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SOME NEW RESULTS RELATED TO THE CONSTANT e AND FUNCTION $(1+1/x)^x$

Xue-Feng Han, Chao-Ping Chen and H. M. Srivastava*

It is known that the constant e has the following series representations: $e = \sum_{k=0}^{\infty} \frac{1}{k!}$ and $e = \sum_{k=0}^{\infty} \frac{9k^2+1}{(3k)!}$. The second series is extremely rapidly convergent. In this paper, we present asymptotic expansions and two-sided inequalities for the remainders R_n and $\mathcal{R}_n$, where $R_n = e - \sum_{k=0}^{n} \frac{1}{k!}$ and $\mathcal{R}_n = e - \sum_{k=0}^{n} \frac{9k^2+1}{(3k)!}$. Also, we present some inequalities and completely monotonic functions involving $(1+1/x)^x$. We also consider a number of related developments on the subject of this paper.

1. INTRODUCTION AND MOTIVATION

Throughout this paper, $\mathbb{N}$ represents the set of positive integers and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. The constant e, which is also known as Euler's number, can be defined by the following limit:

$$e = \lim_{x\to\infty} \left(1 + \frac{1}{x}\right)^x$$

or by the infinite series as follows:

$$e = \sum_{k=0}^{\infty} \frac{1}{k!} \tag{1}$$

*Corresponding author. Chao-Ping Chen
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as first published by Newton [24].

The constant e is the most important constant in mathematics because it appears in countless mathematical contexts involving limits and derivatives. Joost Bürgi seems to have been the first to formulate an approximation to e around 1620, obtaining three-decimal-place accuracy (see [13, p. 31], [19] and [20, pp. 26–27]; see also the Preface in [30]).

For a large value of n in (1), the partial sum:

$$\sum_{k=0}^{n} \frac{1}{k!} = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!}$$

gives a simple and direct approximation to e that is a good way of calculating e to a high accuracy (see [2] and [7]). Numerical values of e are usually derived by using either the optimized versions of the Taylor-Maclaurin series of e^x or the continued-fraction expansion approach initiated by Euler [7].

The constant e has many series representations (see [6, 27]), some of which we reproduce below:

$$e = \sum_{k=0}^{\infty} \frac{k+1}{2\cdot(k!)} = \sum_{k=0}^{\infty} \frac{3-4k^2}{(2k+1)!} = \sum_{k=0}^{\infty} \frac{2k+1}{(2k)!} = \sum_{k=0}^{\infty} \frac{2(k+1)}{(2k+1)!} = \sum_{k=0}^{\infty} \frac{9k^2+1}{(3k)!}. \tag{2}$$

By taking the first twenty terms in the series (1) and (2), we have

$$\left| e - \sum_{k=0}^{19} \frac{1}{k!} \right| \approx 4.315 \times 10^{-19} \qquad \text{and} \qquad \left| e - \sum_{k=0}^{19} \frac{k+1}{2\cdot(k!)} \right| \approx 4.541 \times 10^{-18},$$

$$\left| e - \sum_{k=0}^{19} \frac{3-4k^2}{(2k+1)!} \right| \approx 4.776 \times 10^{-47} \qquad \text{and} \qquad \left| e - \sum_{k=0}^{19} \frac{2k+1}{(2k)!} \right| \approx 5.028 \times 10^{-47}$$

and

$$\left| e - \sum_{k=0}^{19} \frac{2(k+1)}{(2k+1)!} \right| \approx 1.256 \times 10^{-48} \qquad \text{and} \qquad \left| e - \sum_{k=0}^{19} \frac{9k^2+1}{(3k)!} \right| \approx 4.327 \times 10^{-79}.$$

It is observed that, among series representations in (1) and (2), the last series in (2) would prove to be the best one.

We here consider the series (1) and the last series in (2). We set

$$S_n := \sum_{k=0}^{n} \frac{1}{k!} \qquad \text{and} \qquad \mathcal{S}_n = \sum_{k=0}^{n} \frac{9k^2+1}{(3k)!},$$

and we put

$$R_n := e - S_n = \sum_{k=n+1}^{\infty} \frac{1}{k!} \tag{3}$$

and

$$\mathcal{R}_n := e - \mathcal{S}_n = \sum_{k=n+1}^{\infty} \frac{9k^2+1}{(3k)!}. \tag{4}$$

Then, by using the *Maple* software, we find, as $n \to \infty$, that

$$\frac{1}{(n+1)!} \sim \frac{1}{n \cdot n!}\left(1 - \frac{1}{n} + \frac{1}{n^2} - \frac{1}{n^3} + \frac{1}{n^4} - \frac{1}{n^5} + \cdots\right),$$

$$\frac{1}{(n+2)!} \sim \frac{1}{n \cdot n!}\left(\frac{1}{n} - \frac{3}{n^2} + \frac{7}{n^3} - \frac{15}{n^4} + \frac{31}{n^5} - \cdots\right),$$

$$\frac{1}{(n+3)!} \sim \frac{1}{n \cdot n!}\left(\frac{1}{n^2} - \frac{6}{n^3} + \frac{25}{n^4} - \frac{90}{n^5} + \cdots\right),$$

$$\frac{1}{(n+4)!} \sim \frac{1}{n \cdot n!}\left(\frac{1}{n^3} - \frac{10}{n^4} + \frac{65}{n^5} - \cdots\right),$$

$$\frac{1}{(n+5)!} \sim \frac{1}{n \cdot n!}\left(\frac{1}{n^4} - \frac{15}{n^5} + \cdots\right),$$

and so on. Summing these expansions side by side, we obtain the following asymptotic expansion of the remainder R_n:

$$R_n \sim \frac{1}{n \cdot n!}\left(1 - \frac{1}{n^2} + \frac{1}{n^3} + \frac{2}{n^4} - \cdots\right) \qquad (n \to \infty). \tag{5}$$

Also, by using the *Maple* software, we find, as $n \to \infty$, that

$$\frac{9(n+1)^2+1}{\big(3(n+1)\big)!} = \frac{9(n+1)^2+1}{\big(3(n+1)\big)!},$$

$$\frac{9(n+2)^2+1}{\big(3(n+2)\big)!} \sim \frac{9(n+1)^2+1}{\big(3(n+1)\big)!}\Bigg(\frac{1}{27n^3} - \frac{1}{9n^4} + \frac{52}{243n^5} - \frac{80}{243n^6} + \frac{958}{2187n^7} - \frac{394}{729n^8} + \cdots\Bigg),$$

$$\frac{9(n+3)^2+1}{\big(3(n+3)\big)!} \sim \frac{9(n+1)^2+1}{\big(3(n+1)\big)!}\Bigg(\frac{1}{729n^6} - \frac{1}{81n^7} + \frac{428}{6561n^8} - \frac{578}{2187n^9} + \frac{17893}{19683n^{10}} - \cdots\Bigg),$$

$$\frac{9(n+4)^2+1}{\big(3(n+4)\big)!} \sim \frac{9(n+1)^2+1}{\big(3(n+1)\big)!}\left(\frac{1}{19683n^9} - \frac{2}{2187n^{10}} + \cdots\right),$$

and so on. Upon summing these expansions side by side, we obtain the following asymptotic expansion of the remainder $\mathcal{R}_n$,

$$\mathcal{R}_n \sim \frac{9n^2+18n+10}{(3n+3)!}\left(1 + \frac{1}{27n^3} - \frac{1}{9n^4} + \frac{52}{243n^5} - \frac{239}{729n^6} + \frac{931}{2187n^7} - \frac{3118}{6561n^8} + \cdots\right) \qquad (n \to \infty). \tag{6}$$

Even though we can obtain as many coefficients as we please in the right-hand sides of (5) and (6) by using the *Maple* software, we here give a formula for determining the coefficients of each asymptotic expansion, and then establish the lower and upper bounds for the remainders R_n and $\mathcal{R}_n$, which is the first aim of the present paper.

It is well known that

$$\left(1+\frac{1}{n}\right)^n < e \quad \text{for} \quad n \in \mathbb{N}. \tag{7}$$

By using (7), Hardy [18] presented a proof of Carleman's inequality

$$\sum_{n=1}^{\infty}(a_1a_2\cdots a_n)^{1/n} < e\sum_{n=1}^{\infty} a_n, \tag{8}$$

where $a_n \geq 0$ for $n \in \mathbb{N}$ and $0 < \sum_{n=1}^{\infty} a_n < \infty$. By estimating the weight coefficient $(1+1/n)^n$, some strengthened and generalized results of (8) have been given, see two recently published article [10, 11] and the references cited therein.

For $n \in \mathbb{N}$, let

$$I_n = \left(1+\frac{1}{n}\right)^n. \tag{9}$$

The second aim of the present paper is to present some sharp inequalities related to the quantities:

$$\frac{I_{n-1}}{I_n} \quad \text{and} \quad \frac{I_n^2}{I_{n-1}I_{n+1}} \quad \text{for} \quad n \geq 2.$$

Gautschi [15] proved that the harmonic mean of $\Gamma(x)$ and $\Gamma(1/x)$ is greater than or equal to 1, namely,

$$\frac{2\Gamma(x)\Gamma(1/x)}{\Gamma(x)+\Gamma(1/x)} \geq 1 \quad \text{for} \quad x > 0 \tag{10}$$

with equality if $x = 1$. Similarly, Alzer and Jameson [1] proved that

$$\frac{2\psi(x)\psi(1/x)}{\psi(x)+\psi(1/x)} \geq -\gamma \quad \text{for} \quad x > 0 \tag{11}$$

with equality if $x = 1$, where $\psi(x) = \Gamma'(x)/\Gamma(x)$ denotes the psi function and $\gamma = 0.577\ldots$ is the Euler-Mascheroni constant. Nantomah [22] conjectured, then Matejíčka [21] and Nantomah [23] proved that

$$\frac{2\beta(x)\beta(1/x)}{\beta(x)+\beta(1/x)} \leq \ln 2 \quad \text{for} \quad x > 0 \tag{12}$$

with equality if $x = 1$, where $\beta(x)$ is the Nielsen's β-function defined in [25, p. 16] by

$$\beta(x) = \frac{1}{2}\left[\psi\left(\frac{x+1}{2}\right) - \psi\left(\frac{x}{2}\right)\right] = \int_0^\infty \frac{e^{-xt}}{1+e^{-t}}\mathrm{d}t \quad \text{for} \quad x > 0.$$

For $x > 0$, let

$$I(x) = \left(1+\frac{1}{x}\right)^x. \tag{13}$$

Motivated by the harmonic mean inequalities (10), (11) and (12), we here establish the arithmetic mean inequality of $I(x)$ and $I(1/x)$, namely,

$$\frac{I(x)+I(1/x)}{2} \leq 2 \quad \text{for} \quad x > 0$$

with equality if $x = 1$, which is the third aim of the present paper. Thus, we obtain

$$\frac{2I(x)I(1/x)}{I(x)+I(1/x)} \leq \sqrt{I(x)I(1/x)} \leq \frac{I(x)+I(1/x)}{2} \leq 2 \quad \text{for} \quad x > 0$$

with equality if $x = 1$.

The last aim of the present paper is to present completely monotonic functions involving $(1+1/x)^x$.

The numerical values, which we have given in this article, were computed by using the computer program *MAPLE* 11.

2. ASYMPTOTIC EXPANSIONS AND INEQUALITIES OF R_N AND $\mathcal{R}_N$

Theorem 1. *The remainder* R_n, *defined by* (3), *has the following asymptotic expansion*:

$$R_n \sim \frac{1}{n\cdot n!}\sum_{k=0}^\infty \frac{r_k}{n^k} = \frac{1}{n\cdot n!}\left(1 - \frac{1}{n^2} + \frac{1}{n^3} + \frac{2}{n^4} - \frac{9}{n^5} + \frac{9}{n^6} + \frac{50}{n^7} - \cdots\right) \tag{14}$$

as $n \to \infty$, where the coefficients r_k $(k \geq 0)$ are given by the following recursive relation:

(15)

$$r_0 = 1, \quad r_1 = 0, \quad r_k = (-1)^k + \sum_{j=1}^{k} \sum_{\ell=0}^{k-j} (-1)^{k-\ell-1} j r_\ell \binom{k-j-1}{k-j-\ell} \quad \textit{for} \quad k \geq 2.$$

Proof. We begin by setting

$$T_n := \frac{1}{n \cdot n!} \sum_{k=0}^{\infty} \frac{r_k}{n^k},$$

where r_k $(k \geq 0)$ are real numbers to be determined. Then, in view of (5), we can let $R_n \sim T_n$ and

$$\Delta R_n := R_{n+1} - R_n \sim T_{n+1} - T_n =: \Delta T_n \quad \text{as} \quad n \to \infty.$$

Thus, clearly, we have

$$\Delta R_n = -\frac{1}{n \cdot n!} \frac{1}{1+\frac{1}{n}} \sim -\frac{1}{n \cdot n!} \sum_{k=0}^{\infty} (-1)^k \frac{1}{n^k} \qquad (n \to \infty),$$

which can be written for $n \to \infty$ as follows:

$$n \cdot n! \Delta R_n \sim \sum_{k=0}^{\infty} (-1)^{k+1} \frac{1}{n^k}. \tag{16}$$

We also have

$$\begin{aligned} \Delta T_n &= \frac{1}{(n+1)\cdot(n+1)!} \sum_{k=0}^{\infty} \frac{r_k}{(n+1)^k} - \frac{1}{n \cdot n!} \sum_{k=0}^{\infty} \frac{r_k}{n^k} \\ &= \frac{1}{n \cdot n!} \left(\frac{n}{(n+1)^2} \sum_{k=0}^{\infty} \frac{r_k}{(n+1)^k} - \sum_{k=0}^{\infty} \frac{r_k}{n^k} \right). \end{aligned} \tag{17}$$

It is easy to see that

$$\frac{n}{(n+1)^2} \sim \sum_{k=0}^{\infty} \frac{a_k}{n^k} \qquad (n \to \infty),$$

where

$$a_k = (-1)^{k-1} k \quad \text{for} \quad k \geq 0.$$

Direct computation yields

$$\begin{aligned} \sum_{k=0}^{\infty} \frac{r_k}{(n+1)^k} &= \sum_{k=0}^{\infty} \frac{r_k}{n^k} \left(1 + \frac{1}{n}\right)^{-k} = \sum_{k=0}^{\infty} \frac{r_k}{n^k} \sum_{j=0}^{\infty} \binom{-k}{j} \frac{1}{n^j} \\ &= \sum_{k=0}^{\infty} \frac{r_k}{n^k} \sum_{j=0}^{\infty} (-1)^j \binom{k+j-1}{j} \frac{1}{n^j} = \sum_{k=0}^{\infty} \frac{b_k}{n^k}, \end{aligned} \tag{18}$$

where

$$b_k = \sum_{\ell=0}^{k} r_\ell (-1)^{k-\ell} \binom{k-1}{k-\ell}.$$

We then find from (17), a s $n \to \infty$, that

$$n \cdot n! \Delta T_n \sim \sum_{k=0}^{\infty} \frac{a_k}{n^k} \sum_{k=0}^{\infty} \frac{b_k}{n^k} - \sum_{k=0}^{\infty} \frac{r_k}{n^k} = \sum_{k=0}^{\infty} \left(\sum_{j=0}^{k} a_j b_{k-j} - r_k \right) \frac{1}{n^k}. \tag{19}$$

Now, equating the coefficients of the term n^{-k} on the right-hand sides of (16) and (19), we obtain

$$(-1)^{k+1} = \sum_{j=0}^{k} a_j b_{k-j} - r_k \quad \text{for} \quad k \geq 0.$$

For $k = 0$, we obtain $r_0 = a_0 b_0 + 1 = 1$. For $k = 1$, we obtain $r_1 = -1 + r_0 = 0$. And, for $k \geq 2$, we have

$$\begin{aligned} r_k &= (-1)^k + \sum_{j=0}^{k} a_j b_{k-j} = (-1)^k + a_0 b_k + \sum_{j=1}^{k} a_j b_{k-j} \\ &= (-1)^k + \sum_{j=1}^{k} (-1)^{j-1} j b_{k-j} = (-1)^k + \sum_{j=1}^{k} \sum_{\ell=0}^{k-j} (-1)^{k-\ell-1} j r_\ell \binom{k-j-1}{k-j-\ell}. \end{aligned}$$

The proof of Theorem 1 is complete. □

Next, by using the formula (15), we show how easily we can determine r_k $(k \geq 0)$ in (14). We give the first few coefficients r_k as follows:

$$\begin{aligned} &r_0 = 1, \quad r_1 = 0, \\ &r_2 = 1 - 2r_0 + r_1 = -1, \\ &r_3 = -1 + 3r_0 - 3r_1 + r_2 = 1, \\ &r_4 = 1 - 4r_0 + 6r_1 - 4r_2 + r_3 = 2. \end{aligned}$$

We note that the values of r_k (for $k = 0, 1, 2, 3, 4$) here are equal to the coefficients of $1/n^k$ (for $k = 0, 1, 2, 3, 4$) in (5), respectively.

Theorem 2. *For all integers $n \geq 1$, the following two-sided inequality holds true*:

$$L_n < R_n < U_n, \tag{20}$$

where

$$L_n := \frac{1}{n \cdot n!} \left(1 - \frac{1}{n^2} + \frac{1}{n^3} + \frac{2}{n^4} - \frac{9}{n^5} \right)$$

and

$$U_n = \frac{1}{n \cdot n!}\left(1 - \frac{1}{n^2} + \frac{1}{n^3} + \frac{2}{n^4} - \frac{9}{n^5} + \frac{9}{n^6} + \frac{50}{n^7}\right).$$

Proof. For $n \geq 1$, let

$$\xi_n := R_n - L_n \quad \text{and} \quad \eta_n := R_n - U_n.$$

We then have

$$\lim_{n\to\infty} \xi_n = \lim_{n\to\infty} \eta_n = 0.$$

In order to prove (20), it suffices to show that the sequence $\{\xi_n\}$ is strictly decreasing and also that the $\{\eta_n\}$ is strictly increasing for $n \geq 1$. Direct computation yields

$$\begin{aligned}\xi_{n+1} - \xi_n &= \Delta R_n + L_n - L_{n+1} \\ &= -\frac{1}{n \cdot n!}\frac{1}{1+\frac{1}{n}} + \frac{1}{n \cdot n!}\left(1 - \frac{1}{n^2} + \frac{1}{n^3} + \frac{2}{n^4} - \frac{9}{n^5}\right) \\ &\quad - \frac{1}{n \cdot n!}\frac{n}{(n+1)^2}\left(1 - \frac{1}{(n+1)^2} + \frac{1}{(n+1)^3} + \frac{2}{(n+1)^4} - \frac{9}{(n+1)^5}\right) \\ &= -\frac{9 + 174n^2 + 267n^3 + 61n + 231n^4 + 104n^5 + 9n^6}{n^6(n+1)^7 \cdot n!} \\ &< 0\end{aligned}$$

and

$$\begin{aligned}&\eta_{n+1} - \eta_n = \Delta R_n + U_n - U_{n+1} \\ &= -\frac{1}{n \cdot n!}\frac{1}{1+\frac{1}{n}} + \frac{1}{n \cdot n!}\left(1 - \frac{1}{n^2} + \frac{1}{n^3} + \frac{2}{n^4} - \frac{9}{n^5} + \frac{9}{n^6} + \frac{50}{n^7}\right) \\ &\quad - \frac{1}{n \cdot n!}\frac{n}{(n+1)^2}\left(1 - \frac{1}{(n+1)^2} + \frac{1}{(n+1)^3} + \frac{2}{(n+1)^4}\right. \\ &\qquad\qquad \left. - \frac{9}{(n+1)^5} + \frac{9}{(n+1)^6} + \frac{50}{(n+1)^7}\right) \\ &= \frac{50 + 459n + 1872n^2 + 4445n^3 + 6751n^4 + 6758n^5 + 4395n^6 + 1723n^7 + 267n^8}{n^8(n+1)^9 \cdot n!} \\ &> 0.\end{aligned}$$

The proof of Theorem 2 is thus completed. □

Remark 3. We write (20) as follows:

$$P_n < e < Q_n, \tag{21}$$

where

$$P_n = S_n + L_n \quad \text{and} \quad Q_n = S_n + U_n.$$

For $n = 10$ in (21), we have

$$P_{10} = 2.718281828458\cdots$$

and

$$Q_{10} = 2.718281828459\cdots.$$

We then get the approximate value of e, given by

$$e \approx 2.71828182845.$$

The choice $n = 100$ in (21) yields the following approximate value of e:

$$\begin{aligned} e \approx{} & 2.71828182845904523536028747135266249775724709369995 \\ & 95749669676277240766303535475945713821785251664274 \\ & 27466391932003059921817413596629043572900334295260 \\ & 59563073813232862794. \end{aligned}$$

Theorem 4. *The remainder $\mathcal{R}_n$, defined by (4), has the following asymptotic expansion:*

$$\begin{aligned} \mathcal{R}_n &\sim \frac{9n^2+18n+10}{(3n+3)!} \sum_{k=0}^{\infty} \frac{\lambda_k}{n^k} \\ &= \frac{9n^2+18n+10}{(3n+3)!} \left(1 + \frac{1}{27n^3} - \frac{1}{9n^4} + \frac{52}{243n^5} \right. \\ &\qquad \left. - \frac{239}{729n^6} + \frac{931}{2187n^7} - \frac{3118}{6561n^8} + \cdots \right) \qquad (n \to \infty) \end{aligned} \tag{22}$$

with the coefficients λ_k $(k \geq 0)$ given by the following recursive formula for λ_k:

$$\lambda_0 = 1, \quad \lambda_k = \sum_{j=0}^{k-1} \sum_{\ell=0}^{j} \lambda_\ell (-1)^{j-\ell} \binom{j-1}{j-\ell} d_{k-j} \quad \text{for} \quad k \geq 1, \tag{23}$$

where

$$d_k = (-1)^{k-1} \left[\frac{2^k}{120} - \frac{2}{25} \left(\frac{5}{3} \right)^k + \frac{5}{16} \left(\frac{4}{3} \right)^k \right] - \frac{3c_k}{10} \quad \text{for} \quad k \geq 1 \tag{24}$$

and

$$c_1 = 1, \quad c_2 = -\frac{11}{9}, \quad c_k = -2c_{k-1} - \frac{10}{9} c_{k-2} \quad \text{for} \quad k \geq 3. \tag{25}$$

Proof. Let us set

$$I_n := \frac{9n^2+18n+10}{(3n+3)!}\sum_{k=0}^{\infty}\frac{\lambda_k}{n^k},$$

where λ_k $(k \geq 0)$ are real numbers to be determined. Then, in view of (5), we can let $\mathcal{R}_n \sim I_n$ and

$$\Delta\mathcal{R}_n := \mathcal{R}_n - \mathcal{R}_{n+1} \sim I_n - I_{n+1} =: \Delta I_n \quad \text{as} \quad n \to \infty.$$

We thus obtain

$$\Delta\mathcal{R}_n = \frac{9n^2+18n+10}{(3n+3)!} \tag{26}$$

and

$$\begin{aligned}\Delta I_n &= \frac{9n^2+18n+10}{(3n+3)!}\sum_{k=0}^{\infty}\frac{\lambda_k}{n^k} - \frac{9(n+1)^2+18(n+1)+10}{(3n+6)!}\sum_{k=0}^{\infty}\frac{\lambda_k}{(n+1)^k}\\ &= \frac{9n^2+18n+10}{(3n+3)!}\left(\sum_{k=0}^{\infty}\frac{\lambda_k}{n^k}\right.\\ &\qquad \left. - \frac{9n^2+36n+37}{(3n+6)(3n+5)(3n+4)(9n^2+18n+10)}\sum_{k=0}^{\infty}\frac{\lambda_k}{(n+1)^k}\right).\end{aligned} \tag{27}$$

It is easy to see that

$$\begin{aligned}&\frac{9n^2+36n+37}{(3n+6)(3n+5)(3n+4)(9n^2+18n+10)}\\ &\quad = \frac{1}{60n(1+\frac{2}{n})} - \frac{2}{15n(1+\frac{5}{3n})} + \frac{5}{12(1+\frac{4}{3n})} - \frac{3}{10}\frac{9n+7}{9n^2+18n+10}\\ &\quad = \sum_{k=1}^{\infty}(-1)^{k-1}\frac{2^{k-1}}{60n^k} - \sum_{k=1}^{\infty}(-1)^{k-1}\frac{2}{15}\left(\frac{5}{3}\right)^{k-1}\frac{1}{n^k}\\ &\qquad + \sum_{k=1}^{\infty}(-1)^{k-1}\frac{5}{12}\left(\frac{4}{3}\right)^{k-1}\frac{1}{n^k} - \frac{3}{10}\frac{9n+7}{9n^2+18n+10}\\ &\quad = \sum_{k=1}^{\infty}(-1)^{k-1}\left[\frac{2^k}{120} - \frac{2}{25}\left(\frac{5}{3}\right)^k + \frac{5}{16}\left(\frac{4}{3}\right)^k\right]\frac{1}{n^k} - \frac{3}{10}\frac{9n+7}{9n^2+18n+10}.\end{aligned}$$

We now let

$$\frac{9n+7}{9n^2+18n+10} = \sum_{k=1}^{\infty}\frac{c_k}{n^k},$$

where c_k $(k \geq 1)$ are real numbers to be determined. This can be written as follows:

$$9n+7 = (9n^2+18n+10)\sum_{k=1}^{\infty}\frac{c_k}{n^k}$$

and

$$9n+7 = 9c_1 n + 9(c_2+2c_1) + \sum_{k=1}^{\infty}(9c_{k+2}+18c_{k+1}+10c_k)n^{-k}. \tag{28}$$

By equating the coefficients of like powers of n on both sides of (28), we obtain

$$9c_1 = 9, \quad 9(c_2+2c_1) = 7, \quad 9c_{k+2}+18c_{k+1}+10c_k = 0 \quad \text{for} \quad k \geq 1,$$

which yields the following recursive formula for c_k:

$$c_1 = 1, \quad c_2 = -\frac{11}{9}, \quad c_k = -2c_{k-1} - \frac{10}{9}c_{k-2} \quad \text{for} \quad k \geq 3.$$

We then find that

$$\frac{9n^2+36n+37}{(3n+6)(3n+5)(3n+4)(9n^2+18n+10)} = \sum_{k=0}^{\infty}\frac{d_k}{n^k},$$

where

$$d_0 = 0, \quad d_k = (-1)^{k-1}\left[\frac{2^k}{120} - \frac{2}{25}\left(\frac{5}{3}\right)^k + \frac{5}{16}\left(\frac{4}{3}\right)^k\right] - \frac{3c_k}{10} \quad \text{for} \quad k \geq 1.$$

So, by applying (18), we have

$$\sum_{k=0}^{\infty}\frac{\lambda_k}{(n+1)^k} = \sum_{k=0}^{\infty}\frac{\mu_k}{n^k},$$

where

$$\mu_k = \sum_{\ell=0}^{k}\lambda_\ell(-1)^{k-\ell}\binom{k-1}{k-\ell}.$$

We thus find from (27) that

$$\begin{aligned}
\Delta I_n &= \frac{9n^2+18n+10}{(3n+3)!}\left(\sum_{k=0}^{\infty}\frac{\lambda_k}{n^k} - \sum_{k=0}^{\infty}\frac{d_k}{n^k}\sum_{j=0}^{\infty}\frac{\mu_j}{n^j}\right)\\
&= \frac{9n^2+18n+10}{(3n+3)!}\sum_{k=0}^{\infty}\left(\lambda_k - \sum_{j=0}^{k}\mu_j d_{k-j}\right)\frac{1}{n^k}\\
&= \frac{9n^2+18n+10}{(3n+3)!}\left[\lambda_0 + \sum_{k=1}^{\infty}\left(\lambda_k - \sum_{j=0}^{k}\mu_j d_{k-j}\right)\frac{1}{n^k}\right]. \qquad (29)
\end{aligned}$$

By $\Delta\mathcal{R}_n \sim \Delta I_n$, we find from (26) and (29) that

$$\lambda_0 = 1 \quad \text{and}$$
$$\lambda_k = \sum_{j=0}^{k} \mu_j d_{k-j} = \sum_{j=0}^{k-1} \mu_j d_{k-j} = \sum_{j=0}^{k-1}\sum_{\ell=0}^{j} \lambda_\ell (-1)^{j-\ell}\binom{j-1}{j-\ell} d_{k-j} \quad \text{for} \quad k \geq 1,$$

where we have also noted that $d_0 = 0$. The proof of Theorem 4 is now complete. □

Next, by using the formula (23), we show how easily we can determine λ_k's in (22). Indeed, we see from (25) that

$$c_1 = 1, \quad c_2 = -\frac{11}{9}, \quad c_3 = \frac{4}{3}, \quad c_4 = -\frac{106}{81},$$
$$c_5 = \frac{92}{81}, \quad c_6 = -\frac{596}{729}, \quad c_7 = \frac{272}{729}, \quad c_8 = \frac{1064}{6561}.$$

We note that the sequence $\{c_k\}$ has the following explicit formula:

$$c_k = \left(\frac{1}{2} + \frac{1}{3}i\right)\left(-1 + \frac{1}{3}i\right)^{k-1} + \left(\frac{1}{2} - \frac{1}{3}i\right)\left(-1 - \frac{1}{3}i\right)^{k-1}$$

$$(k \geq 1;\ i = \sqrt{-1}).$$

We thus find from (24) that

$$d_0 = d_1 = d_2 = 0, \quad d_3 = \frac{1}{27}, \quad d_4 = -\frac{1}{9}, \quad d_5 = \frac{52}{243},$$
$$d_6 = -\frac{80}{243}, \quad d_7 = \frac{958}{2187}, \quad d_8 = -\frac{394}{729}.$$

We give the first few coefficients λ_k as follows:

$$\begin{aligned}
&\lambda_0 = 1,\\
&\lambda_1 = \lambda_0 d_1 = 0,\\
&\lambda_2 = \lambda_0 d_2 + \lambda_1 d_1 = 0,\\
&\lambda_3 = \lambda_0 d_3 + \lambda_1 d_2 - \lambda_1 d_1 + \lambda_2 d_1 = \lambda_0 d_3 = \frac{1}{27},\\
&\lambda_4 = \lambda_0 d_4 + \lambda_1 d_3 - \lambda_1 d_2 + \lambda_2 d_2 + \lambda_1 d_1 - 2\lambda_2 d_1 + \lambda_3 d_1 = \lambda_0 d_4 = -\frac{1}{9},\\
&\lambda_5 = \lambda_0 d_5 + \lambda_1 d_4 - \lambda_1 d_3 + \lambda_2 d_3 + \lambda_1 d_2 - 2\lambda_2 d_2 + \lambda_3 d_2 - \lambda_1 d_1 + 3\lambda_2 d_1\\
&\qquad - 3\lambda_3 d_1 + \lambda_4 d_1 = \lambda_0 d_5 = \frac{52}{243},\\
&\lambda_6 = \lambda_0 d_6 + \lambda_1 d_5 - \lambda_1 d_4 + \lambda_2 d_4 + \lambda_1 d_3 - 2\lambda_2 d_3 + \lambda_3 d_3 - \lambda_1 d_2 + 3\lambda_2 d_2 - 3\lambda_3 d_2\\
&\qquad + \lambda_4 d_2 + \lambda_1 d_1 - 4\lambda_2 d_1 + 6\lambda_3 d_1 - 4\lambda_4 d_1 + \lambda_5 d_1 = \lambda_0 d_6 + \lambda_3 d_3 = -\frac{239}{729},\\
&\lambda_7 = \lambda_0 d_7 + \lambda_3 d_4 - 3\lambda_3 d_3 + \lambda_4 d_3 = \frac{931}{2187},\\
&\lambda_8 = \lambda_0 d_8 + \lambda_3 d_5 + \lambda_4 d_4 - 3\lambda_3 d_4 + \lambda_5 d_3 - 4\lambda_4 d_3 + 6\lambda_3 d_3 = -\frac{3118}{6561}.
\end{aligned}$$

We note that the values of λ_k (for $k = 0,1,2,3,4,5,6,7,8$) here are equal to the coefficients of $1/n^k$ (for $k = 0,1,2,3,4,5,6,7,8$) in (6), respectively.

Theorem 5. *Let the remainder $\mathcal{R}_n$ be defined in* (4). *Then, for all integers $n \ge 1$,*

$$\mathcal{L}_n < \mathcal{R}_n < \mathcal{U}_n, \tag{30}$$

where

$$\mathcal{L}_n := \frac{9n^2+18n+10}{(3n+3)!}\left(1 + \frac{1}{27n^3} - \frac{1}{9n^4}\right)$$

and

$$\mathcal{U}_n := \frac{9n^2+18n+10}{(3n+3)!}\left(1 + \frac{1}{27n^3} - \frac{1}{9n^4} + \frac{52}{243n^5}\right).$$

Proof. For $n \ge 1$, let

$$x_n := \mathcal{R}_n - \mathcal{L}_n \quad \text{and} \quad y_n := \mathcal{R}_n - \mathcal{U}_n.$$

Then, clearly, we have

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = 0.$$

In order to prove (30), it suffices to show that the sequence $\{x_n\}$ is strictly decreasing and that the sequence $\{y_n\}$ is strictly increasing for $n \ge 1$. Direct computation yields

$$\begin{aligned}
x_n - x_{n+1} &= \frac{9n^2+18n+10}{(3n+3)!} - \mathcal{L}_n + \mathcal{L}_{n+1} \\
&= \frac{9n^2+18n+10}{(3n+3)!} - \frac{9n^2+18n+10}{(3n+3)!}\left(1+\frac{1}{27n^3}-\frac{1}{9n^4}\right) \\
&\quad + \frac{9(n+1)^2+18(n+1)+10}{(3(n+1)+3)!}\left(1+\frac{1}{27(n+1)^3}-\frac{1}{9(n+1)^4}\right) \\
&= \frac{1}{27n^4(n+1)^4(3n+6)!}\Big(1404n^8+13293n^7+54012n^6+123121n^5 \\
&\quad + 172750n^4+153300n^3+84258n^2+26340n+3600\Big) > 0
\end{aligned}$$

and

$$\begin{aligned}
&y_n - y_{n+1} = \frac{9n^2+18n+10}{(3n+3)!} - \mathcal{U}_n + \mathcal{U}_{n+1} \\
&= \frac{9n^2+18n+10}{(3n+3)!} - \frac{9n^2+18n+10}{(3n+3)!}\left(1+\frac{1}{27n^3}-\frac{1}{9n^4}+\frac{52}{243n^5}\right) \\
&\quad + \frac{9(n+1)^2+18(n+1)+10}{(3(n+1)+3)!}\left(1+\frac{1}{27(n+1)^3}-\frac{1}{9(n+1)^4}+\frac{52}{243(n+1)^5}\right) \\
&= -\frac{1}{243n^5(n+1)^5(3n+6)!}\Big(19359n^9+207171n^8+971847n^7+2624469n^6 \\
&\quad + 4504202n^5+5103030n^4+3821274n^3+1827492n^2+507360n+62400\Big) < 0.
\end{aligned}$$

We thus have completed the proof of Theorem 5. □

Remark 6. We write (30) as follows:

$$\mathcal{P}_n < e < \mathcal{Q}_n, \tag{31}$$

where

$$\mathcal{P}_n := \mathcal{S}_n + \mathcal{L}_n \quad \text{and} \quad \mathcal{Q}_n := \mathcal{S}_n + \mathcal{U}_n.$$

For $n = 10$ in (31), we have

$$\mathcal{P}_{10} = 2.71828182845904523536028747135266249775 70\cdots$$

and

$$\mathcal{Q}_{10} = 2.71828182845904523536028747135266249775 72\cdots.$$

We then get the following approximate value of e:

$$e \approx 2.718281828459045235360287471352662497757.$$

The choice $n = 100$ in (31) yields the approximate value of e as follows:

$$
\begin{aligned}
e \approx\, & 2.71828182845904523536028747135266249775724709369995\\
& 95749669676277240766303535475945713821785251664274\\
& 27466391932003059921817413596629043572900334295260\\
& 59563073813232862794349076323382988075319525101901\\
& 15738341879307021540891499348841675092447614606680\\
& 82264800168477411853742345442437107539077744992069\\
& 55170276183860626133138458300075204493382656029760\\
& 67371132007093287091274437470472306969772093101416\\
& 92836819025515108657463772111252389784425056953696\\
& 77078544996996794686445490598793163688923009879312\\
& 77361782154249992295763514822082698951936680331825\\
& 28869398496465105820939239829488793320362509443117\\
& 30123819706841614039701983 7.
\end{aligned}
$$

Clearly, the two-sided inequality (31) is much better than the two-sided inequality (21).

3. INEQUALITIES FOR I_n AND $I(x)$

Theorem 7. *Let I_n be defined by* (9)*. Then, for $n \geq 2$,*

$$
1 - \frac{1}{2n^2 + \frac{2}{3}n - \frac{1}{3}} \leq \frac{I_{n-1}}{I_n} < 1 - \frac{1}{2n^2 + \frac{2}{3}n - \frac{5}{18}}, \tag{32}
$$

where the constants $\frac{1}{3}$ and $\frac{5}{18}$ are the best possible.

Proof. First of all, we show that the left-hand side of (32) is valid for $n = 2$. Direct computation yields

$$
\left[\frac{I_{n-1}}{I_n}\right]_{n=2} = \frac{8}{9} \quad \text{and} \quad \left[1 - \frac{1}{2n^2 + \frac{2}{3}n - \frac{1}{3}}\right]_{n=2} = \frac{8}{9}.
$$

Hence, for $n = 2$, the equal sign on the left-hand side of (32) holds. In order to prove the double inequality (32) for $n \geq 2$, it suffices to show that

$$
U(n) > 0 \quad \text{for} \quad n \geq 3 \quad \text{and} \quad V(n) < 0 \quad \text{for} \quad n \geq 2,
$$

where

$$
U(x) = (x-1)\ln\left(1 + \frac{1}{x-1}\right) - x\ln\left(1 + \frac{1}{x}\right) - \ln\left(1 - \frac{1}{2x^2 + \frac{2}{3}x - \frac{1}{3}}\right)
$$

and

$$V(x) = (x-1)\ln\left(1+\frac{1}{x-1}\right) - x\ln\left(1+\frac{1}{x}\right) - \ln\left(1-\frac{1}{2x^2+\frac{2}{3}x-\frac{5}{18}}\right).$$

Differentiation yields

$$U'(x) = \ln\left(1+\frac{1}{x-1}\right) - \ln\left(1+\frac{1}{x}\right) - \frac{2(9x^2-3x+1)}{x(3x-2)(6x^2+2x-1)}$$

and

$$U''(x) = \frac{2\left[109+1171(x-3)+1152(x-3)^2+393(x-3)^3+45(x-3)^4\right]}{x^2(x^2-1)(3x-2)^2(6x^2+2x-1)^2} > 0$$

for $x \geq 3$. We then obtain that, for $x \geq 3$,

$$U'(x) < \lim_{t\to\infty} U'(t) = 0 \Longrightarrow U(x) > \lim_{t\to\infty} U(t) = 0.$$

Therefore, the left-hand side of (32) is valid for $n \geq 2$.

Differentiation yields

$$\begin{aligned} V'(x) =& \ln\left(1+\frac{1}{x-1}\right) - \ln\left(1+\frac{1}{x}\right) \\ &- \frac{1296x^4+2160x^3+648x^2-120x+115}{x(x+1)(36x^2+12x-23)(36x^2+12x-5)} \end{aligned}$$

and

$$V''(x) = -\frac{P_6(x-2)}{x^2(x-1)(36x^2+12x-5)^2(36x^2+12x-23)^2(x+1)^2},$$

where

$$\begin{aligned} P_6(x) =& 155930911+400928667x+422130384x^2+232978896x^3 \\ &+ 71102448x^4+11380176x^5+746496x^6. \end{aligned}$$

We then obtain that, for $x \geq 2$,

$$V''(x) < 0 \Longrightarrow V'(x) > \lim_{t\to\infty} V'(t) = 0 \Longrightarrow V(x) < \lim_{t\to\infty} V(t) = 0.$$

Therefore, the right-hand side of (32) is valid for $n \geq 2$.

If we write the double inequality (32) as

$$\frac{5}{18} < x_n \leq \frac{1}{3},$$

where

$$x_n = 2n^2 + \frac{2}{3}n - \frac{1}{1 - \frac{I_{n-1}}{I_n}}, \tag{33}$$

we find that

$$x_2 = \frac{1}{3}$$

and

$$\begin{aligned}\lim_{n\to\infty} x_n &= \lim_{n\to\infty}\left\{2n^2 + \frac{2}{3}n - \frac{1}{1-\frac{I_{n-1}}{I_n}}\right\}\\ &= \lim_{n\to\infty}\left\{2n^2 + \frac{2}{3}n - \frac{1}{\frac{1}{2n^2} - \frac{1}{6n^3} + \frac{1}{8n^4} - \frac{1}{20n^5} + O(\frac{1}{n^6})}\right\}\\ &= \lim_{n\to\infty}\left\{\frac{5}{18} + O\left(\frac{1}{n}\right)\right\} = \frac{5}{18}.\end{aligned}$$

Hence, the double inequality (32) holds for $n \ge 2$, and the constants $\frac{1}{3}$ and $\frac{5}{18}$ are the best possible. The proof of Theorem 7 is complete. □

Remark 8. *Let the sequence $\{x_n\}$ be defined by* (33)*. In order to prove Theorem 7 , it suffices to show that the sequence $\{x_n\}$ is strictly decreasing for $n \ge 2$.*

Remark 9. *From the right-hand side of* (32)*, we obtain*

$$I_n < I_{n+1} \quad \textit{for} \quad n \in \mathbb{N}.$$

This shows that the sequence $\{I_n\}_{n\in\mathbb{N}}$ is strictly increasing.

Theorem 10. *Let I_n be defined by* (9)*. Then, for $n \ge 2$,*

$$1 + \frac{1}{n^3 + 2n^2 - \frac{7}{6}} < \frac{I_n^2}{I_{n-1}I_{n+1}} \le 1 + \frac{1}{n^3 + 2n^2 - \frac{176}{139}}, \tag{34}$$

where the constants $\frac{7}{6}$ and $\frac{176}{139}$ are the best possible.

Proof. First of all, we show that the right-hand side of (34) is valid for $n = 2$. Direct computation yields

$$\left[\frac{I_n^2}{I_{n-1}I_{n+1}}\right]_{n=2} = \frac{2187}{2048} \quad \text{and} \quad \left[1 + \frac{1}{n^3 + 2n^2 - \frac{176}{139}}\right]_{n=2} = \frac{2187}{2048}.$$

Hence, for $n = 2$, the equal sign on the right-hand side of (34) holds. In order to prove the double inequality (34) for $n \ge 2$, it suffices to show that

$$p(n) > 0 \quad \text{for} \quad n \ge 2 \quad \text{and} \quad q(n) < 0 \quad \text{for} \quad n \ge 3,$$

where

$$p(x) = 2x\ln\left(1+\frac{1}{x}\right) - (x-1)\ln\left(1+\frac{1}{x-1}\right) - (1+x)\ln\left(1+\frac{1}{x+1}\right) - \ln\left(1+\frac{1}{x^3+2x^2-\frac{7}{6}}\right)$$

and

$$q(x) = 2x\ln\left(1+\frac{1}{x}\right) - (x-1)\ln\left(1+\frac{1}{x-1}\right) - (1+x)\ln\left(1+\frac{1}{x+1}\right) - \ln\left(1+\frac{1}{x^3+2x^2-\frac{176}{139}}\right).$$

Differentiation yields

$$p'(x) = \ln\left(\frac{\left(1+\frac{1}{x}\right)^2}{\left(1+\frac{1}{x-1}\right)\left(1+\frac{1}{x+1}\right)}\right) + \frac{2(7+48x^2+276x^3+378x^4+198x^5+36x^6)}{x(1+x)(2+x)(6x^3+12x^2-1)(6x^3+12x^2-7)}$$

and

$$p''(x) = \frac{P_{10}(x-2)}{x^2(6x^3+12x^2-1)^2(6x^3+12x^2-7)^2(2+x)^2(x-1)(1+x)^2},$$

where

$$\begin{aligned} P_{10}(x) = & 56238695 + 219901505x + 382905747x^2 + 391107816x^3 \\ & + 259578240x^4 + 116999226x^5 + 36276750x^6 + 7641846x^7 \\ & + 1046898x^8 + 84240x^9 + 3024x^{10}. \end{aligned}$$

We then obtain that, for $x \geq 2$,

$$p''(x) > 0 \Longrightarrow p'(x) < \lim_{t\to\infty} p'(t) = 0 \Longrightarrow p(x) > \lim_{t\to\infty} p(t) = 0.$$

Therefore, the left-hand side of (34) is valid for $n \geq 2$.

Differentiation yields

$$q'(x) = \ln\left(\frac{\left(1+\frac{1}{x}\right)^2}{\left(1+\frac{1}{x-1}\right)\left(1+\frac{1}{x+1}\right)}\right) + \frac{38642x^6+212531x^5+405741x^4+288564x^3+36140x^2+13024}{x(x+1)(x+2)(139x^3+278x^2-37)(139x^3+278x^2-176)}$$

and

$$q''(x) = -\frac{P_{11}(x-3)}{x^2(x-1)(x+1)^2(x+2)^2(139x^3+278x^2-176)^2(139x^3+278x^2-37)^2},$$

where

$$\begin{aligned} P_{11}(x) =& 231145991851952 + 1449139445548576x + 3015626308590458x^2 \\ &+ 3311135822124859x^3 + 2255215232619316x^4 + 1025646710875072x^5 \\ & 322045070339960x^6 + 70367089380622x^7 + 10539007553692x^8 \\ &+ 1034049254808x^9 + 59969872270x^{10} + 1560344639x^{11}. \end{aligned}$$

We then obtain that, for $x \geq 3$,

$$q''(x) < 0 \Longrightarrow q'(x) > \lim_{t\to\infty} q'(t) = 0 \Longrightarrow q(x) < \lim_{t\to\infty} q(t) = 0.$$

Therefore, the right-hand side of (34) is valid for $n \geq 2$.

If we write the double inequality (34) as

$$\frac{7}{6} < y_n \leq \frac{176}{139},$$

where

(35) $$y_n = n^3 + 2n^2 - \frac{1}{\frac{I_n^2}{I_{n-1}I_{n+1}} - 1},$$

we find that

$$y_2 = \frac{176}{139}$$

and

$$\begin{aligned} \lim_{n\to\infty} y_n &= \lim_{n\to\infty} \left\{ n^3 + 2n^2 - \frac{1}{\frac{I_n^2}{I_{n-1}I_{n+1}} - 1} \right\} \\ &= \lim_{n\to\infty} \left\{ n^3 + 2n^2 - \frac{1}{\frac{1}{n^3} - \frac{2}{n^4} + \frac{4}{n^5} - \frac{41}{6n^6} + \frac{23}{2n^7} - \frac{56}{30n^8} + O(\frac{1}{n^9})} \right\} \\ &= \lim_{n\to\infty} \left\{ \frac{7}{6} + O\left(\frac{1}{n}\right) \right\} = \frac{7}{6}. \end{aligned}$$

Hence, the double inequality (34) holds for $n \geq 2$, and the constants $\frac{7}{6}$ and $\frac{176}{139}$ are the best possible. The proof of Theorem 10 is complete. □

Remark 11. *Let the sequence* $\{y_n\}$ *be defined by* (35)*. In order to prove Theorem 10, it suffices to show that the sequence* $\{y_n\}$ *is strictly decreasing for* $n \geq 2$.

Remark 12. *A sequence* $\{a_n\}_{n\in\mathbb{N}}$ *of real numbers is called strictly log-convex (log-concave), if it is positive and*

$$a_{n+1}^2 < (>) a_n a_{n+2} \quad \textit{for} \quad n \in \mathbb{N}.$$

By the arithmetic-geometric mean inequality, the log-convexity implies the convexity, and the concavity implies the log-concavity. From the left-hand side of (34)*, we obtain*

$$I_{n+1}^2 > I_n I_{n+2} \quad \textit{for} \quad n \in \mathbb{N}.$$

This shows that the sequence $\{I_n\}_{n\in\mathbb{N}}$ *is strictly log-concave.*

Turán [28] proved that, for $|x| \leq 1$ and $n \in \mathbb{N}$,

$$0 \leq P_n(x)^2 - P_{n-1}(x)P_{n+1}(x), \tag{36}$$

where $P_n(x)$ denotes the Legendre polynomial of degree n. The inequality (36) has attracted much interest from many mathematicians, various inequalities of the same type were presented for other special functions. From the left-hand side of (34), we obtain the following Turán-type inequality:

$$0 < I_n^2 - I_{n-1}I_{n+1} \quad \text{for} \quad n \geq 2.$$

Some computer experiments led us to pose the following conjecture.

Conjecture 13. *Let* I_n *be defined by* (9)*. Then, for* $n \geq 2$*,*

$$\frac{e^2}{n^3 + 3n^2 + \frac{11}{6}n - a} < I_n^2 - I_{n-1}I_{n+1} \leq \frac{e^2}{n^3 + 3n^2 + \frac{11}{6}n - b},$$

with the best possible constants

$$a = \frac{1}{2} \quad \textit{and} \quad b = \frac{71}{3} - \frac{432e^2}{139} = 0.702118\ldots.$$

Theorem 14. *Let* $I(x)$ *be defined by* (13)*. Then, for* $x > 0$*,*

$$\frac{I(x) + I(1/x)}{2} \leq 2 \tag{37}$$

with equality if $x = 1$*.*

Proof. It suffices to show that the inequality (37) holds for $x \geq 1$. Consider the function $F(x)$ defined by

$$F(x) = \frac{I(x) + I(1/x)}{2}, \qquad x \geq 1.$$

Differentiation yields

$$2xF'(x) = G(x) - G\left(\frac{1}{x}\right), \tag{38}$$

where

$$G(x) = xI'(x) = x\left(1+\frac{1}{x}\right)^x\left[\ln\left(1+\frac{1}{x}\right) - \frac{1}{x+1}\right].$$

Differentiation yields

$$-\frac{G'(x)}{x\left(1+\frac{1}{x}\right)^x} = \frac{x+2}{x(x+1)^2} - \frac{1}{x}\ln\left(1+\frac{1}{x}\right) - \left[\ln\left(1+\frac{1}{x}\right) - \frac{1}{x+1}\right]^2. \tag{39}$$

We consider two cases to prove $G'(x) < 0$ for $x \geq 1$.

Case 1. $1 \leq x \leq 2$.

Using the techniques in [8], we now prove $G'(x) < 0$ for $1 \leq x \leq 2$. Write (39) as

$$-\frac{G'(x)}{x\left(1+\frac{1}{x}\right)^x} = Q(x) - P(x),$$

where

$$P(x) = \frac{1}{x}\ln\left(1+\frac{1}{x}\right) + \left[\ln\left(1+\frac{1}{x}\right) - \frac{1}{x+1}\right]^2 \quad \text{and} \quad Q(x) = \frac{x+2}{x(x+1)^2}.$$

Clearly, $Q(x)$ is strictly decreasing on $(0,\infty)$. Noting that the functions

$$\frac{1}{x}\ln\left(1+\frac{1}{x}\right) \quad \text{and} \quad \ln\left(1+\frac{1}{x}\right) - \frac{1}{x+1}$$

are both strictly decreasing and $\ln\left(1+\frac{1}{x}\right) - \frac{1}{x+1} > 0$ on $(0,\infty)$, we see that $P(x)$ is strictly decreasing on $(0,\infty)$. We divide the interval $[1,2]$ into 100 subintervals:

$$[1,2] = \bigcup_{k=0}^{99}\left[1+\frac{k}{100}, 1+\frac{k+1}{100}\right].$$

By direct computation we get

$$Q\left(1+\frac{k+1}{100}\right) > P\left(1+\frac{k}{100}\right) \quad \text{for} \quad k = 0,1,2,\ldots,99.$$

Hence,

$$Q(x) > P(x) \quad \text{for} \quad x \in \left[1+\frac{k}{100}, 1+\frac{k+1}{100}\right] \quad \text{and} \quad k = 0,1,2,\ldots,99.$$

This implies that $Q(x) > P(x)$ and $G'(x) < 0$ hold for $1 \leq x \leq 2$.

Case 2. $x \geq 2$.

It is well known that

$$\sum_{j=1}^{2m} \frac{(-1)^{j-1}}{j} t^j < \ln(1+t) < \sum_{j=1}^{2m-1} \frac{(-1)^{j-1}}{j} t^j \tag{40}$$

for $-1 < t \leq 1$ and $m \in \mathbb{N}$. Using the right-hand side of (40), we obtain that for $x \geq 2$,

$$\begin{aligned} -\frac{G'(x)}{x\left(1+\frac{1}{x}\right)^x} &= \frac{x+2}{x(x+1)^2} - \frac{1}{x}\ln\left(1+\frac{1}{x}\right) - \left[\ln\left(1+\frac{1}{x}\right) - \frac{1}{x+1}\right]^2 \\ &> \frac{x+2}{x(x+1)^2} - \frac{1}{x}\left(\frac{1}{x} - \frac{1}{2x^2} + \frac{1}{3x^3}\right) - \left[\left(\frac{1}{x} - \frac{1}{2x^2} + \frac{1}{3x^3}\right) - \frac{1}{x+1}\right]^2 \\ &= \frac{144 + 352(x-2) + 315(x-2)^2 + 123(x-2)^3 + 18(x-2)^4}{36x^6(x+1)} > 0. \end{aligned}$$

Hence, $G'(x) < 0$ holds for $x \geq 1$. Therefore, the function $G(x)$ is strictly decreasing for $x > 1$. We see from (38) that $F'(x) < 0$ for $x \in (1, \infty)$. Therefore, the function $F(x)$ strictly decreasing on $(1, \infty)$, and we have, for $x \geq 1$,

$$\frac{I(x) + I(1/x)}{2} = F(x) \leq F(1) = 2$$

with equality if $x = 1$. The proof of Theorem 10 is complete. □

4. COMPLETELY MONOTONIC FUNCTIONS

A function f is said to be completely monotonic on an open interval (a, b) $(-\infty \leq a < b \leq \infty)$ if

$$(-1)^n f^{(n)}(x) \geq 0 \quad \text{for} \quad a < x < b \quad \text{and} \quad n \in \mathbb{N}_0. \tag{41}$$

If, in addition, f is continuous at $x = a$, then it is called completely monotonic on $[a, b)$, with similar definitions for $(a, b]$ and $[a, b]$.

Dubourdieu [14, p. 98] pointed out that if a non-constant function f is completely monotonic over (a, ∞), then the strict inequality in (41) holds true. See also [17] for a simpler proof of this result. It is known (Bernstein's Theorem) that f is completely monotonic on $(0, \infty)$ if and only if

$$f(x) = \int_0^\infty e^{-xt}\, \mathrm{d}\mu(t),$$

where μ is a nonnegative measure on $[0, \infty)$ such that the integral converges for all $x > 0$ (see [29, p. 161]). This means that a completely monotonic function f on

$[0, \infty)$ is a Laplace transform with respect to the measure μ. The main properties of completely monotonic functions are given in [29, Chapter IV].

Recall [16] that a positive function f is said to be logarithmically completely monotonic on an interval I if its logarithm $\ln f$ satisfies

$$(-1)^k[\ln f(x)]^{(k)} \geq 0 \quad \text{for} \quad x \in I \quad \text{and} \quad k \in \mathbb{N}.$$

Recall that a function f is said to be a Bernstein function on an interval I if $f > 0$ and f' is completely monotonic on I.

Lemma 15. *Denote $I = (a, b)$ with $a \leq 0$ and $0 < b \leq \infty$. Let the function ϕ have derivatives of all orders on I and $\phi(0) = 0$. Define the function f_1 by*

$$f_1(x) = \begin{cases} \dfrac{\phi(x)}{x}, & x \neq 0 \\ \phi'(0), & x = 0. \end{cases}$$

If ϕ' is completely monotonic on I, then f_1 is completely monotonic on I.

Lemma 15 follows from Theorem 2.1 and Remark 2.2 (i) of [9].

Theorem 16. *For $x > -1$, let*

$$f_1(x) = \frac{\ln(1+x)}{x} \quad x \neq 0 \quad \text{and} \quad f_1(0) = 1. \tag{42}$$

Then the function $f_1(x)$ is completely monotonic on $(-1, \infty)$.

Proof. Let $\phi(x) = \ln(1+x)$. Then

$$f_1(x) = \begin{cases} \dfrac{\phi(x)}{x}, & x \neq 0, \\ 1, & x = 0. \end{cases}$$

We have $\phi(0) = 0$ and

$$(-1)^n \phi^{(n+1)}(x) = \frac{n!}{(1+x)^{n+1}} > 0 \quad \text{for} \quad x > -1 \quad \text{and} \quad n \in \mathbb{N}_0.$$

Hence, the function ϕ' is completely monotonic on $(-1, \infty)$. We see from Lemma 15 that the function $f_1(x)$, defined by (42), is completely monotonic on $(-1, \infty)$. □

Remark 17. *Theorem 16 shows that the function*

$$f(x) = (1+x)^{1/x}, \quad x \neq 0 \quad \text{and} \quad f(0) = e \tag{43}$$

is logarithmically completely monotonic on $(-1, \infty)$. It was shown in [3–5, 26] that a logarithmically completely monotonic function f on I must be completely monotonic on I. Hence, the function $f(x)$, defined by (43), is completely monotonic on $(-1, \infty)$.

Theorem 18. *The function*

$$g(x) = x \ln(1 + 1/x)$$

is a Bernstein function on $(0, \infty)$.

Proof. By using the integral representation

$$\ln x = \int_0^\infty \frac{e^{-t} - e^{-xt}}{t} \mathrm{d}t, \qquad x > 0,$$

we get

$$g(x) = x \ln\left(1 + \frac{1}{x}\right) = x \int_0^\infty \frac{e^{-xt} - e^{-(x+1)t}}{t} \mathrm{d}t = x \int_0^\infty \varphi(t) e^{-xt} \mathrm{d}t,$$

where

$$\varphi(t) = \frac{1 - e^{-t}}{t} = \frac{e^t - 1}{te^t}.$$

Differentiation yields

$$\varphi'(t) = -\frac{e^t - 1 - t}{t^2 e^t} = -\frac{1}{t^2 e^t} \sum_{n=2}^\infty \frac{t^n}{n!} < 0, \qquad t > 0.$$

Hence, the function $\varphi(t)$ is strictly decreasing on $(0, \infty)$, and we have

$$\varphi(t) > \lim_{u \to \infty} \varphi(u) = 0, \qquad t > 0.$$

Direct computation yields

$$\begin{aligned}
(-1)^n g^{(n)}(x) &= (-1)^n \sum_{k=0}^n \binom{n}{k} x^{(k)} \left(\int_0^\infty \varphi(t) e^{-xt} \mathrm{d}t\right)^{(n-k)} \\
&= x \int_0^\infty \varphi(t) e^{-xt} t^n \mathrm{d}t - n \int_0^\infty \varphi(t) e^{-xt} t^{n-1} \mathrm{d}t \\
&= \int_0^{n/x} \varphi(t) e^{-xt} t^{n-1} (xt - n) \mathrm{d}t + \int_{n/x}^\infty \varphi(t) e^{-xt} t^{n-1} (xt - n) \mathrm{d}t \\
&< \varphi(n/x) \int_0^{n/x} e^{-xt} t^{n-1} (xt - n) \mathrm{d}t + \varphi(n/x) \int_{n/x}^\infty e^{-xt} t^{n-1} (xt - n) \mathrm{d}t \\
&= \varphi(n/x) \int_0^\infty e^{-xt} t^{n-1} (xt - n) \mathrm{d}t. \qquad (44)
\end{aligned}$$

Noting that

$$\frac{m!}{x^{m+1}} = \int_0^\infty t^m e^{-xt} \mathrm{d}t \quad \text{for} \quad x > 0 \quad \text{and} \quad m \in \mathbb{N}_0,$$

we find

$$\int_0^\infty e^{-xt}t^{n-1}(xt-n)\mathrm{d}t = 0.$$

We then obtain from (44) that

$$(-1)^n g^{(n)}(x) < 0 \quad \text{for} \quad x > 0 \quad \text{and} \quad n \in \mathbb{N},$$

which can be written as

$$(-1)^n (g'(x))^{(n)} > 0 \quad \text{for} \quad x > 0 \quad \text{and} \quad n \in \mathbb{N}_0.$$

Clearly, the function $g'(x)$ is completely monotonic on $(0,\infty)$. Therefore, the function $g(x)$ is a Bernstein function on $(0,\infty)$. □

Remark 19. *Chen* et al. *[12, Theorem 3] proved that if f is a Bernstein function on an interval I, then $1/f$ is completely monotonic on I. We see from Theorem 18 that the function $1/g(x)$ is completely monotonic on $(0,\infty)$.*

5. CONCLUSION

Here, in our present investigation, we have first revisited several rapidly convergent and not-so-rapidy convergent series representations for the familiar constant e, which is also known as Euler's number. We have then presented asymptotic expansions and two-sided inequalities for the remainders R_n and $\mathcal{R}_n$, which are given by

$$R_n = e - \sum_{k=0}^{n} \frac{1}{k!} \quad \text{and} \quad \mathcal{R}_n = e - \sum_{k=0}^{n} \frac{9k^2+1}{(3k)!}.$$

We present some sharp inequalities related to the quantities:

$$\frac{I_{n-1}}{I_n} \quad \text{and} \quad \frac{I_n^2}{I_{n-1}I_{n+1}} \quad \text{for} \quad n \geq 2.$$

We establish the arithmetic mean inequality of $I(x)$ and $I(1/x)$, namely,

$$\frac{I(x)+I(1/x)}{2} \leq 2 \quad \text{for} \quad x > 0$$

with equality if $x = 1$. Finally, we present completely monotonic functions involving $(1+1/x)^x$. We have also considered a number of related developments on the subject of this paper.

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Xue-Feng Han
School of Mathematics and Informatics,
Henan Polytechnic University,
Jiaozuo City 454000, Henan Province,
People's Republic of China.
E-Mail: *hanxuefeng8110@sohu.com*

Chao-Ping Chen
School of Mathematics and Informatics,
Henan Polytechnic University,
Jiaozuo City 454000, Henan Province,
People's Republic of China.
E-Mail: *chenchaoping@sohu.com*

H. M. Srivastava
Department of Mathematics and Statistics,
University of Victoria,
Victoria, British Columbia V8W 3R4, Canada
Department of Medical Research,
China Medical University Hospital,
China Medical University, Taichung 40402, Taiwan, Republic of China.
Department of Mathematics and Informatics,
Azerbaijan University,
71 Jeyhun Hajibeyli Street, AZ1007 Baku, Azerbaijan.
Section of Mathematics, International Telematic
University Uninettuno, I-00186 Rome, Italy
E-Mail: *harimsri@math.uvic.ca*