4a_1)\\&\quad+\tfrac{1}{90}(s-t)^6a_1(1+a_1)(6+a_1(-31+4a_1(9+16a_1)))\Big)x^{-5}+\mathcal{O}(x^{-6}).\end{aligned}$$

3.2 Stabilizable means

Theorem 12. *Let homogeneous symmetric means M, N and K have the asymptotic expansions* (7), (8) *and* (9). *Suppose K and M are stable means. Then the coefficients $(a_m^N)_{m\in\mathbb{N}_0}$ in the asymptotic expansion* (8) *of (K,M)-stabilizable mean N are given by:*

$$a_0^N = 1,$$

$$\begin{aligned}a_m^N &= \frac{2^{2m}}{2^{2m}-1}\Bigg[\frac{1}{2}\sum_{n=0}^{m-1}a_n^N\sum_{k=0}^{2m-2n}P[k,2n,\mathbf{g}^M]P[2m-2n-k,-2n+1,\mathbf{h}^M]\\&\quad+\sum_{n=1}^{m}a_n^K\sum_{k=0}^{m-n}P[k,2n,\mathbf{d}]P[m-n-k,-2n+1,\mathbf{s}]\Bigg],\qquad m\in\mathbb{N},\end{aligned}\tag{35}$$

where $\mathbf{g}^M$ and $\mathbf{h}^M$ are defined in (10) *with $a_m = a_m^M$ and $\mathbf{d}$ and $\mathbf{s}$ are defined by* (19) *and* (20).

Proof. From Theorem 7, with $R = N$ and thereby $a_m^R = a_m^N$, $m \in \mathbb{N}_0$, we have

$$a_m^N = \sum_{n=0}^{m}a_n^K\sum_{k=0}^{m-n}P[k,2n,\mathbf{d}]P[m-n-k,-2n+1,\mathbf{s}],\quad m\in\mathbb{N}_0.\tag{36}$$

For $m = 0$ in equation (36) we obtain that $a_0^N = 1$ as expected. Now let $m > 0$. First, we divide the sum into two parts, for $n = 0$ and $n \in \{1,\dots,m\}$:

$$a_m^N = a_0^K\sum_{k=0}^{m}P[k,0,\mathbf{d}]P[m-k,1,\mathbf{s}]+\sum_{n=1}^{m}a_n^K\sum_{k=0}^{m-n}P[k,2n,\mathbf{d}]P[m-n-k,-2n+1,\mathbf{s}].$$

According to Remark 5, the first part reduces and we have

$$a_m^N = s_m+\sum_{n=1}^{m}a_n^K\sum_{k=0}^{m-n}P[k,2n,\mathbf{d}]P[m-n-k,-2n+1,\mathbf{s}].$$

Now we use formula (20) for s_m

$$a_m^N = \frac{1}{2}\sum_{n=0}^{m} a_n^N \sum_{k=0}^{2m-2n} P[k, 2n, \mathbf{g}^M] P[2m-2n-k, -2n+1, \mathbf{h}^M]$$
$$+ \sum_{n=1}^{m} a_n^K \sum_{k=0}^{m-n} P[k, 2n, \mathbf{d}] P[m-n-k, -2n+1, \mathbf{s}],$$

then divide the sum into two parts, for $n = m$ and $n \in \{0, \dots, m-1\}$, and apply conclusions from Remark 5. That is,

$$a_m^N = \frac{1}{2} a_m^N P[0, 2m, \mathbf{g}^M] P[0, -2m+1, \mathbf{h}^M]$$
$$+ \frac{1}{2}\sum_{n=0}^{m-1} a_n^N \sum_{k=0}^{2m-2n} P[k, 2n, \mathbf{g}^M] P[2m-2n-k, -2n+1, \mathbf{h}^M]$$
$$+ \sum_{n=1}^{m} a_n^K \sum_{k=0}^{m-n} P[k, 2n, \mathbf{d}] P[m-n-k, -2n+1, \mathbf{s}],$$

which with $g_0 = 1$ and $h_0 = 2$ reads as

$$a_m^N = 2^{-2m} a_m^N + \frac{1}{2}\sum_{n=0}^{m-1} a_n^N \sum_{k=0}^{2m-2n} P[k, 2n, \mathbf{g}^M] P[2m-2n-k, -2n+1, \mathbf{h}^M]$$
$$+ \sum_{n=1}^{m} a_n^K \sum_{k=0}^{m-n} P[k, 2n, \mathbf{d}] P[m-n-k, -2n+1, \mathbf{s}]$$

wherefrom (35) follows easily. □

Using formula (35) from Theorem 12 we obtain the coefficients of a mean N such that (3) holds for given means K and M. Here is a list of the first few such coefficients:

$$\begin{aligned}
a_0^N &= 1,\\
a_1^N &= \tfrac{1}{3}\big(a_1^K + 2a_1^M\big),\\
a_2^N &= \tfrac{1}{45}\Big(-2a_1^K(6a_1^M + 5)a_1^M - (a_1^K)^2(8a_1^M + 3) + a_1^K + 3a_2^K\\
&\qquad + 8(a_1^M)^3 + 4(a_1^M)^2 + 2a_1^M + 24a_2^M\Big)\\
a_3^N &= \tfrac{1}{2835}\Big((a_1^K)^3\big(8a_1^M(26a_1^M + 23) + 41\big)\\
&\qquad + 2(a_1^K)^2\big(a_1^M(4a_1^M(40a_1^M + 81) + 61) - 5(48a_2^M + 7)\big)\\
&\qquad - a_1^K\big(18a_2^K(16a_1^M + 7) + 408a_2^M + 16a_1^M(2a_1^M(a_1^M(3a_1^M - 7) - 1)\\
&\qquad + 54a_2^M + 11) - 21\big) - 6a_2^K\big(a_1^M(68a_1^M + 71) - 3\big) + 45a_3^K
\end{aligned}$$

$$+ 6a_1^M\big(32(a_1^M)^4 - 12(a_1^M)^2 + 16(16a_1^M + 7)a_2^M + 7\big) + 144(a_2^M + 10a_3^M)\Big).$$

If we require that mean N is (K,M)-stabilizable then we must take into account stability of K and M. When means K and M are stable, coefficients a_m^K and a_m^M, $m \geq 2$, can be expressed as polynomials in variables a_1^K and a_1^M respectively. From expansion (34) used with coefficients a_m^K and a_m^M we obtain coefficients a_m^N with K and M stable:

$$\begin{aligned}
a_0^N &= 1,\\
a_1^N &= \tfrac{1}{3}\left(a_1^K + 2a_1^M\right),\\
a_2^N &= \tfrac{1}{90}\Big(-(a_1^K)^2(16a_1^M + 9) + a_1^K\big(3 - 4a_1^M(6a_1^M + 5)\big)\\
&\qquad - 4(a_1^K)^3 - 4a_1^M\big(4a_1^M(a_1^M + 1) - 3\big)\Big)\\
a_3^N &= \tfrac{1}{5670}\Big(64(a_1^K)^5 + 4(a_1^K)^4(67 + 96a_1^M)\\
&\qquad + 12a_1^M(-1 + 2a_1^M)(3 + 2a_1^M)\big(-9 + 8a_1^M(1 + a_1^M)\big)\\
&\qquad + 3(a_1^K)^3\big(63 + 8a_1^M(51 + 40a_1^M)\big)\\
&\qquad + (a_1^K)^2\big(-225 + 2a_1^M(207 + 4a_1^M(273 + 160a_1^M))\big)\\
&\qquad + 2a_1^K\big(27 + a_1^M\left(-315 + 8a_1^M\left(3 + 4a_1^M(29 + 15a_1^M)\right)\right)\big)\Big).
\end{aligned}\tag{37}$$

3.3 Stabilized means

Theorem 13. *Let homogeneous symmetric means M, N and K have the asymptotic expansions* (7), (8) *and* (9). *Suppose K and N are stable means. Then the coefficients $(a_m^M)_{m\in\mathbb{N}_0}$ in the asymptotic expansion* (7) *of (K,N)-stabilized mean M are given by:*

$$\begin{aligned}
a_0^M &= 1,\\
a_m^M &= \sum_{n=1}^{m} a_n^N \sum_{k=0}^{2m-2n} P[k,2n,\mathbf{g}^M]P[2m-2n-k,-2n+1,\mathbf{h}^M]\\
&\qquad + 2\sum_{n=1}^{m} a_n^K \sum_{k=0}^{m-n} P[k,2n,\mathbf{d}]P[m-n-k,-2n+1,\mathbf{s}], \quad m\in\mathbb{N},
\end{aligned}\tag{38}$$

where $\mathbf{g}^M$ and $\mathbf{h}^M$ are given by (10) *with $a_m = a_m^M$ and $\mathbf{d}$ and $\mathbf{s}$ are defined by* (19) *and* (20).

Proof. Using formula (18) from Theorem 7, with $R = M$ and thereby $a_m^R = a_m^M$, $m \in \mathbb{N}_0$, we obtain

$$a_m^M = \sum_{n=0}^{m} a_n^K \sum_{k=0}^{m-n} P[k,2n,\mathbf{d}]P[m-n-k,-2n+1,\mathbf{s}].$$

For $m = 0$ we obtain $a_0^M = 1$ as expected. Now let $m > 0$. On the right hand side for $n = 0$ we have $a_0^K \sum_{k=0}^{m} P[k, 0, \mathbf{d}]P[m-k, 1, \mathbf{s}]$ where $a_0^K = 1$, $P[k, 0, \mathbf{d}] = \delta_k$ and $P[m, 1, \mathbf{s}] = s_m$. Hence, the sum can be written as

$$a_m^M = s_m + \sum_{n=1}^{m} a_n^K \sum_{k=0}^{m-n} P[k, 2n, \mathbf{d}]P[m-n-k, -2n+1, \mathbf{s}].$$

After replacing s_m according to definition (20) and dividing it into two parts, for $n = 0$ and $n \geq 1$, we obtain

$$\begin{aligned}
a_m^M &= \frac{1}{2}\sum_{n=0}^{m} a_n^N \sum_{k=0}^{2m-2n} P[k, 2n, \mathbf{g}^M]P[2m-2n-k, -2n+1, \mathbf{h}^M] \\
&\quad + \sum_{n=1}^{m} a_n^K \sum_{k=0}^{m-n} P[k, 2n, \mathbf{d}]P[m-n-k, -2n+1, \mathbf{s}] \\
&= \frac{1}{2}a_m^M + \frac{1}{2}\sum_{n=1}^{m} a_n^N \sum_{k=0}^{2m-2n} P[k, 2n, \mathbf{g}^M]P[2m-2n-k, -2n+1, \mathbf{h}^M] \\
&\quad + \sum_{n=1}^{m} a_n^K \sum_{k=0}^{m-n} P[k, 2n, \mathbf{d}]P[m-n-k, -2n+1, \mathbf{s}]
\end{aligned}$$

which after subtracting $\frac{1}{2}a_m^M$ and multiplying by 2 gives the desired result (38). $\square$

Using formula (38) from Theorem 13 we obtain coefficients in the asymptotic expansion of a mean M such that equation (4) holds for a given means K and N. Here are the first few of them:

$$\begin{aligned}
a_0^M &= 1, \\
a_1^M &= \tfrac{1}{2}\big(a_1^K + a_1^N\big), \\
a_2^M &= \tfrac{1}{8}\big(a_2^K + a_1^N + (a_1^N)^2 + (a_1^N)^3 - a_1^K a_1^N(3 + 2a_1^N) - (a_1^K)^2(1 + 3a_1^N) + a_2^N\big), \\
a_3^M &= \tfrac{1}{32}\Big(a_3^K + (a_1^K)^3(1 + 2a_1^N)(2 + 5a_1^N) + (a_1^K)^2\left(a_1^N\left(5 + 6a_1^N(3 + a_1^N)\right) - 2a_2^N\right) \\
&\quad - a_1^K\left(2a_2^K(2 + 5a_1^N) + a_1^N\left(5 + a_1^N\left(2 + a_1^N + 2(a_1^N)^2\right) - 2a_2^N\right) - a_2^N\right) + 6a_2^N \\
&\quad + a_1^N\left(1 - 3a_2^K(3 + 2a_1^N) + 10a_2^N + a_1^N\left(a_1^N + 2(a_1^N)^2(1 + a_1^N) + 8a_2^N\right)\right) + a_3^N\Big).
\end{aligned}$$

In order for M to be (K, N)-stabilized, means K and N need to be stable and therefore their coefficients obey the rule of stable means coefficients (34) and can be expressed through a_1^K and a_1^N. Here are the first few coefficients in the asymptotic expansion of (K, N)-stabilized mean M:

$$
\begin{aligned}
a_0^M &= 1,\\
a_1^M &= \tfrac{1}{2}(a_1^K + a_1^N),\\
a_2^M &= \tfrac{1}{48}\Big(-4(a_1^K)^3 - 9(a_1^K)^2(1+2a_1^N)\\
&\quad + a_1^K\left(1 - 6a_1^N(3+2a_1^N)\right) + a_1^N\left(7 + a_1^N(3+2a_1^N)\right)\Big),\\
a_3^M &= \tfrac{1}{2880}\Big(64(a_1^K)^5 + 20(a_1^K)^4(17+30a_1^N)\\
&\quad + 5(a_1^K)^3\left(73 + 36a_1^N(10+7a_1^N)\right)\\
&\quad + 5(a_1^K)^2\left(-17 + 3a_1^N\left(45 + 44a_1^N(3+a_1^N)\right)\right)\\
&\quad - 3a_1^K\left(-2 + 5a_1^N\left(38 + a_1^N\left(19 + 4a_1^N(4+5a_1^N)\right)\right)\right)\\
&\quad - a_1^N\left(-186 + a_1^N\left(145 + a_1^N\left(595 + 4a_1^N(170+59a_1^N)\right)\right)\right)\Big).
\end{aligned}
\tag{39}
$$

4. SIMULTANEOUSLY STABILIZABLE AND STABILIZED

In this section we will show how Theorems from Section 3 may be used to derive some necessary conditions in the following interesting cases.

4.1 Simultaneously (K, M)-stabilizable and (M, K)-stabilizable

Let K and M be two stable means. If N is (K, M)-stabilizable and (M, K)-stabilizable, then

$$\tfrac{1}{3}(a_1^K + 2a_1^M) = a_1^N = \tfrac{1}{3}(a_1^M + 2a_1^K)$$

wherefrom it follows that $a_1^N = a_1^M = a_1^K$ and, because asymptotic expansion of stable mean is completely determined by the coefficient with index 1, it holds that $a_m^M = a_m^K$ for all $m \in \mathbb{N}_0$. Calculating the next coefficients from the list (37) reveals that

$$
\begin{aligned}
a_2^N &= \tfrac{1}{6}a_1^N(1+a_1^N)(1-4a_1^N),\\
a_3^N &= \tfrac{1}{90}a_1^N(1+a_1^N)\left(6 - 31a_1^N + 36(a_1^N)^2 + 64(a_1^N)^3\right),
\end{aligned}
$$

which have the form of stable mean coefficients (34) suggesting N should also be stable.

4.2 Simultaneously (K, N)-stabilized and (N, K)-stabilized

Let K and N be two stable means. Assume mean M is simultaneously (K, N)-stabilized and (N, K)-stabilized. Observe the list of coefficients (39). Coefficient

$a_1^M = \frac{1}{2}(a_1^K + a_1^N)$ is symmetric in K and N. Furthermore, for $m = 2$ the following equality must hold

$$\begin{aligned} a_2^M &= \tfrac{1}{48}\Big(-4(a_1^K)^3 - 9(a_1^K)^2(1+2a_1^N) + a_1^K\big(1 - 6a_1^N(3+2a_1^N)\big) \\ &\quad + a_1^N\big(7 + a_1^N(3+2a_1^N)\big)\Big) \\ &= \tfrac{1}{48}\Big(-4(a_1^N)^3 - 9(a_1^N)^2(1+2a_1^K) + a_1^N\big(1 - 6a_1^K(3+2a_1^K)\big) \\ &\quad + a_1^K\big(7 + a_1^K(3+2a_1^K)\big)\Big) \end{aligned}$$

wherefrom it follows

$$(a_1^K - a_1^N)(1 + a_1^K + a_1^N)^2 = 0.$$

We have two possibilities. If $a_1^K = a_1^N$, then $a_1^M = a_1^N = a_1^K$ and a_2^M has the form of stable mean coefficient from (34) which suggests that M should be stable and therefore (asymptotically) equal to K and N. If $a_1^K + a_1^N = -1$, then simple computation yields

$$a_2^M = -\tfrac{1}{2},\ a_2^M = -\tfrac{1}{8},\ a_3^M = -\tfrac{1}{16},\ a_4^M = -\tfrac{5}{128},\ a_5^M = -\tfrac{7}{256}.$$

These coefficients correspond to the first coefficients in the asymptotic expansion of geometric mean which can be found in [**12**]. This correspondence suggests that geometric mean is the only simultaneously (K, N)-stabilized and (N, K)-stabilized mean for different stable means K and N. In that case an interesting equation appears. If G is (K, N)-stabilized, using homogeneity of N and K

$$\begin{aligned} G(s,t) &= K\big(N(s,\sqrt{st}), N(\sqrt{st},t)\big) = K\big(\sqrt{s}N(\sqrt{s},\sqrt{t}), \sqrt{t}N(\sqrt{s},\sqrt{t})\big) \\ &= N(\sqrt{s},\sqrt{t})K(\sqrt{s},\sqrt{t}), \end{aligned}$$

which implies that G is also (N, K)-stabilized. Additionally, by including s^2 and t^2 instead of s and t, for $s, t > 0$, we obtain

$$st = G(s^2, t^2) = N(s,t)K(s,t)$$

and taking the square root the following equation follows

$$G = G(N, K)$$

which is also known as Gauss functional equation which defines compound mean obtained with Gauss iterative procedure ([**5**, Ch.VI.3]) indicating that geometric mean G is also compound mean of N and K, denoted by $G = N \otimes K$.

4.3 Simultaneously (K, N)-stabilizable and (K, N)-stabilized

Modifying formula (3) by interchanging M and N, from the list (37) we read the first few coefficients of (K, N)-stabilizable mean M and by comparison with coefficients from the list (39) we obtain the necessary conditions for mean M to be (K, N)-stabilizable and also (K, N)-stabilized. For $m = 1$ we have

$$\tfrac{1}{3}\left(a_1^K + 2a_1^N\right) = a_1^M = \tfrac{1}{2}\left(a_1^K + a_1^N\right)$$

and hence $a_1^K = a_1^N = a_1^M$. Since K and N are stable means and their asymptotic expansions are completely determined by a_1^K and a_1^N it follows that $a_m^K = a_m^N$ for all $m \in \mathbb{N}_0$. Then, either by using that M is (K, N)-stabilizable or M is (K, N)-stabilized, after replacing a_1^K and a_1^N by a_1^M, we obtain that coefficients a_2^M and a_3^M have the form of stable mean coefficients (34) indicating that M should also be stable and (asymptotically) equal to K and N.

5. EXAMPLES AND APPLICATIONS

Based on the results of Theorem 9, we may easily see which are the necessary conditions for a mean to be stable. This can be useful especially in the case of parametric means such as Stolarsky and Gini means mentioned in the Introduction. In the paper [**11**] we have derived asymptotic expansion of some one and two parameter classes of means which will be used to solve the open problem of Raïssouli and to demonstrate the application of the main Theorem. More about the mathematical means can be found in [**5**].

5.1 Power means

The r-th power mean is defined for all $s, t > 0$ by

$$B_r(s,t) = \begin{cases} \left(\dfrac{s^r + t^r}{2}\right)^{1/r}, & r \neq 0, \\ \sqrt{st}, & r = 0. \end{cases}$$

For example, the special cases of this mean are arithmetic mean $A = B_1$, quadratic mean $Q = B_2$ and harmonic mean $H = B_{-1}$. Geometric mean $G = B_0$ is obtained as limit case of B_r as $r \to 0$. Easy computations reveal that power mean M_r is stable for every r.

In paper [**11**], with $\alpha = 0$ and $\beta = t$, we find the asymptotic expansion of the r-th power mean:

$$B_r(x-t, x+t) = x + \tfrac{1}{2}(r-1)t^2x^{-1} - \tfrac{1}{24}(r-1)(r+1)(2r-3)t^4x^{-3} + \mathcal{O}(x^{-5}). \tag{40}$$

Asymptotic behavior of n-variable power means has been studied in [**9**]. Since power mean is stable these coefficients satisfy the recursive formula (27).

5.2 Gini means

The Gini means are defined for all $s, t > 0$ by

$$G_{p,r}(s,t) = \begin{cases} \left(\dfrac{s^p + t^p}{s^r + t^r}\right)^{\frac{1}{p-r}}, & p \neq r, \\ \exp\left(\dfrac{s^p \log s + t^p \log t}{s^p + t^p}\right), & p = r \neq 0, \\ \sqrt{st}, & p = r = 0. \end{cases}$$

for parameters p and r. These means were first introduced by Gini ([**16**]). Power means belong to the class of power means as $G_{0,r} = B_r$ and also Lehmer mean $G_{r+1,r}$ is a special case of Gini mean. Gini means are increasing with respect to both p and r, logarithmically convex with respect to both p and r if $(p, r) \in \langle -\infty, 0\rangle \times \langle -\infty, 0\rangle$ and logarithmically concave if $(p, r) \in \langle 0, \infty\rangle \times \langle 0, \infty\rangle$ ([**39**]).

In paper [**11**], with $\alpha = 0$ and $\beta = t$, we find the asymptotic expansion of Gini means:

$$\begin{aligned} G_{p,r}(x-t, x+t) = x + \tfrac{1}{2}(p + r - 1)t^2 x^{-1} + \tfrac{1}{24}\big[\big(-3 - 2p^3 + p^2(3 - 2r) \\ + 2r + (3 - 2r)r^2 + p\big(2 - 2(-3 + r)r\big)\big)\big]t^4 x^{-3} + \mathcal{O}(x^{-5}). \end{aligned}$$

By equating known coefficients of the asymptotic expansion of Gini means with coefficients of stable mean (34), we see that

$$\begin{aligned} a_1 &= \tfrac{1}{2}(p + r - 1) \\ \tfrac{1}{6}a_1(1 + a_1)(1 - 4a_1) &= \tfrac{1}{24}\big[\big(-3 - 2p^3 + p^2(3 - 2r) + 2r \\ &\qquad + (3 - 2r)r^2 + p\big(2 - 2(-3 + r)r\big)\big)\big] \end{aligned}$$

which is equivalent to

$$pr(p + r) = 0.$$

Since $G_{0,r} = B_r$, $G_{p,0} = B_p$ and $G_{p,-p} = B_0 = G$, we may conclude that the only stable Gini means are power means.

5.3 Stolarsky means

The Stolarsky means, also called the extended means or difference mean values, is a class of two-parameter means introduced by Stolarsky in [**36**]. Their properties were studied by Leach and Sholander in [**20, 21**] and further by Páles [**25**] and others. Interesting results regarding properties, monotonicity, Schur-convexity, logarithmic convexity and comparison of Stolarsky means can also be found in [**18, 26, 27, 28, 29, 35**].

The Stolarsky mean of order p, r is defined for all $s, t > 0$ by

$$E_{p,r}(s,t) = \begin{cases} \left[\frac{r(t^p - s^p)}{p(t^r - s^r)}\right]^{1/(p-r)}, & p \neq r,\ p, r \neq 0, \\ \frac{1}{e^{1/r}}\left(\frac{t^{t^r}}{s^{s^r}}\right)^{1/(t^r - s^r)}, & r = p \neq 0, \\ \left[\frac{t^r - s^r}{r(\log t - \log s)}\right]^{1/r}, & p = 0,\ r \neq 0, \\ \sqrt{st}, & p = r = 0. \end{cases}$$

Special cases are obtained by limit procedure. It is symmetric both on t and s as well as on p and r. $E_{p,r}(s,t)$ increases with increase in either s or t and also with increase in either p or r ([**20**]).

From paper [**11**], with $\alpha = 0$ and $\beta = t$, we have:

$$\begin{aligned} E_{p,r}(x-t, x+t) = x &+ \tfrac{1}{6}(p + r - 3)t^2 x^{-1} + \tfrac{1}{360}\big[-45 - 2p^3 + p^2(5 - 2r) \\ &+ r\big(10 + (5 - 2r)r\big) - 2p\big(-5 + (-5 + r)r\big)\big]t^4 x^{-3} + \mathcal{O}(x^{-5}). \end{aligned}$$

By comparing corresponding coefficients of Stolarsky with coefficients of stable mean, we see that

$$\begin{aligned} a_1 &= \tfrac{1}{6}(p + r - 3) \\ \tfrac{1}{6}a_1(1 + a_1)(1 - 4a_1) &= \tfrac{1}{360}\big[-45 - 2p^3 + p^2(5 - 2r) \\ &\qquad + r\big(10 + (5 - 2r)r\big) - 2p\big(-5 + (-5 + r)r\big)\big] \end{aligned}$$

which reduces to

$$(p - 2r)(2r - q)(p + r) = 0.$$

Since $E_{2r,r} = B_r$, $E_{p,2p} = B_p$ and $E_{p,-p} = B_0 = G$, we may conclude that the only stable Stolarsky means are again power means.

5.4 Generalized logarithmic mean

Let r be a real number. The generalized logarithmic mean ([**37**]) is defined for $s, t > 0$ $(s \neq t)$ by

$$L_r(s,t) = \begin{cases} \left(\frac{t^{r+1} - s^{r+1}}{(r+1)(t-s)}\right)^{1/r}, & r \neq -1, 0, \\ \frac{t - s}{\log t - \log s}, & r = -1, \\ \frac{1}{e}\left(\frac{t^t}{s^s}\right)^{1/(t-s)}, & r = 0. \end{cases}$$

with L_{-1} being the logarithmic and L_0 the identric mean. It is Schur-convex for $r > 1$ and Schur-concave for $r < 1$ ([**10**]).

The beginning of the asymptotic expansion of generalized logarithmic mean ([**11**]) reads as

$$L_r(x-t, x+t) = x + \tfrac{1}{6}(r-1)t^2 x^{-1} - \tfrac{1}{360}(r-1)(2r^2 + 5r - 13)t^4 x^{-3} + \mathcal{O}(x^{-5}).$$

In order for L_r to be stable its coefficients must coincide with stable mean coefficients (34), that is, the following conditions must hold

$$\begin{aligned} a_1 &= \tfrac{1}{6}(r-1), \\ \tfrac{1}{6}a_1(1+a_1)(1-4a_1) &= -\tfrac{1}{360}(r-1)(2r^2+5r-13), \end{aligned}$$

which leads to the cubic equation with solutions $r=1$, $r=-\frac{1}{2}$ and $r=-2$. In each of these three cases we have one of the power means, that is, $L_1 = A$, $L_{-\frac{1}{2}} = B_{\frac{1}{2}}$ and $L_{-2} = G$ which are all stable.

Remark 14. Within classes of Gini, Stolarsky and generalized logarithmic mean only power means are stable.

5.5 Stability and stabilizability with power means

Let us find coefficients for (K,M)-stabilizable mean N, where $K = B_p$ and $M = B_q$. Combining coefficients of stabilizable mean (37) with those from (40) we find that coefficients in the expansion of mean N which is stabilizable with pair of power means (B_p, B_q) are:

$$\begin{aligned} a_0^N &= 1, \\ a_1^N &= \tfrac{1}{6}(p+2q-3), \\ a_2^N &= \tfrac{1}{360}\Big(-45-2p^3+p^2(5-8q)+2p\big(5+2(5-3q)q\big) \\ &\qquad +4q\big(5+(5-2q)q\big)\Big). \end{aligned} \tag{41}$$

Observe the list of coefficients (39), where $K = B_p$ $N = B_q$ and whose coefficients can be obtained from (40). Then we obtain the coefficients of mean M which is stabilized with pair of power means (B_p, B_q):

$$\begin{aligned} a_0^M &= 1, \\ a_1^M &= \tfrac{1}{4}(p+q-2), \\ a_2^M &= \tfrac{1}{192}\Big(-24-2p^3+p^2(6-9q)+q(2+q)(4+q)+p\big(8-6(-2+q)q\big)\Big). \end{aligned}$$

5.6 Seiffert and Neuman-Sándor means

Let $s,t>0$. The first, the second Seiffert mean ([**5**]) and the Neuman-Sándor mean ([**24**]) are defined by

$$P(s,t) = \frac{t-s}{2\arcsin\frac{t-s}{t+s}}, \qquad T(s,t) = \frac{t-s}{2\arctan\frac{t-s}{t+s}}, \qquad NS(s,t) = \frac{t-s}{2\operatorname{arcsinh}\frac{t-s}{t+s}}.$$

These means have been subject of investigation by many authors who explored properties and found various bounds such can be seen in [**22, 23**] and references therein.

Asymptotic expansions of Seiffert means can be found in [**38**], with $\alpha = 0$ and $\beta = t$:

$$(42)\qquad \begin{aligned} P(x-t,x+t) &= x - \tfrac{1}{6}tx^{-1} - \tfrac{17}{360}t^4x^{-3} - \tfrac{367}{15120}t^6x^{-5} + \mathcal{O}(x^{-7}),\\ T(x-t,x+t) &= x + \tfrac{1}{3}tx^{-1} - \tfrac{4}{45}t^4x^{-3} + \tfrac{44}{945}t^6x^{-5} + \mathcal{O}(x^{-7}), \end{aligned}$$

and the asymptotic expansion of the Neuman-Sándor mean was given in [**13**]:

$$(43)\qquad NS(x-t,x+t) = x + \tfrac{1}{6}tx^{-1} - \tfrac{17}{360}t^4x^{-3} + \tfrac{367}{15120}t^6x^{-5} + \mathcal{O}(x^{-7}).$$

Assume P is (K, M)-stabilizable mean. Then comparing coefficients (42) with those from list (37) yields

$$-\tfrac{1}{6} = \tfrac{1}{3}(a_1^K + 2a_1^M),$$

and with $a_1^K = -\frac{1}{2} - 2a_1^M$, comparison of the second coefficients yields

$$-\tfrac{17}{360} = -\tfrac{1}{360}(13 + 32(a_1^M)^2)$$

or equivalently $a_1^M = \pm\frac{1}{\sqrt{8}}$. Each of these two values leads to contradiction when comparing the third coefficients.

By the same procedure we may obtain that the second Seiffert and the Neuman-Sándor mean are also not stabilizable.

5.7 Asymptotic inequalities and sub-stabilizability with power means

Let us observe the difference between the first Seiffert mean P and the resultant mean-map $\mathcal{R}(K, N, M)$ with $K = B_p$ and $M = B_q$. According to (26), (40) and (42), its asymptotic expansion reads as

$$\begin{aligned} P(x-t,x+t) - \mathcal{R}(B_p,P,B_q)(x-t,x+t) &= -\tfrac{1}{8}\,(p+2q-2)\,t^2x^{-1}\\ &+ \tfrac{1}{384}\big(2p^3-3p^2+4p(q-3)+4q(q-2)(4q+3)+24\big)t^4x^{-3} + \mathcal{O}(x^{-5}). \end{aligned}$$

The best approximation is obtained when the coefficient in the first parentheses is equal to zero, i.e. when $p = 2 - 2q$. For such relation between q and p the asymptotic expansion of a difference reads as

$$\begin{aligned} P(x-t,x+t) - \mathcal{R}(B_p,P,B_q)(x-t,x+t) &= \tfrac{1}{96}(1-4q+2q^2)t^4x^{-3}\\ &+ \tfrac{1}{2880}\big(23 - q(q(20(q-3)q+9)+72)\big)t^6x^{-5} + \mathcal{O}(x^{-7}). \end{aligned}$$

Again, from equating the first coefficient with zero, we obtain two solutions of quadratic equation: $q_{1,2} = 1 \pm \frac{\sqrt{2}}{2}$. For either one of these solutions, the asymptotic expansion of a difference is

$$P(x-t,x+t) - \mathcal{R}(B_p,P,B_q)(x-t,x+t) = \tfrac{1}{1152}t^6x^{-5} + \mathcal{O}(x^{-7}),$$

and thereby the difference is asymptotically greater than 0:

$$P - \mathcal{R}(B_p, P, B_q) \succ 0,$$

which means that the necessary condition for the inequality $P > \mathcal{R}(B_p, P, B_q)$ is fulfilled. If P is (B_p, B_q)-sub-stabilizable for $(p, q) = (\pm\sqrt{2}, 1 \mp \frac{\sqrt{2}}{2})$, then those parameters are optimal. Numerical experiments and plotting the graph of a difference $P(s, 1-s) - \mathcal{R}(B_p, P, B_q)(s, 1-s)$, $s \in [0, 1]$, indicate that mean P should be (B_p, B_q)-sub-stabilizable for these p and q.

Observe the difference between the Neuman-Sándor mean NS and the resultant mean-map $\mathcal{R}(K, NS, M)$ with $K = B_p$ and $M = B_q$. Similarly as before, using (26), (40) and (43), its asymptotic expansion reads as

$$\begin{aligned} NS(x-t, x+t) - \mathcal{R}(B_p, NS, B_q)(x-t, x+t) &= \tfrac{1}{8}(4 - p - 2q)t^2x^{-1} \\ &+ \tfrac{1}{384}\big(2p^3 - 3p^2 + 4p(5q-4) + 4q(q(4q-7) - 8) + 20\big)t^4x^{-3} + \mathcal{O}(x^{-5}). \end{aligned}$$

The best approximation is obtained when the coefficient in the first parentheses is equal to zero, i.e. when $p = 4 - 2q$. For such relation between q and p the asymptotic expansion of a difference reads as

$$\begin{aligned} NS(x-t, x+t) - \mathcal{R}(B_p, NS, B_q)(x-t, x+t) &= \tfrac{1}{96}(9 - 16q + 4q^2)t^4x^{-3} \\ &+ \tfrac{1}{2880}\big(-40q^4 + 240q^3 - 459q^2 + 376q - 51\big)t^6x^{-5} + \mathcal{O}(x^{-7}). \end{aligned}$$

Again, from equating the next coefficient with zero, we obtain two solutions of quadratic equation: $q_{1,2} = 2 \pm \frac{\sqrt{7}}{2}$. For either one of these solutions, the following holds

$$NS(x-t, x+t) - \mathcal{R}(B_p, NS, B_q)(x-t, x+t) = \tfrac{79}{3840}t^6x^{-5} + \mathcal{O}(x^{-7}),$$

which means that difference is asymptotically greater than 0:

$$NS - \mathcal{R}(B_p, NS, B_q) \succ 0.$$

The necessary condition for the inequality $NS > \mathcal{R}(B_p, NS, B_q)$ is fulfilled. If NS is (B_p, B_q)-sub-stabilizable for $(p, q) = (\pm\sqrt{7}, 2 \mp \frac{\sqrt{7}}{2})$, then those parameters are the best possible. Numerical experiments and plotting the graph of a difference $NS - \mathcal{R}(B_p, NS, B_q)$ on line $(s, 1-s)$, $s \in [0, 1]$, indicate that NS should be (B_p, B_q)-sub-stabilizable for these p and q.

Similar procedure for the second Seiffert mean T, i.e. equating two coefficients with zero, does not give the difference that is always greater than 0. If only one coefficient is equated with zero, or equivalently $p = 5 - 2q$, then the next coefficient is $a_2 = \frac{1}{96}(5q^2 - 25q + 22)$. The problem of finding best parameters is reduced to finding q which minimizes the expression a_2, with $|10q - 25| \geq \sqrt{185}$, and that inequality $T - \mathcal{R}(B_p, T, B_q) > 0$ still holds.

Notice that this approach enables us to treat similar problems with superstability and with other parametric means involved as well.

6. CONCLUSION

In this paper we have derived the complete asymptotic expansions of the resultant mean-map and consequently obtained the asymptotic expansion of stable (balanced) mean. Furthermore, we obtained the asymptotic expansion of stabilizable and stabilized means. All the asymptotic expansions were given in a form of recursive relations for their coefficients. Besides the form of Theorems presented in Section 3, given asymptotic expansions may be used to obtain any unknown of three means involved in stabilizability problem. Significance of the main results has been shown by examples of various types.

Based on the asymptotic equalities and observations from Section 4 we may state the following.

Conjecture 15. 1. If mean N is simultaneously (K, M) and (M, K)-stabilizable, then $N = K = M$.

2. If mean M is simultaneously (K, N) and (N, K)-stabilized, then either $M = K = N$ or $M = G = K \otimes N$.

3. If mean M is simultaneously (K, N)-stabilizable and (K, N)-stabilized, then $M = K = N$.

Notice we have proved the asymptotic equality between K and M in 1., and between K and N in parts 2. and 3.

Regarding questions from the Introduction, based on reasoning from Section 5, we may state the following.

a. All pairs (p, q) such that Gini means are stable are $\{(0, q), (p, 0), (p, -p)\}$. All pairs (p, q) such that Stolarsky means are stable are $\{(2q, q), (p, 2p), (p, -p)\}$. In addition, all parameters r such that generalized logarithmic means are stable are $\{-2, -\frac{1}{2}, 1\}$.

b. The first Seiffert mean P is not stabilizable as well as the second Seiffert mean T and the Neuman-Sándor mean NS.

c. If P is (B_p, B_q)-sub-stabilizable for $(p, q) = (\pm\sqrt{2}, 1 \mp \frac{\sqrt{2}}{2})$, then those parameters are the best possible.

d. If NS is (B_p, B_q)-sub-stabilizable for $(p, q) = (\pm\sqrt{7}, 2 \mp \frac{\sqrt{7}}{2})$, then those parameters are the best possible.

Methods presented in this paper can be used to obtain valuable information regarding means involved in other similar problems defined through functional equations, especially when the explicit solution is not easy to find.

Acknowledgements. The author is grateful to the anonymous referees for their valuable comments.

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(Received 10. 03. 2023.)
(Revised 18. 01. 2024.)