Applicable Analysis and Discrete Mathematics
available online at http://pefmath.etf.rs

Appl. Anal. Discrete Math. **18** (2024), 092–115.
https://doi.org/10.2298/AADM230902020Q

SERIES EXPANSIONS FOR POWERS OF SINC FUNCTION AND CLOSED-FORM EXPRESSIONS FOR SPECIFIC PARTIAL BELL POLYNOMIALS

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Dedicated to Professor Dr. Jen-Chih Yao at China Medical University in Taiwan

In the paper, with the aid of the Faà di Bruno formula, in terms of the central factorial numbers and the Stirling numbers of the second kinds, the authors derive several series expansions for any positive integer powers of the sinc and sinhc functions, discover several closed-form expressions for partial Bell polynomials of all derivatives of the sinc function, establish several series expansions for any real powers of the sinc and sinhc functions, and present several identities for central factorial numbers of the second kind and for the Stirling numbers of the second kind.

1. MOTIVATIONS

According to common knowledge in complex analysis, the principal value of the number α^β for $\alpha, \beta \in \mathbb{C}$ with $\alpha \ne 0$ is defined by $\alpha^\beta = \mathrm{e}^{\beta \ln \alpha}$, where $\ln \alpha = \ln |\alpha| + \mathrm{i} \arg \alpha$ and $\arg \alpha$ are principal values of the logarithm and argument of $\alpha \ne 0$ respectively. In what follows, we always consider principal values of real or complex functions.

In mathematical sciences, ones commonly consider elementary functions

$$\mathrm{e}^z, \quad \ln(1+z), \quad \sin z, \quad \csc z, \quad \cos z, \quad \sec z, \quad \tan z, \quad \cot z,$$

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2020 Mathematics Subject Classification. 41A58, 05A19, 11B73, 11B83, 11C08, 33B10
Keywords and Phrases. Faà di Bruno formula, Series expansion, Sinc function, Power, Sinhc function, Partial Bell polynomial, Closed-form expression, Stirling number of the second kind, Central factorial number of the second kind, Weighted Stirling number of the second kind, Combinatorial proof

$$\arcsin z,\quad \arccos z,\quad \arctan z,\quad \sinh z,\quad \operatorname{csch} z,\quad \cosh z,\quad \operatorname{sech} z,$$
$$\tanh z,\quad \coth z,\quad \operatorname{arcsinh} z,\quad \operatorname{arccosh} z,\quad \operatorname{arctanh} z$$

and their series expansions at the point $z = 0$. Their series expansions can be found in mathematical handbooks such as [**1, 18, 39**].

What are the series expansions of positive integer powers or real powers of these functions at the origin $z = 0$?

It is combinatorial knowledge [**13, 16**] that the coefficients in the power series expansion of the power function $(\mathrm{e}^z - 1)^k$ for $k \in \mathbb{N} = \{1, 2, \dots\}$ are the Stirling numbers of the second kind, while the coefficients in the series expansion of the power function $[\ln(1+z)]^k$ for $k \in \mathbb{N}$ are the Stirling numbers of the first kind. In other words, the power functions $(\mathrm{e}^z - 1)^k$ and $[\ln(1+z)]^k$ for $k \in \mathbb{N}$ are generating functions of the Stirling numbers of the first and second kinds.

In the paper [**12**], among other things, Carlitz introduced the notion of weighted Stirling numbers of the second kind $R(n, k, r)$. Carlitz also proved in [**12**] that the numbers $R(n, k, r)$ can be generated by

$$\frac{(\mathrm{e}^z - 1)^k}{k!}\,\mathrm{e}^{rz} = \sum_{n=k}^{\infty} R(n, k, r)\frac{z^n}{n!} \tag{1}$$

and can be explicitly expressed by

$$R(n, k, r) = \frac{1}{k!}\sum_{j=0}^{k} (-1)^{k-j}\binom{k}{j}(r+j)^n \tag{2}$$

for $r \in \mathbb{R}$ and $n \ge k \in \mathbb{N}_0 = \{0, 1, 2, \dots\}$. Specially, when $r = 0$, the quantities $R(n, k, 0)$ become the Stirling numbers of the second kind $S(n, k)$. By the way, the notion $\left\{ n \atop k \right\}_r = R(n-r, k-r, r)$ is called the r-Stirling numbers of the second kind in [**8**] by Broder.

The central factorial numbers of the second kind $T(n, \ell)$ for $n \ge \ell \in \mathbb{N}_0$ can be generated [**10, 34**] by

$$\frac{1}{\ell!}\left(2\sinh\frac{z}{2}\right)^{\ell} = \sum_{n=\ell}^{\infty} T(n, \ell)\frac{z^n}{n!}. \tag{3}$$

In [**59**, Chapter 6, Eq. (26)], it was established that

$$T(n, \ell) = \frac{1}{\ell!}\sum_{j=0}^{\ell} (-1)^j\binom{\ell}{j}\left(\frac{\ell}{2} - j\right)^n. \tag{4}$$

Note that $T(0, 0) = 1$ and $T(n, 0) = 0$ for $n \in \mathbb{N}$. See also [**10**, Proposition 2.4, (xii)] and [**51, 56**]. Comparing (3) with (1) or comparing (4) with (2) gives the relation

$$R\left(n, \ell, -\frac{\ell}{2}\right) = T(n, \ell) \tag{5}$$

between weighted Stirling numbers of the second kind and central factorial numbers of the second kind. See also [**56**, Theorem 3.1].

In the handbook [**18**], series expansions at $z = 0$ of the functions $\arcsin^2 z$, $\arcsin^3 z$, $\sin^2 z$, $\cos^2 z$, $\sin^3 z$, and $\cos^3 z$ are collected.

In the papers [**7, 19, 20, 35, 49, 55**] and plenty of references collected therein, the series expansions at $z = 0$ of the functions $\arcsin^m z$, $\operatorname{arcsinh}^m z$, $\arctan^m z$, $\operatorname{arctanh}^m z$ for $m \in \mathbb{N}$ have been established, applied, reviewed, and surveyed.

In the papers [**9, 45**], explicit series expansions at $z = 0$ of the functions $\tan^2 z$, $\tan^3 z$, $\cot^2 z$, $\cot^3 z$, $\sin^m z$, $\cos^m z$ for $m \in \mathbb{N}$ were written down.

In the papers [**3, 4, 24, 25, 36, 63, 69**], series expansions of the functions $I_\mu(z)I_\nu(z)$ and $[I_\nu(z)]^2$ were explicitly written out, while the series expansion of the power function $[I_\nu(z)]^r$ for $\nu \in \mathbb{C} \setminus \{-1, -2, \dots\}$ and $r, z \in \mathbb{C}$ was recursively formulated, where $I_\nu(z)$ denotes modified Bessel functions of the first kind.

In the paper [**46**], series expansions at $z = 0$ of the functions $(\arccos z)^r$ and $\bigl(\frac{\arcsin z}{z}\bigr)^r$ were established for real $r \in \mathbb{R}$. In [**49**], a series expansion at $z = 1$ of the function $\bigl[\frac{(\arccos z)^2}{2(1-z)}\bigr]^r$ was invented for real $r \in \mathbb{R}$.

For $z \in \mathbb{C}$, the functions

$$\operatorname{sinc} z = \begin{cases} \dfrac{\sin z}{z}, & z \ne 0 \\ 1, & z = 0 \end{cases} \quad \text{and} \quad \operatorname{sinhc} z = \begin{cases} \dfrac{\sinh z}{z}, & z \ne 0 \\ 1, & z = 0 \end{cases}$$

are called the sinc function and hyperbolic sinc function respectively. The function $\operatorname{sinc} z$ is also called the sine cardinal or sampling function, as well as the function $\operatorname{sinhc} z$ is also called hyperbolic sine cardinal, see [**60**]. The sinc function $\operatorname{sinc} z$ arises frequently in signal processing, the theory of the Fourier transforms, and other areas in mathematics, physics, and engineering. It is easy to see that these two functions $\operatorname{sinc} z$ and $\operatorname{sinhc} z$ are analytic on $\mathbb{C}$, that is, they are entire functions.

In [**13**, Theorem 11.4] and [**16**, p. 139, Theorem C], the Faà di Bruno formula is given for $n \in \mathbb{N}$ and $z \in \mathbb{C}$ by

$$\frac{\mathrm{d}^n}{\mathrm{d}\, z^n} f \circ h(z) = \sum_{k=1}^{n} f^{(k)}(h(z)) B_{n,k}\bigl(h'(z), h''(z), \dots, h^{(n-k+1)}(z)\bigr), \tag{6}$$

where partial Bell polynomials $B_{n,k}$ are defined for $n \ge k \in \mathbb{N}_0$ by

$$B_{n,k}(z_1, z_2, \dots, z_{n-k+1}) = \sum_{\substack{1 \le i \le n-k+1,\, \ell_i \in \{0\} \cup \mathbb{N} \\ \sum_{i=1}^{n-k+1} i\ell_i = n,\, \sum_{i=1}^{n-k+1} \ell_i = k}} \frac{n!}{\prod_{i=1}^{n-k+1} \ell_i!} \prod_{i=1}^{n-k+1} \left(\frac{z_i}{i!}\right)^{\ell_i}$$

in [**13**, Definition 11.2] and [**16**, p. 134, Theorem A].

In this paper, with the help of the Faà di Bruno formula (6), in terms of central factorial numbers of the second kind $T(n, k)$ and the Stirling numbers of the second kind $S(n,k)$, we will derive several series expansions at $z=0$ of the positive integer power functions $\operatorname{sinc}^\ell z$ and $\operatorname{sinhc}^\ell z$ for $\ell\in\mathbb{N}$ and $z\in\mathbb{C}$, we will deduce several closed-form expressions for central factorial numbers of the second kind $T(j+\ell,\ell)$ with $j,\ell\in\mathbb{N}$ in terms of the Stirling numbers of the second kind $S(n,k)$, we will discover several closed-form expressions of specific partial Bell polynomials

$$B_{n,k}\left(0,-\frac{1}{3},0,\frac{1}{5},\dots,\frac{(-1)^{n-k}}{n-k+2}\sin\frac{(n-k)\pi}{2}\right)$$

for $n\ge k\in\mathbb{N}$, we will establish series expansions at $z=0$ of the real power functions $\operatorname{sinc}^r z$ and $\operatorname{sinhc}^r z$ for $z\in\mathbb{C}$ and $r\in\mathbb{R}$, and we will present several identities for central factorial numbers of the second kind $T(n,k)$ and for the Stirling numbers of the second kind $S(n,k)$.

2. SERIES EXPANSIONS OF POSITIVE INTEGER POWERS

In this section, we derive several series expansions at $z=0$ of the positive integer power functions $\operatorname{sinc}^\ell z$ and $\operatorname{sinhc}^\ell z$ for $\ell\in\mathbb{N}$ and $z\in\mathbb{C}$ in terms of central factorial numbers of the second kind $T(n,k)$ and the Stirling numbers of the second kind $S(n,k)$, we deduce several closed-form expressions of $T(j+\ell,\ell)$ for $j\in\mathbb{N}_0$ and $\ell\in\mathbb{N}$ in terms of the Stirling numbers of the second kind $S(n,k)$, and we present several identities for central factorial numbers of the second kind $T(n,k)$.

Theorem 1. *For $\ell\in\mathbb{N}_0$ and $z\in\mathbb{C}$, we have*

$$\operatorname{sinc}^\ell z=1+\sum_{j=1}^{\infty}(-1)^j\frac{T(\ell+2j,\ell)}{\binom{\ell+2j}{\ell}}\frac{(2z)^{2j}}{(2j)!}. \tag{7}$$

Proof. For $\ell\in\mathbb{N}$, the formula

$$\sin^\ell z=\frac{(-1)^\ell}{2^\ell}\sum_{q=0}^{\ell}(-1)^q\binom{\ell}{q}\cos\left[(2q-\ell)z-\frac{\ell\pi}{2}\right] \tag{8}$$

is given in [**22**, Corollary 2.1]. Applying the identity

$$\cos(z-y)=\cos z\cos y+\sin z\sin y$$

to the formula (8) leads to

$$\begin{aligned}\sin^\ell z&=\frac{(-1)^\ell}{2^\ell}\sum_{q=0}^{\ell}(-1)^q\binom{\ell}{q}\left(\cos[(2q-\ell)z]\cos\frac{\ell\pi}{2}+\sin[(2q-\ell)z]\sin\frac{\ell\pi}{2}\right)\\&=\frac{(-1)^\ell}{2^\ell}\cos\frac{\ell\pi}{2}\sum_{q=0}^{\ell}(-1)^q\binom{\ell}{q}\left[1+\sum_{j=1}^{\infty}(-1)^j(2q-\ell)^{2j}\frac{z^{2j}}{(2j)!}\right]\end{aligned}$$

$$
\begin{aligned}
&+\frac{(-1)^\ell}{2^\ell}\sin\frac{\ell\pi}{2}\sum_{q=0}^{\ell}(-1)^q\binom{\ell}{q}\sum_{j=0}^{\infty}(-1)^j(2q-\ell)^{2j+1}\frac{z^{2j+1}}{(2j+1)!}\\
&=\frac{(-1)^\ell}{2^\ell}\cos\frac{\ell\pi}{2}\sum_{j=1}^{\infty}(-1)^j\left[\sum_{q=0}^{\ell}(-1)^q\binom{\ell}{q}(2q-\ell)^{2j}\right]\frac{z^{2j}}{(2j)!}\\
&+\frac{(-1)^\ell}{2^\ell}\sin\frac{\ell\pi}{2}\sum_{j=0}^{\infty}(-1)^j\left[\sum_{q=0}^{\ell}(-1)^q\binom{\ell}{q}(2q-\ell)^{2j+1}\right]\frac{z^{2j+1}}{(2j+1)!}.
\end{aligned}
$$

Replacing ℓ by $2\ell-1$ and by 2ℓ and simplifying result in the series expansions

$$
\sin^{2\ell-1} z=\frac{(-1)^\ell}{2^{2\ell-1}}\sum_{j=\ell}^{\infty}\left[\sum_{q=0}^{2\ell-1}(-1)^q\binom{2\ell-1}{q}(2q-2\ell+1)^{2j-1}\right]\frac{(-1)^{j-1}z^{2j-1}}{(2j-1)!} \tag{9}
$$

and

$$
\sin^{2\ell} z=\frac{(-1)^\ell}{2^{2\ell}}\sum_{j=\ell}^{\infty}(-1)^j2^{2j}\left[\sum_{q=0}^{2\ell}(-1)^q\binom{2\ell}{q}(q-\ell)^{2j}\right]\frac{z^{2j}}{(2j)!}. \tag{10}
$$

The series expansions (9) and (10) can be reformulated as

$$
\operatorname{sinc}^{2\ell-1} z=\frac{1}{2^{2\ell-1}}\sum_{j=0}^{\infty}\left[\sum_{q=0}^{2\ell-1}(-1)^q\binom{2\ell-1}{q}(2q-2\ell+1)^{2\ell+2j-1}\right]\frac{(-1)^{j-1}z^{2j}}{(2\ell+2j-1)!}
$$

and

$$
\operatorname{sinc}^{2\ell} z=\frac{1}{2^{2\ell}}\sum_{j=0}^{\infty}\frac{(-1)^j}{(2\ell+2j)!}\left[\sum_{q=0}^{2\ell}(-1)^q\binom{2\ell}{q}(2q-2\ell)^{2\ell+2j}\right]z^{2j}.
$$

for $\ell\in\mathbb{N}$ and $z\in\mathbb{C}$. These two series expansions can be unified and rearranged as the series expansion (7). Theorem 1 is thus proved. □

Corollary 2. *For $\ell\in\mathbb{N}$, we have*

$$
T(2j-1,2\ell-1)=\begin{cases}0, & 1\le j\le \ell-1\\ 1, & j=\ell\end{cases} \quad\textit{and}\quad T(2j,2\ell)=\begin{cases}0, & 1\le j\le \ell-1\\ 1, & j=\ell\end{cases}
$$

Proof. This follows from Theorem 1 and its proof. □

Theorem 3. *For $j,\ell\in\mathbb{N}_0$, we have*

$$
T(2j+\ell+1,\ell)=0. \tag{11}
$$

For $\ell\in\mathbb{N}_0$ and $z\in\mathbb{C}$, the series expansions

$$
\operatorname{sinhc}^\ell z=1+\sum_{j=1}^{\infty}\frac{T(2j+\ell,\ell)}{\binom{2j+\ell}{\ell}}\frac{(2z)^{2j}}{(2j)!} \tag{12}
$$

and (7) *are valid.*

Proof. Replacing z by $2z$ in (3) and rearranging yield

$$\left(\frac{\sinh z}{z}\right)^{\ell}=1+\sum_{n=1}^{\infty}\frac{T(n+\ell,\ell)}{\binom{n+\ell}{\ell}}\frac{(2z)^n}{n!}.$$

Considering that the function $\frac{\sinh x}{x}$ is even on $\mathbb{R}$, we conclude that the identity (11) and the series (12) are valid.

Substituting $z\,\mathrm{i}$ for z in (12) and employing the relation $\sinh(z\,\mathrm{i})=\mathrm{i}\sin z$ give the series expansion (7) in Theorem 1. □

Theorem 4. *For $j,\ell\in\mathbb{N}_0$ and $z\in\mathbb{C}$, we have*

$$\sum_{k=0}^{2j+1}(-1)^k\binom{2j+1}{k}\left(\frac{2}{\ell}\right)^k\frac{S(k+\ell,\ell)}{\binom{k+\ell}{\ell}}=0 \tag{13}$$

and

$$\operatorname{sinc}^{\ell} z=1+\sum_{j=1}^{\infty}(-1)^j\left[\sum_{k=0}^{2j}(-1)^k\binom{2j}{k}\left(\frac{2}{\ell}\right)^k\frac{S(k+\ell,\ell)}{\binom{k+\ell}{\ell}}\right]\frac{(\ell z)^{2j}}{(2j)!}. \tag{14}$$

Proof. Taking $r=0$ in (1) and reformulating give

$$\left(\frac{\mathrm{e}^z-1}{z}\right)^k=\sum_{n=0}^{\infty}\frac{S(n+k,k)}{\binom{n+k}{k}}\frac{z^n}{n!}. \tag{15}$$

Since

$$\sin z=\frac{\mathrm{e}^{z\,\mathrm{i}}-\mathrm{e}^{-z\,\mathrm{i}}}{2\,\mathrm{i}}=\frac{\mathrm{e}^{2z\,\mathrm{i}}-1}{2\,\mathrm{i}}\,\mathrm{e}^{-z\,\mathrm{i}},$$

by (15) and the Cauchy product of two series, we obtain

$$\begin{aligned}
\operatorname{sinc}^{\ell} z&=\left(\frac{\sin z}{z}\right)^{\ell}=\left(\frac{\mathrm{e}^{2z\,\mathrm{i}}-1}{2z\,\mathrm{i}}\right)^{\ell}\mathrm{e}^{-\ell z\,\mathrm{i}}\\
&=\left[\sum_{n=0}^{\infty}\frac{S(n+\ell,\ell)}{\binom{n+\ell}{\ell}}\frac{(2z\,\mathrm{i})^n}{n!}\right]\left[\sum_{n=0}^{\infty}\frac{(-\ell z\,\mathrm{i})^n}{n!}\right]\\
&=\sum_{j=0}^{\infty}\left[\sum_{n=0}^{j}\frac{S(n+\ell,\ell)}{\binom{n+\ell}{\ell}}\frac{(2\,\mathrm{i})^n}{n!}\frac{(-\ell\,\mathrm{i})^{j-n}}{(j-n)!}\right]z^j\\
&=\sum_{j=0}^{\infty}(-1)^j\frac{\ell^j}{j!}\left[\sum_{k=0}^{j}(-1)^k\binom{j}{k}\left(\frac{2}{\ell}\right)^k\frac{S(k+\ell,\ell)}{\binom{k+\ell}{\ell}}\right]\left(\cos\frac{j\pi}{2}+\mathrm{i}\sin\frac{j\pi}{2}\right)z^j\\
&=\sum_{j=0}^{\infty}(-1)^j\frac{\ell^{2j}}{(2j)!}\left[\sum_{k=0}^{2j}(-1)^k\binom{2j}{k}\left(\frac{2}{\ell}\right)^k\frac{S(k+\ell,\ell)}{\binom{k+\ell}{\ell}}\right]z^{2j}
\end{aligned}$$

$$+\,\mathrm{i}\sum_{j=1}^{\infty}(-1)^j\frac{\ell^{2j-1}}{(2j-1)!}\left[\sum_{k=0}^{2j-1}(-1)^k\binom{2j-1}{k}\left(\frac{2}{\ell}\right)^k\frac{S(k+\ell,\ell)}{\binom{k+\ell}{\ell}}\right]z^{2j-1}$$

$$=\sum_{j=0}^{\infty}(-1)^j\frac{\ell^{2j}}{(2j)!}\left[\sum_{k=0}^{2j}(-1)^k\binom{2j}{k}\left(\frac{2}{\ell}\right)^k\frac{S(k+\ell,\ell)}{\binom{k+\ell}{\ell}}\right]z^{2j}.$$

The proof of Theorem 4 is complete. □

Corollary 5. *For $j\in\mathbb{N}_0$ and $\ell\in\mathbb{N}$, we have*

$$\frac{T(2j+\ell,\ell)}{\binom{2j+\ell}{\ell}}=\sum_{m=0}^{2j}(-1)^m\binom{2j}{m}\left(\frac{\ell}{2}\right)^m\frac{S(2j+\ell-m,\ell)}{\binom{2j+\ell-m}{\ell}}. \tag{16}$$

Proof. This follows from comparing the series expansion (7) in Theorem 1 with the series expansion (14) in Theorem 4 and simplifying. □

Corollary 6. *For $j\in\mathbb{N}_0$ and $\ell\in\mathbb{N}$, we have*

$$\frac{T(j+\ell,\ell)}{\binom{j+\ell}{\ell}}=\sum_{m=0}^{j}(-1)^m\binom{j}{m}\left(\frac{\ell}{2}\right)^m\frac{S(j+\ell-m,\ell)}{\binom{j+\ell-m}{\ell}}. \tag{17}$$

Proof. This follows from combining the identities (11), (13), and (16). □

3. CLOSED-FORM EXPRESSIONS FOR BELL POLYNOMIALS

In this section, with the help of Theorem 1 and other results mentioned above, we will establish closed-form expressions for special partial Bell polynomials $B_{n,k}$ of all derivatives at $z=0$ of the sinc function $\operatorname{sinc} z$.

Theorem 7. *For $n\ge k\ge 1$, partial Bell polynomials $B_{n,k}$ satisfy*

$$B_{2m-1,k}\left(0,-\frac{1}{3},0,\frac{1}{5},\dots,\frac{(-1)^m}{2m-k+1}\cos\frac{k\pi}{2}\right)=0,\quad 1\le k\le 2m-1$$

and

$$B_{2m,k}\left(0,-\frac{1}{3},0,\frac{1}{5},\dots,\frac{(-1)^m}{2m-k+2}\sin\frac{k\pi}{2}\right)=(-1)^{m+k}\frac{2^{2m}}{k!}\sum_{j=1}^{k}(-1)^j\binom{k}{j}\frac{T(2m+j,j)}{\binom{2m+j}{j}},\quad 1\le k\le 2m.$$

Proof. From

$$\operatorname{sinc} z=\sum_{j=0}^{\infty}\frac{(-1)^j}{2j+1}\frac{z^{2j}}{(2j)!},\quad z\in\mathbb{C}, \tag{18}$$

it follows that

$$(\operatorname{sinc} z)^{(2j)}\big|_{z=0}=\frac{(-1)^j}{2j+1}\quad\text{and}\quad(\operatorname{sinc} z)^{(2j-1)}\big|_{z=0}=0,\quad j\in\mathbb{N}. \tag{19}$$

On [**16**, p. 133], the identity

$$\frac{1}{m!}\left(\sum_{\ell=1}^{\infty}z_\ell\frac{t^\ell}{\ell!}\right)^m=\sum_{n=m}^{\infty}B_{n,m}(z_1,z_2,\dots,z_{n-m+1})\frac{t^n}{n!} \tag{20}$$

is given for $m\in\mathbb{N}_0$. The formula (20) implies that

$$B_{n+k,k}(z_1,z_2,\dots,z_{n+1})=\binom{n+k}{k}\lim_{t\to0}\frac{\mathrm{d}^n}{\mathrm{d}\,t^n}\left[\sum_{\ell=0}^{\infty}\frac{z_{\ell+1}}{(\ell+1)!}t^\ell\right]^k \tag{21}$$

for $n\ge k\in\mathbb{N}_0$. Substituting $z_{2j}=\frac{(-1)^j}{2j+1}$ and $z_{2j-1}=0$, that is, $z_j=\frac{1}{j+1}\cos\left(\frac{j}{2}\pi\right)$, for $j\in\mathbb{N}$ into (21) results in

$$\begin{aligned}
&B_{n+k,k}\left(0,-\frac{1}{3},0,\frac{1}{5},\dots,\frac{1}{n+2}\cos\left(\frac{n+1}{2}\pi\right)\right)\\
&=\binom{n+k}{k}\lim_{t\to0}\frac{\mathrm{d}^n}{\mathrm{d}\,t^n}\left[\sum_{\ell=0}^{\infty}\frac{1}{(\ell+2)!}\cos\left(\frac{\ell+1}{2}\pi\right)t^\ell\right]^k\\
&=\binom{n+k}{k}\lim_{t\to0}\frac{\mathrm{d}^n}{\mathrm{d}\,t^n}\left(\frac{\operatorname{sinc} t-1}{t}\right)^k\\
&=\binom{n+k}{k}\lim_{t\to0}\frac{\mathrm{d}^n}{\mathrm{d}\,t^n}\left[\frac{(-1)^k}{t^k}+\frac{(-1)^k}{t^k}\sum_{j=1}^{k}(-1)^j\binom{k}{j}(\operatorname{sinc} t)^j\right]\\
&=\binom{n+k}{k}\lim_{t\to0}\frac{\mathrm{d}^n}{\mathrm{d}\,t^n}\left(\frac{(-1)^k}{t^k}\sum_{\ell=1}^{\infty}(-1)^\ell\left[\sum_{j=1}^{k}\binom{k}{j}\frac{1}{2^j}\frac{1}{(j+2\ell)!}\right.\right.\\
&\quad\left.\left.\times\sum_{q=0}^{j}(-1)^q\binom{j}{q}(2q-j)^{j+2\ell}\right]t^{2\ell}\right)\\
&=(-1)^k\binom{n+k}{k}\lim_{t\to0}\frac{\mathrm{d}^n}{\mathrm{d}\,t^n}\sum_{\ell=k}^{\infty}(-1)^\ell\left[\sum_{j=1}^{k}\binom{k}{j}\frac{1}{2^j}\frac{1}{(j+2\ell)!}\right.\\
&\quad\left.\times\sum_{q=0}^{j}(-1)^q\binom{j}{q}(2q-j)^{j+2\ell}\right]t^{2\ell-k}
\end{aligned}$$

$$
\begin{aligned}
&= (-1)^k \binom{n+k}{k} \lim_{t\to0} \sum_{\ell=k}^{\infty} (-1)^\ell \Biggl[\sum_{j=1}^{k} \binom{k}{j} \frac{1}{2^j} \frac{1}{(j+2\ell)!} \\
&\quad \times \sum_{q=0}^{j} (-1)^q \binom{j}{q} (2q-j)^{j+2\ell}\Biggr] \langle 2\ell-k\rangle_n t^{2\ell-k-n} \\
&= \begin{cases} 0, & n+k=2m+1 \\ \dfrac{(-1)^{k+m}(2m)!}{k!} \displaystyle\sum_{j=1}^{k} \frac{\binom{k}{j}}{2^j (j+2m)!} \sum_{q=0}^{j} (-1)^q \binom{j}{q} (2q-j)^{j+2m}, & n+k=2m \end{cases}
\end{aligned}
$$

for $m \in \mathbb{N}$ and $n \ge k \ge 1$, where we used the series expansion (7) in Theorem 1. The proof of Theorem 7 is complete. □

Corollary 8. *For $k \ge 2$ and $1 \le \ell \le k-1$, we have*

$$
\sum_{j=1}^{k} (-1)^j \binom{k}{j} \frac{T(2\ell+j,j)}{\binom{2\ell+j}{j}} = 0 \tag{22}
$$

and

$$
\sum_{j=1}^{k} (-1)^j \binom{k}{j} \sum_{m=0}^{2\ell} (-1)^m \binom{2\ell}{m} \left(\frac{j}{2}\right)^m \frac{S(2\ell+j-m,j)}{\binom{2\ell+j-m}{j}} = 0.
$$

Proof. This follows from the proof of Theorem 7 and further making use of the formula (16). □

Corollary 9. *For $n \ge k \ge 1$, partial Bell polynomials $B_{n,k}$ satisfy*

$$
\begin{aligned}
B_{n,k}&\left(0, -\frac{1}{3}, 0, \frac{1}{5}, \dots, \frac{1}{n-k+2} \cos\left(\frac{n-k+1}{2}\pi\right)\right) \\
&= (-1)^k \cos\left(\frac{n\pi}{2}\right) \frac{2^n}{k!} \sum_{j=1}^{k} (-1)^j \binom{k}{j} \frac{T(n+j,j)}{\binom{n+j}{j}}
\end{aligned}
$$

and

$$
\begin{aligned}
B_{n,k}&\left(0, -\frac{1}{3}, 0, \frac{1}{5}, \dots, \frac{1}{n-k+2} \cos\left(\frac{n-k+1}{2}\pi\right)\right) \\
&= (-1)^k \cos\left(\frac{n\pi}{2}\right) \frac{2^n}{k!} \sum_{j=1}^{k} (-1)^j \binom{k}{j} \sum_{m=0}^{n} (-1)^m \binom{n}{m} \left(\frac{j}{2}\right)^m \frac{S(n+j-m,j)}{\binom{n+j-m}{j}}.
\end{aligned}
$$

Proof. This follows from combining the identity (11) with Theorem 7 and the formula (17). □

Applying Theorem 7 and Corollary 9, we deduce the following corollary.

Corollary 10. *For $z \in \mathbb{C}$, we have*

$$\begin{aligned}
e^{\operatorname{sinc} z - 1} &= 1 + \sum_{k=1}^{\infty} (-1)^k \left[\sum_{j=1}^{2k} \frac{(-1)^j}{j!} \sum_{\ell=1}^{j} (-1)^\ell \binom{j}{\ell} \frac{T(2k+\ell,\ell)}{\binom{2k+\ell}{\ell}}\right] \frac{(2z)^{2k}}{(2k)!} \\
&= 1 - \frac{z^2}{6} + \frac{z^4}{45} - \frac{107z^6}{45360} + \frac{1189z^8}{5443200} - \frac{1633z^{10}}{89812800} + \cdots
\end{aligned}$$

and

$$\begin{aligned}
e^{\operatorname{sinc} z - 1} = 1 + \sum_{k=1}^{\infty} (-1)^k \Bigg[\sum_{j=1}^{2k} \frac{(-1)^j}{j!} \sum_{\ell=1}^{j} (-1)^\ell \binom{j}{\ell} \\
\times \sum_{m=0}^{2k} (-1)^m \binom{2k}{m} \left(\frac{\ell}{2}\right)^m \frac{S(2k+\ell-m,\ell)}{\binom{2k+\ell-m}{\ell}}\Bigg] \frac{(2z)^{2k}}{(2k)!}.
\end{aligned}$$

Proof. Making use of the Faà di Bruno formula (6), the derivatives in (19), and Theorem 7, we obtain

$$\begin{aligned}
e^{\operatorname{sinc} z} &= \sum_{k=0}^{\infty} \left(\lim_{z\to 0} \frac{d^k e^{\operatorname{sinc} z}}{d z^k}\right) \frac{z^k}{k!} \\
&= e + \sum_{k=1}^{\infty} \left[\lim_{z\to 0} \sum_{j=1}^{k} e^{\operatorname{sinc} z} B_{k,j}\big((\operatorname{sinc} z)', (\operatorname{sinc} z)'', \dots, (\operatorname{sinc} z)^{(k-j+1)}\big)\right] \frac{z^k}{k!} \\
&= e + e \sum_{k=1}^{\infty} \left[\sum_{j=1}^{k} B_{k,j}\big((\operatorname{sinc} z)'\big|_{z=0}, (\operatorname{sinc} z)''\big|_{z=0}, \dots, (\operatorname{sinc} z)^{(k-j+1)}\big|_{z=0}\big)\right] \frac{z^k}{k!} \\
&= e + e \sum_{k=1}^{\infty} \left[\sum_{j=1}^{k} B_{k,j}\left(0, -\frac{1}{3}, \dots, \frac{1}{k-j+2} \cos\left(\frac{k-j+1}{2}\pi\right)\right)\right] \frac{z^k}{k!} \\
&= e + e \sum_{k=1}^{\infty} \left[\sum_{j=1}^{2k} B_{2k,j}\left(0, -\frac{1}{3}, \dots, \frac{1}{2k-j+2} \cos\left(\frac{2k-j+1}{2}\pi\right)\right)\right] \frac{z^{2k}}{(2k)!} \\
&= e + e \sum_{k=1}^{\infty} \left[\sum_{j=1}^{2k} (-1)^{k+j} \frac{2^{2k}}{j!} \sum_{\ell=1}^{j} (-1)^\ell \binom{j}{\ell} \frac{T(2k+\ell,\ell)}{\binom{2k+\ell}{\ell}}\right] \frac{z^{2k}}{(2k)!}.
\end{aligned}$$

Further considering (16), we prove Corollary 10. □

4. SERIES EXPANSIONS OF REAL POWERS

In this section, with the aid of Theorem 7 and other results in the above sections, we establish series expansions at the point $z = 0$ of the power functions $\operatorname{sinc}^r z$ and $\operatorname{sinhc}^r z$ for real $r \in \mathbb{R}$.

Theorem 11. *Let $r \in \mathbb{R}$. When $r \ge 0$, the series expansions*

$$\operatorname{sinc}^r z = 1 + \sum_{q=1}^{\infty}(-1)^q\left[\sum_{k=1}^{2q}\frac{(-r)_k}{k!}\sum_{j=1}^{k}(-1)^j\binom{k}{j}\frac{T(2q+j,j)}{\binom{2q+j}{j}}\right]\frac{(2z)^{2q}}{(2q)!} \tag{23}$$

and

$$\begin{aligned}\operatorname{sinc}^r z = 1 + \sum_{q=1}^{\infty}(-1)^q\Bigg[\sum_{k=1}^{2q}\frac{(-r)_k}{k!}\sum_{j=1}^{k}(-1)^j\binom{k}{j} \\ \times\sum_{m=0}^{2q}(-1)^m\binom{2q}{m}\left(\frac{j}{2}\right)^m\frac{S(2q+j-m,j)}{\binom{2q+j-m}{j}}\Bigg]\frac{(2z)^{2q}}{(2q)!}\end{aligned} \tag{24}$$

are convergent in $z \in \mathbb{C}$, where the rising factorial $(r)_k$ is defined by

$$(r)_k = \prod_{\ell=0}^{k-1}(r+\ell) = \begin{cases} r(r+1)\cdots(r+k-1), & k \ge 1; \\ 1, & k = 0. \end{cases}$$

When $r < 0$, the series expansions (23) *and* (24) *are convergent in $|z| < \pi$.*

Proof. By virtue of the Faà di Bruno formula (6), we obtain

$$\begin{aligned}\frac{\mathrm{d}^j(\operatorname{sinc}^r z)}{\mathrm{d}\,z^j} &= \sum_{k=1}^{j}\frac{\mathrm{d}^k u^r}{\mathrm{d}\,u^k}B_{j,k}\big((\operatorname{sinc} z)', (\operatorname{sinc} z)'', \dots, (\operatorname{sinc} z)^{(j-k+1)}\big) \\ &= \sum_{k=1}^{j}\langle r\rangle_k \operatorname{sinc}^{r-k} z B_{j,k}\big((\operatorname{sinc} z)', (\operatorname{sinc} z)'', \dots, (\operatorname{sinc} z)^{(j-k+1)}\big) \\ &\to \sum_{k=1}^{j}\langle r\rangle_k B_{j,k}\left(0, -\frac{1}{3}, 0, \frac{1}{5}, \dots, \frac{1}{j-k+2}\sin\frac{(j-k)\pi}{2}\right), \quad z \to 0 \\ &= \begin{cases} 0, & j = 2m-1 \\ \displaystyle\sum_{k=1}^{2m}\langle r\rangle_k B_{2m,k}\left(0, -\frac{1}{3}, 0, \frac{1}{5}, \dots, \frac{1}{j-k+2}\sin\frac{(2m-k)\pi}{2}\right), & j = 2m \end{cases}\end{aligned}$$

for $m \in \mathbb{N}$, where $u = u(z) = \operatorname{sinc} z$, the notation

$$\langle r\rangle_k = \prod_{k=0}^{k-1}(r-k) = \begin{cases} r(r-1)\cdots(r-k+1), & k \ge 1 \\ 1, & k = 0 \end{cases}$$

for $r \in \mathbb{R}$ is called the falling factorial, and we used derivatives in (19). Therefore, with the help of Theorem 7, we arrive at

$$\operatorname{sinc}^r z - 1 = \sum_{j=1}^{\infty}\left[\lim_{z\to 0}\frac{\mathrm{d}^j(\operatorname{sinc}^r z)}{\mathrm{d}\,z^j}\right]\frac{z^j}{j!} = \sum_{m=1}^{\infty}\left[\lim_{z\to 0}\frac{\mathrm{d}^{2m}(\operatorname{sinc}^r z)}{\mathrm{d}\,z^{2m}}\right]\frac{z^{2m}}{(2m)!}$$

$$= \sum_{m=1}^{\infty}\left[\sum_{k=1}^{2m}\langle r\rangle_k B_{2m,k}\left(0,-\frac{1}{3},0,\frac{1}{5},\dots,\frac{1}{j-k+2}\sin\frac{(2m-k)\pi}{2}\right)\right]\frac{z^{2m}}{(2m)!}$$
$$= \sum_{m=1}^{\infty}\left[\sum_{k=1}^{2m}(-1)^{m+k}\frac{\langle r\rangle_k}{k!}\sum_{j=1}^{k}\binom{k}{j}\frac{1}{2^j}\frac{1}{(2m+j)!}\sum_{q=0}^{j}(-1)^q\binom{j}{q}(2q-j)^{2m+j}\right]z^{2m}$$
$$= \sum_{m=1}^{\infty}(-1)^m\left[\sum_{k=1}^{2m}\frac{(-r)_k}{k!}\sum_{j=1}^{k}(-1)^j\binom{k}{j}\frac{T(2m+j,j)}{\binom{2m+j}{j}}\right]\frac{(2z)^{2m}}{(2m)!}.$$

By virtue of (16), the proof of Theorem 11 is thus complete. □

Corollary 12. *For $r\in\mathbb{R}$ and $z\in\mathbb{C}$, we have*

$$\operatorname{sinhc}^r z = 1+\sum_{q=1}^{\infty}\left[\sum_{k=1}^{2q}\frac{(-r)_k}{k!}\sum_{j=1}^{k}(-1)^j\binom{k}{j}\frac{T(2q+j,j)}{\binom{2q+j}{j}}\right]\frac{(2z)^{2q}}{(2q)!} \tag{25}$$

and

$$\operatorname{sinhc}^r z = 1+\sum_{q=1}^{\infty}\left[\sum_{k=1}^{2q}\frac{(-r)_k}{k!}\sum_{j=1}^{k}(-1)^j\binom{k}{j}\times\sum_{m=0}^{2q}(-1)^m\binom{2q}{m}\left(\frac{j}{2}\right)^m\frac{S(2q+j-m,j)}{\binom{2q+j-m}{j}}\right]\frac{(2z)^{2q}}{(2q)!}. \tag{26}$$

Proof. The series expansions (25) and (26) follow from replacing $\operatorname{sinc} z$ by $\operatorname{sinhc}(z\,\mathrm{i})$ in (23) and (24) and then substituting $z\,\mathrm{i}$ for z. □

5. COMBINATORIAL PROOFS OF TWO IDENTITIES

We now modify combinatorial proofs at the website `https://mathoverflow.net/a/420309/` for the identities (11) and (22) as follows.

First combinatorial proof of the identity (11). Theorem 8 in [**8**, p. 247] states that weighted Stirling numbers of the second kind $\left\{ {n+k \atop k} \right\}_r = R(n+k-r,n-r,r)$ are the monomial symmetric functions of degree k of the integers $r,\dots,n$. This means that $R(n,k,r)=h_{n-k}(r,r+1,r+2,\dots,r+k)$, where h_m is the complete homogenous symmetric function. So it follows that

$$R\left(2m+j-1,j,-\frac{j}{2}\right)=h_{2m-1}\left(-\frac{j}{2},1-\frac{j}{2},2-\frac{j}{2},\dots,\frac{j}{2}\right)$$

and we can pair up each monomial $x_{a_1}x_{a_2}x_{a_3}\cdots$ with

$$x_{2m-a_1}x_{2m-a_2}x_{2m-a_3}\cdots=(-1)^{2m-1}x_{a_1}x_{a_2}x_{a_3}\cdots,$$

which gives cancellation. The only terms which don't pair up like this with a different term are those which include $x_m = 0$ and pair with themselves, but by virtue of including a zero multiplicand they do not contribute anything to the sum. Consequently, by the relation (5), the identity (11) is proved. □

Second combinatorial proof of the identity (11). Let

$$\mathcal{T}(n,k) = 2^{n-k}T(n,k), \quad n \ge k \ge 0 \tag{27}$$

with $\mathcal{T}(0,0) = 1$. This scaled central triangle number $\mathcal{T}(n,k)$ counts set partitions of n elements into k odd-sized blocks. See the references [**17, 38**]. This immediately gives the identity (11), since an even-sized set cannot be partitioned into an odd number of odd-sized blocks, nor an odd-sized set partitioned into an even number of odd-sized blocks. □

A combinatorial proof of the identity (22). Since

$$\begin{aligned}\sum_{j=1}^{k}(-1)^j\binom{k}{j}\frac{T(2\ell+j,j)}{\binom{2\ell+j}{j}} &= \sum_{j=1}^{k}(-1)^j\frac{k!(2\ell)!}{(k-j)!(2\ell+j)!}T(2\ell+j,j)\\ &= \frac{(-1)^k}{2^{2\ell}\binom{2\ell+k}{k}}\sum_{j=1}^{k}(-1)^{k-j}\binom{2\ell+k}{2\ell+j}\mathcal{T}(2\ell+j,j),\end{aligned}$$

the identity (22) is equivalent to

$$\sum_{j=1}^{k}(-1)^{k-j}\binom{2\ell+k}{2\ell+j}\mathcal{T}(2\ell+j,j) = 0, \quad 1 \le \ell < k, \tag{28}$$

where $\mathcal{T}(2\ell+j,j)$ is defined by (27). The equality (28) has the following combinatorial proof.

Consider set partitions of $2\ell+k$ elements into k odd-sized blocks where blocks of size 3 or greater are coloured red and singleton blocks can be coloured red or blue. Then the sum counts such set partitions weighted by $(-1)^{\text{number of blue partitions}}$. Note that j is the number of red partitions. Observe that partitions containing at least one singleton can be paired with the partition which differs only in the colour assigned to the singleton with the smallest element, so that the sum counts the number of partitions of $2\ell + k$ elements into k odd-sized blocks of at least 3 elements each. But, if $k > \ell$, there are no such partitions. The required proof is complete. □

6. POSITIVITY OF COEFFICIENTS

In this section, we discuss positivity of the coefficients in the Maclaurin power series expansions (23) and (24).

Theorem 13. *For $r = -\alpha < 0$ and $q \ge 1$, the coefficients in the Maclaurin power series expansions* (23) *and* (24) *are nonnegative, that is,*

$$(-1)^q \sum_{k=1}^{2q} \frac{(\alpha)_k}{k!} \sum_{j=1}^{k} (-1)^j \binom{k}{j} \frac{T(2q+j,j)}{\binom{2q+j}{j}} \ge 0 \tag{29}$$

and

$$(-1)^q \sum_{k=1}^{2q} \frac{(\alpha)_k}{k!} \sum_{j=1}^{k} (-1)^j \binom{k}{j} \sum_{m=0}^{2q} (-1)^m \binom{2q}{m} \left(\frac{j}{2}\right)^m \frac{S(2q+j-m,j)}{\binom{2q+j-m}{j}} \ge 0. \tag{30}$$

Proof. The idea of this proof comes from the site `https://math.stackexchange.com/a/4662350` where the first author proved the inequality

$$\frac{x}{\sin x} < \frac{\tan x}{x}, \quad x \in \left(0, \frac{\pi}{2}\right). \tag{31}$$

When $x \in \left(\frac{\pi}{2}, \pi\right)$, the inequality (31) is reversed.

In the handbook [**18**, p. 55, Item 1.518], the power series expansion

$$\ln \sin x = \ln x + \sum_{k=1}^{\infty} \frac{2^{2k-1}}{k} \left[(-1)^k B_{2k}\right] \frac{x^{2k}}{(2k)!}, \quad 0 < x < \pi$$

is listed, where B_k denotes the Bernoulli numbers which can be generated by

$$\frac{z}{\mathrm{e}^z - 1} = \sum_{n=0}^{\infty} B_n \frac{z^n}{n!} = 1 - \frac{z}{2} + \sum_{n=1}^{\infty} B_{2n} \frac{z^{2n}}{(2n)!}, \quad |z| < 2\pi.$$

For properties of the Bernoulli numbers B_n, please see [**1**, p. 804], [**18**, p. xxx], and the papers [**15, 42, 47, 62**]. One of the properties of the Bernoulli numbers B_n is $(-1)^{k+1} B_{2k} < 0$ for all $k \ge 1$. Therefore, we derive that the derivatives

$$\begin{aligned}[\alpha(\ln x - \ln \sin x)]^{(n)} &= \left[\ln\left(\frac{x}{\sin x}\right)^{\alpha}\right]^{(n)} \\ &= \alpha \sum_{k=1}^{\infty} \frac{2^{2k-1}}{k} \left[(-1)^{k+1} B_{2k}\right] \langle 2k \rangle_n \frac{x^{2k-n}}{(2k)!}\end{aligned}$$

is positive for any real number $\alpha > 0$, any integer $n \ge 0$, and $0 < x < \pi$. This means that the function $\alpha(\ln x - \ln \sin x)$ is absolutely monotonic functions on $(0, \pi)$ for any real number $\alpha > 0$. In other words, for any real number $\alpha > 0$, the even function

$$f_\alpha(x) = (\operatorname{sinc} x)^{-\alpha} = \begin{cases} \left(\dfrac{x}{\sin x}\right)^{\alpha}, & x \ne 0 \\ 1, & x = 0 \end{cases} \tag{32}$$

is logarithmically absolutely monotonic on $[0, \pi)$ and is logarithmically completely monotonic on $(-\pi, 0]$. An infinitely differentiable real function $h(x)$ defined on an interval I is

i) absolutely monotonic on I if and only if $h^{(n)} \geq 0$ for $n \geq 0$ on I;

ii) completely monotonic on I if and only if $(-1)^n h^{(n)} \geq 0$ for $n \geq 0$ on I;

iii) logarithmically absolutely monotonic on I if and only if $h(x)$ is positive and $[\ln h(x)]^{(n+1)} \geq 0$ for $n \geq 0$ on I;

iv) logarithmically completely monotonic on I if and only if $h(x)$ is positive and $(-1)^{n+1}[\ln h(x)]^{(n+1)} \geq 0$ for $n \geq 0$ on I.

The relations among these four kinds of monotonic functions are listed as follows:

1) a logarithmically absolutely monotonic function $h(x)$ on I implies that $h(x)$ is an absolutely monotonic function on I, but not conversely;

2) a logarithmically completely monotonic function $h(x)$ on I implies that $h(x)$ is a completely monotonic function on I, but not conversely;

3) a function $h(x)$ is (logarithmically) absolutely monotonic on I if and only if $h(-x)$ is (logarithmically) completely monotonic on $-I = \{-x : x \in I\}$.

These notions and relations can be found in the papers [**5, 21, 50**] and in the monographs [**41, 61, 67**]. There have been many papers such as [**23, 27, 28, 33, 40, 43, 44, 57, 64, 65, 66, 68**] dedicating to the investigation of (logarithmically) (absolutely) completely monotonic functions.

Basing on the above notions and relations, we see that, for any real number $\alpha > 0$, the function $f_\alpha(x)$ is absolutely monotonic on $[0, \pi)$ and is completely monotonic on $(-\pi, 0]$. This implies that

$$f_\alpha^{(2n+1)}(0) = 0 \quad \text{and} \quad f_\alpha^{(2n)}(0) \geq 0 \tag{33}$$

for $n \geq 0$. Consequently, for $r = -\alpha < 0$, all of the coefficients in the power series expansions (23) and (24) are nonnegative, that is, the inequalities (29) and (30) are valid for $q \geq 1$. □

7. REMARKS

Finally we list several remarks about our main results and related things.

Remark 14. *The formulation of the series expansions* (7) *and* (14) *in Theorems 1 and 4 are better and simpler than corresponding ones in [***9***, pp. 798–799].*

The formula (8) *can also be found at* `https://math.stackexchange.com/a/4331451/` *and* `https://math.stackexchange.com/a/4332549/`.

Remark 15. *After reading the preprint [***48***] of this paper, Jacques Gélinas, a retired mathematician at Ottawa in Canada, pointed out that the series expansion* (7) *in Theorem 1, or say, the series expansion* (14) *in Theorem 4, has been considered by John Blissard in [***6***, pp. 50–51] with different and old notations.*

Remark 16. *The series expansion* (7) *in Theorem 1 has been applied to answer questions at the sites* `https://math.stackexchange.com/a/4429078/`, `https://math.stackexchange.com/a/4332549/`, *and* `https://math.stackexchange.com/a/4331451/`.

The series expansion (7) *in Theorem 1 or the series expansion* (23) *in Theorem 11 can be used to answer questions at* `https://math.stackexchange.com/q/2267836/` *and* `https://math.stackexchange.com/q/3673133/`.

The series expansion (23) *in Theorem 11 has been employed to answer questions at the websites* `https://math.stackexchange.com/a/4427504/`, `https://math.stackexchange.com/a/4426821/`, *and* `https://math.stackexchange.com/a/4428010/`.

The series expansion (23) *in Theorem 11 has been employed in [***15***, Theorem 3] to derive two closed-form expressions for the Bernoulli numbers B_{2m} in terms of central factorial numbers of the second kind $T(2m+j,j)$.*

Remark 17. *The first identity in Theorem 7 is a special case of the following general conclusion in [***20***, Theorem 1.1]: For $k,n\in\mathbb{N}_0$ and $x_m\in\mathbb{C}$ with $m\in\mathbb{N}$, we have*

$$B_{2n+1,k}\left(0,x_2,0,x_4,\dots,\frac{1+(-1)^k}{2}x_{2n-k+2}\right)=0.$$

Remark 18. *As done in Corollary 10, as long as the function $f(u)$ is infinitely differentiable at the point $u=1$, Theorem 7 can be utilized to compute series expansions at $x=0$ of the functions $f(\operatorname{sinc}x)$ and $f(\operatorname{sinhc}x)$.*

Remark 19. *Let $r>0$ and $k\in\mathbb{N}_0$. Making use of the Faà di Bruno formula* (6) *and employing the formula*

$$B_{n,k}(x,1,0,\dots,0)=\frac{1}{2^{n-k}}\frac{n!}{k!}\binom{k}{n-k}x^{2k-n}$$

*collected in [***53***, Section 1.4], we obtain*

$$\begin{aligned}\left[\frac{1}{(1+x^2)^r}\right]^{(k)}&=\sum_{j=0}^{k}\frac{\mathrm{d}^j}{\mathrm{d}\,u^j}\left(\frac{1}{u^r}\right)B_{k,j}(2x,2,0,\dots,0)\\&=\sum_{j=0}^{k}\frac{\langle -r\rangle_j}{u^{r+j}}2^jB_{k,j}(x,1,0,\dots,0)\\&=\sum_{j=0}^{k}\frac{\langle -r\rangle_j}{(1+x^2)^{r+j}}2^j\frac{1}{2^{k-j}}\frac{k!}{j!}\binom{j}{k-j}x^{2j-k}\\&=\frac{k!}{2^kx^k(1+x^2)^r}\sum_{j=0}^{k}\langle -r\rangle_j\frac{2^{2j}}{j!}\binom{j}{k-j}\frac{x^{2j}}{(1+x^2)^j},\end{aligned}$$

where $u=u(x)=1+x^2$. See also texts at the site `https://math.stackexchange.com/a/4418636/`.

Remark 20. *We would like to mention the papers [**14, 30, 58, 70**], in which the power function* $\operatorname{sinc}^r z$ *for some specific ranges of* $r, x \in \mathbb{R}$ *is bounded from both sides, and to mention the papers [**26, 37, 52**], in which many bounds of the sinc function* $\operatorname{sinc} x$ *for* $x \in \left(0, \frac{\pi}{2}\right)$ *are established, reviewed, and surveyed.*

Remark 21. *Taking* $\alpha = 1$ *in* (33)*, we see that*

$$\frac{1}{x} - \frac{1}{\sin x} = \frac{1}{x}\left(1 - \frac{x}{\sin x}\right) = -\frac{1}{x}\sum_{k=1}^{\infty} f_1^{(k)}(0)\frac{x^k}{k!} = -\sum_{k=1}^{\infty} f_1^{(2k)}(0)\frac{x^{2k-1}}{(2k)!} < 0$$

and

$$\left(\frac{1}{x} - \frac{1}{\sin x}\right)' = -\frac{1}{x^2} + \frac{1}{\tan x \sin x} = -\sum_{k=1}^{\infty} f_1^{(2k)}(0)(2k-1)\frac{x^{2k-2}}{(2k)!} < 0$$

for $x \in (0, \pi)$*. Consequently, we arrive at* $\frac{1}{\tan x \sin x} < \frac{1}{x^2}$ *for* $x \in (0, \pi)$*, which can be rearranged as the inequality* (31) *on* $\left(0, \frac{\pi}{2}\right)$ *and its reversed version on* $\left(\frac{\pi}{2}, \pi\right)$*.*

Furthermore, from

$$\left(\frac{1}{x} - \frac{1}{\sin x}\right)'' = \frac{1}{x^3} - \frac{\cos^2 x + 1}{\sin^3 x} = -\sum_{k=2}^{\infty} f_1^{(2k)}(0)(2k-1)(2k-2)\frac{x^{2k-3}}{(2k)!} < 0$$

for $x \in (0, \pi)$*, we arrive at* $\left(\frac{\sin x}{x}\right)^3 < \frac{1+\cos^2 x}{2}$ *for* $x \in (-\pi, \pi)$*.*

Repeatedly differentiating as above, we can discover more similar inequalities.

Remark 22. *When* $\alpha = -1$*, the function* $f_{-1}(x) = \operatorname{sinc} x$*, whose power series expansion is* (18)*. This shows that the coefficients in power series expansion of the function* $f_\alpha(x)$ *are neither always positive nor always negative for some* $\alpha < 0$*.*

At the site `https://math.stackexchange.com/q/4445114`*, it was claimed that the Maclaurin series of the function* $1 - \left(\frac{\sin x}{x}\right)^{2/5}$ *has all coefficients positive.*

Denote the coefficients in the power series expansion (23) *by*

$$F_q(r) = (-1)^q \sum_{k=1}^{2q} \frac{(-r)_k}{k!} \sum_{j=1}^{k} (-1)^j \binom{k}{j} \frac{T(2q+j, j)}{\binom{2q+j}{j}}$$

for $r \in \mathbb{R}$ *and* $q \in \mathbb{N}$*, where the rising factorial* $(-r)_k$ *can be expressed as*

$$(-r)_k = (-1)^k \sum_{\ell=1}^{k} s(k, \ell) r^\ell.$$

Accordingly, we have

$$F_q(r) = (-1)^q \sum_{k=1}^{2q} \frac{(-1)^k}{k!} \left[\sum_{\ell=1}^{k} s(k, \ell) r^\ell\right] \left[\sum_{j=1}^{k} (-1)^j \binom{k}{j} \frac{T(2q+j, j)}{\binom{2q+j}{j}}\right]$$

$$= (-1)^q \sum_{\ell=1}^{2q} \left[\sum_{k=\ell}^{2q} (-1)^k \frac{s(k,\ell)}{k!} \sum_{j=1}^{k} (-1)^j \binom{k}{j} \frac{T(2q+j,j)}{\binom{2q+j}{j}}\right] r^\ell$$

for $r \in \mathbb{R}$ *and* $q \in \mathbb{N}$. *Therefore, we can regard* $F_q(r)$ *as a polynomials in* r *of degree* $2q$. *Theorem 13 means that the polynomial* $F_q(r)$ *in* r *is nonnegative on* $(-\infty, 0)$. *By discussing the zeros of the polynomial* $F_q(r)$ *in* $r \in (0, \infty)$, *one can obtain an answer to the positivity of the coefficients in the Maclaurin series of the function* $1 - \left(\frac{\sin x}{x}\right)^{2/5}$.

Remark 23. *Theorem 13 reveals that, two inequalities*

$$\left(\frac{x}{\sin x}\right)^\alpha > 1 + \sum_{q=1}^{n} (-1)^q \left[\sum_{k=1}^{2q} \frac{(-r)_k}{k!} \sum_{j=1}^{k} (-1)^j \binom{k}{j} \frac{T(2q+j,j)}{\binom{2q+j}{j}}\right] \frac{(2x)^{2q}}{(2q)!} \tag{34}$$

and

$$\begin{aligned} \left(\frac{x}{\sin x}\right)^\alpha > 1 + \sum_{q=1}^{n} (-1)^q &\left[\sum_{k=1}^{2q} \frac{(-r)_k}{k!} \sum_{j=1}^{k} (-1)^j \binom{k}{j} \right.\\ &\left. \times \sum_{m=0}^{2q} (-1)^m \binom{2q}{m} \left(\frac{j}{2}\right)^m \frac{S(2q+j-m,j)}{\binom{2q+j-m}{j}}\right] \frac{(2x)^{2q}}{(2q)!} \end{aligned} \tag{35}$$

are valid for any real number $\alpha > 0$, *any positive integer* $n \ge 1$, *and* $x \in (-\pi, \pi)$. *As three special cases of the inequalities* (34) *and* (35) *with* $n = 3$ *and* $\alpha = \frac{1}{3}, \frac{1}{2}, 1, 2, 3$, *the inequalities*

$$\sqrt[3]{\frac{\sin x}{x}} < \frac{1}{1 + \frac{x^2}{18} + \frac{11x^4}{3240} + \frac{61x^6}{244944}}, \qquad \sqrt{\frac{\sin x}{x}} < \frac{1}{1 + \frac{x^2}{12} + \frac{x^4}{160} + \frac{61x^6}{120960}},$$

$$\frac{\sin x}{x} < \frac{1}{1 + \frac{x^2}{6} + \frac{7x^4}{360} + \frac{31x^6}{15120}}, \qquad \left(\frac{\sin x}{x}\right)^2 < \frac{1}{1 + \frac{x^2}{3} + \frac{x^4}{15} + \frac{2x^6}{189}},$$

and

$$\left(\frac{\sin x}{x}\right)^3 < \frac{1}{1 + \frac{x^2}{2} + \frac{17x^4}{120} + \frac{457x^6}{15120}}$$

hold true for $x \in (-\pi, \pi)$.

Remark 24. *From the series expansions* (23) *and* (24) *in Theorem 11, we can obtain*

$$\begin{aligned} \frac{1}{1-\cos(z^4)} &= \frac{1}{2\sin^2 \frac{z^4}{2}} = \frac{1}{2}\left(\frac{2}{z^4}\right)^2 \left(\frac{\frac{z^4}{2}}{\sin \frac{z^4}{2}}\right)^2 \\ &= \frac{2}{z^8}\left\{1 + \sum_{q=1}^{\infty} (-1)^q \left[\sum_{k=1}^{2q} \frac{(2)_k}{k!} \sum_{j=1}^{k} (-1)^j \binom{k}{j} \frac{T(2q+j,j)}{\binom{2q+j}{j}}\right] \frac{z^{8q}}{(2q)!}\right\} \end{aligned}$$

for $x^4 < 2\pi$ and

$$\frac{1}{1-\cos x}=\frac{1}{2\sin^2\frac{x}{2}}=\frac{1}{2}\left(\frac{2}{x}\right)^2\left(\frac{\frac{x}{2}}{\sin\frac{x}{2}}\right)^2=\frac{2}{x^2}\left\{1+\sum_{q=1}^{\infty}(-1)^q\left[\sum_{k=1}^{2q}\frac{(2)_k}{k!}\right.\right.$$
$$\left.\left.\times\sum_{j=1}^{k}(-1)^j\binom{k}{j}\sum_{m=0}^{2q}(-1)^m\binom{2q}{m}\left(\frac{j}{2}\right)^m\frac{S(2q+j-m,j)}{\binom{2q+j-m}{j}}\right]\frac{x^{2q}}{(2q)!}\right\}$$

for $|x| < 2\pi$. See the websites `https://math.stackexchange.com/a/4672242` *and* `https://math.stackexchange.com/a/4672268`.

Remark 25. *This paper is a revised version of the electronic arXiv preprints [***48, 54***] which have been cited in the articles [***2, 11, 15, 29, 30, 31, 32, 46, 49***]. This means that the results in this paper are applicable, significant, interesting, and novel.*

Acknowledgements. The authors sincerely thank Dr. Jacques Gélinas, a retired mathematician at Ottawa in Canada, for his hard efforts to look up closely-related references and for his valuable discussions, including those mentioned in Remark 15 of this paper. The authors are also grateful to anonymous referees for their helpful suggestions, careful corrections, and valuable comments on the original version of this paper.

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(Received 02.09.2023)
(Revised 30.09.2023)

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