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EQUIVALENT CONDITIONS OF A REVERSE HILBERT - TYPE INTEGRAL INEQUALITY IN THE WHOLE PLANE AND APPLICATIONS

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In the present paper we establish a few equivalent conditions of a reverse Hilbert-type integral inequality with a general non-homogeneous kernel in the whole plane. In the form of applications, we deduce a few equivalent conditions of a reverse Hilbert-type integral inequality with a general homogeneous kernel in the whole plane. We also consider some particular cases and examples.

1. INTRODUCTION

If $f(x), g(y) \geq 0$,

$$0 < \int_0^\infty f^2(x)dx < \infty \ \ and \ \ 0 < \int_0^\infty g^2(y)dy < \infty \,,$$

we have the following well-known Hilbert integral inequality (cf. [**2**]):

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y} dxdy < \pi \left(\int_0^\infty f^2(x)dx \int_0^\infty g^2(y)dy \right)^{\frac{1}{2}} , \tag{1}$$

with the best possible constant factor π.
Recently, by the use of weight functions, several extensions of (1) were presented in

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books of B. Yang (cf. [**18**], [**17**]). Some Hilbert-type inequalities with homogenous kernels of degree 0 and with non-homogenous kernels were established in [**19**]-[**1**]. Some other kinds of Hilbert-type inequalities were obtained in [**8**]-[**?**]. Several of them are constructed in the quarter plane of the first quadrant.

In 2007, Yang [**22**] presented the following Hilbert-type integral inequality in the whole plane:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{f(x)g(y)}{(1+e^{x+y})^{\lambda}}dxdy$$
$$< B\left(\frac{\lambda}{2},\frac{\lambda}{2}\right)\left(\int_{-\infty}^{\infty}e^{-\lambda x}f^2(x)dx\int_{-\infty}^{\infty}e^{-\lambda y}g^2(y)dy\right)^{\frac{1}{2}}, \tag{2}$$

with the best possible constant factor $B(\frac{\lambda}{2},\frac{\lambda}{2})$, $\lambda > 0$, where $B(u,v)$ stands for the beta function (cf. [**12**]). He et al. proved in [**3**]-[**?**] some new Hilbert-type integral inequalities in the whole plane with the best possible constant factors.

In this paper, using weight functions, we obtain a few equivalent conditions of a reverse Hilbert-type integral inequality with a general non-homogeneous kernel in the whole plane. In the form of applications, a few equivalent conditions of a reverse Hilbert-type integral inequality with the general homogeneous kernel in the whole plane are deduced. We also consider some particular cases and examples.

2. TWO LEMMAS

In the sequel we shall assume that $0 < p < 1$, $\frac{1}{p}+\frac{1}{q}=1$, $\sigma_1,\sigma \in \mathbf{R} = (-\infty,\infty)$, and $h(u)$ is a non-negative measurable function in $\mathbf{R}$, with

$$K(\sigma) := \int_{-\infty}^{\infty} h(u)|u|^{\sigma-1}du = \int_0^{\infty}(h(-u)+h(u))u^{\sigma-1}du\,. \tag{3}$$

For $n \in \mathbf{N} = \{1,2,\dots\}$, we define the following two expressions:

$$I_1 := \int_{\{y;|y|\geq 1\}}\left(\int_{\{x;|x|\leq 1\}} h(xy)|x|^{\sigma+\frac{1}{pn}-1}dx\right)|y|^{\sigma_1-\frac{1}{qn}-1}dy, \tag{4}$$

$$I_2 := \int_{\{y;|y|\leq 1\}}\left(\int_{\{x;|x|\geq 1\}} h(xy)|x|^{\sigma-\frac{1}{pn}-1}dx\right)|y|^{\sigma_1+\frac{1}{qn}-1}dy. \tag{5}$$

Setting $u = xy$ in (4), by Fubini's theorem (cf. [**7**]), it follows that

$$\begin{aligned}
I_1 &= 2\int_1^{\infty}\left[\int_0^1 (h(xy)+h(-xy))x^{\sigma+\frac{1}{pn}-1}dx\right]y^{\sigma_1-\frac{1}{qn}-1}dy \\
&= 2\int_1^{\infty}\left[\int_0^y (h(u)+h(-u))u^{\sigma+\frac{1}{pn}-1}du\right]y^{\sigma_1-\sigma-\frac{1}{n}-1}dy \\
&\leq 2\int_1^{\infty}\left[\int_0^{\infty} (h(-u)+h(u))u^{\sigma+\frac{1}{pn}-1}du\right]y^{\sigma_1-\sigma-\frac{1}{n}-1}dy \\
&\leq 2\int_1^{\infty} y^{\sigma_1-\sigma-\frac{1}{n}-1}dy\left[\int_0^1 (h(-u)+h(u))u^{\sigma-1}du\right] \\
&\quad +2\int_1^{\infty} y^{\sigma_1-\sigma-\frac{1}{n}-1}dy\left[\int_1^{\infty} (h(-u)+h(u))u^{\sigma+\frac{1}{pn}-1}du\right]. \qquad (6)
\end{aligned}$$

We similarly obtain that

$$\begin{aligned}
I_2 &= \int_{\{y;|y|\leq 1\}}\left(\int_{\{x;|x|\geq 1\}} h(xy)|x|^{\sigma-\frac{1}{pn}-1}dx\right)|y|^{\sigma_1+\frac{1}{qn}-1}dy \\
&= 2\int_0^1\left[\int_1^{\infty} (h(xy)+h(-xy))x^{\sigma-\frac{1}{pn}-1}dx\right]y^{\sigma_1+\frac{1}{qn}-1}dy \\
&\leq 2\int_0^1\left[\int_0^{\infty} (h(u)+h(-u))u^{\sigma-\frac{1}{pn}-1}du\right]y^{\sigma_1-\sigma+\frac{1}{n}-1}dy \\
&\leq 2\int_0^1 y^{\sigma_1-\sigma+\frac{1}{n}-1}dy\left[\int_0^1 (h(-u)+h(u))u^{\sigma-\frac{1}{pn}-1}du\right] \\
&\quad +2\int_0^1 y^{\sigma_1-\sigma+\frac{1}{n}-1}dy\left[\int_1^{\infty} (h(-u)+h(u))u^{\sigma-1}du\right]. \qquad (7)
\end{aligned}$$

Lemma 1. *If there exist constants $\delta_0, M > 0$, such that $K(\sigma \pm \delta_0) < \infty$, and for any non-negative measurable functions $f(x)$ and $g(y)$ in $\mathbf{R}$, the following inequality*

$$\begin{aligned}
I &:= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(xy)f(x)g(y)dxdy \\
&\geq M\left[\int_{-\infty}^{\infty}|x|^{p(1-\sigma)-1}f^p(x)dx\right]^{\frac{1}{p}}\left[\int_{-\infty}^{\infty}|y|^{q(1-\sigma_1)-1}g^q(y)dy\right]^{\frac{1}{q}} \qquad (8)
\end{aligned}$$

holds true, then we have $\sigma_1 = \sigma$.

Proof. We have

$$\begin{aligned}
K(\sigma) &= \int_0^1 (h(-u)+h(u))u^{\sigma-1}du + \int_1^{\infty}(h(-u)+h(u))u^{\sigma-1}du \\
&\leq \int_0^1 (h(-u)+h(u))u^{\sigma-\delta_0-1}du + \int_1^{\infty}(h(-u)+h(u))u^{\sigma+\delta_0-1}du \\
&\leq K(\sigma-\delta_0)+K(\sigma+\delta_0) < \infty.
\end{aligned}$$

If $\sigma_1 > \sigma$, then for $n > \frac{1}{p\delta_0}$ $(n \in \mathbf{N})$, we set the functions

$$f_n(x) := \begin{cases} 0, |x| < 1 \\ |x|^{\sigma - \frac{1}{pn} - 1}, |x| \geq 1 \end{cases}, \quad g_n(y) := \begin{cases} |y|^{\sigma_1 + \frac{1}{qn} - 1}, |y| \leq 1 \\ 0, |y| > 1 \end{cases},$$

and obtain

$$\begin{aligned} J_2 \quad : \quad &= \left[\int_{-\infty}^{\infty} |x|^{p(1-\sigma)-1} f_n^p(x)dx\right]^{\frac{1}{p}} \left[\int_{-\infty}^{\infty} |y|^{q(1-\sigma_1)-1} g_n^q(y)dy\right]^{\frac{1}{q}} \\ &= \left(\int_{\{x;|x|\geq 1\}} |x|^{-\frac{1}{n}-1}dx\right)^{\frac{1}{p}} \left(\int_{\{y;|y|\leq 1\}} |y|^{\frac{1}{n}-1}dy\right)^{\frac{1}{q}} = 2n. \end{aligned}$$

By (7), we have

$$\begin{aligned} & \frac{2}{\sigma_1 - \sigma + \frac{1}{n}}(K(\sigma - \delta_0) + K(\sigma)) \\ \geq \quad & 2\int_0^1 \left[\int_0^1 (h(-u) + h(u))u^{\sigma-\delta_0-1}du\right] y^{\sigma_1-\sigma+\frac{1}{n}-1}dy \\ & +2\int_0^1 \left[\int_1^{\infty} (h(-u) + h(u))u^{\sigma-1}du\right] y^{\sigma_1-\sigma+\frac{1}{n}-1}dy \\ \geq \quad & I_2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(xy)f_n(x)g_n(y)dxdy \geq MJ_2 = 2Mn. \qquad (9) \end{aligned}$$

For $n \to \infty$ in (9), we deduce that

$$\infty > \frac{2}{\sigma_1 - \sigma}(K(\sigma - \delta_0) + K(\sigma)) \geq \infty,$$

which is a contradiction.

If $\sigma_1 < \sigma$, then for $n > \frac{1}{p\delta_0}$ $(n \in \mathbf{N})$, we set the functions

$$\widetilde{f}_n(x) := \begin{cases} |x|^{\sigma + \frac{1}{pn} - 1}, |x| \leq 1 \\ 0, |x| > 1 \end{cases}, \quad \widetilde{g}_n(y) := \begin{cases} 0, |y| < 1 \\ |y|^{\sigma_1 - \frac{1}{qn} - 1}, |y| \geq 1 \end{cases},$$

and find

$$\begin{aligned} \widetilde{J}_2 \quad := \quad & \left[\int_{-\infty}^{\infty} |x|^{p(1-\sigma)-1} \widetilde{f}_n^p(x)dx\right]^{\frac{1}{p}} \left[\int_{-\infty}^{\infty} |y|^{q(1-\sigma_1)-1} \widetilde{g}_n^q(y)dy\right]^{\frac{1}{q}} \\ = \quad & \left(\int_{-1}^{1} |x|^{\frac{1}{n}-1}dx\right)^{\frac{1}{p}} \left(\int_{\{y;|y|\geq 1\}} |y|^{-\frac{1}{n}-1}dy\right)^{\frac{1}{q}} = 2n. \end{aligned}$$

By (6), we have

$$\begin{aligned}
&\frac{2}{\sigma-\sigma_1}(K(\sigma)+K(\sigma+\delta_0)) \\
\geq\ & 2\int_1^\infty y^{\sigma_1-\sigma-1}dy\left[\int_0^1 (h(-u)+h(u))u^{\sigma-1}du\right] \\
&+2\int_1^\infty y^{\sigma_1-\sigma-1}dy\left[\int_1^\infty (h(-u)+h(u))u^{\sigma+\delta_0-1}du\right] \\
\geq\ & I_1=\int_0^\infty\int_0^\infty h(xy)\widetilde{f}_n(x)\widetilde{g}_n(y)dxdy \geq M\widetilde{J}_2=2Mn. \qquad (10)
\end{aligned}$$

For $n\to\infty$ in (10), we derive that

$$\infty > \frac{2}{\sigma-\sigma_1}(K(\sigma)+K(\sigma+\delta_0)) \geq \infty,$$

which is a contradiction.

Hence, we conclude that $\sigma_1=\sigma$.

This completes the proof of the lemma. □

Lemma 2. *If there exist constants δ_0, $M>0$, such that $K(\sigma+\delta_0)<\infty$ and for any non-negative measurable functions $f(x)$ and $g(y)$ in* $\mathbf{R}$, *the following inequality*

$$\begin{aligned}
&\int_{-\infty}^\infty\int_{-\infty}^\infty h(xy)f(x)g(y)dxdy \\
\geq\ & M\left[\int_{-\infty}^\infty |x|^{p(1-\sigma)-1}f^p(x)dx\right]^{\frac{1}{p}}\left[\int_{-\infty}^\infty |y|^{q(1-\sigma)-1}g^q(y)dy\right]^{\frac{1}{q}} \qquad (11)
\end{aligned}$$

holds true, then we have $K(\sigma)\geq M>0$.

Proof. For $\sigma_1=\sigma$, we reduce (6) and then use inequality $I_1\geq M\widetilde{J}_2$ (when $\sigma_1=\sigma$) as follows

$$\int_0^1 (h(-u)+h(u))u^{\sigma-1}du+\int_1^\infty (h(-u)+h(u))u^{\sigma+\frac{1}{pn}-1}du \geq \frac{I_1}{2n}\geq M. \qquad (12)$$

Since

$$(h(-u)+h(u))u^{\sigma+\frac{1}{pn}-1}\leq (h(-u)+h(u))u^{\sigma+\delta_0-1}(u\geq 1)$$

and

$$\int_1^\infty (h(-u)+h(u))u^{\sigma+\delta_0-1}du\leq K(\sigma+\delta_0)<\infty,$$

by the Lebesgue control convergence theorem (cf. [**7**]) and (12), we have

$$\begin{aligned}
K(\sigma) &= \int_0^1 (h(-u)+h(u))u^{\sigma-1}du+\int_1^\infty \lim_{n\to\infty}(h(-u)+h(u))u^{\sigma+\frac{1}{pn}-1}du \\
&= \lim_{n\to\infty}\left[\int_0^1 (h(-u)+h(u))u^{\sigma-1}du+\int_1^\infty (h(-u)+h(u))u^{\sigma+\frac{1}{pn}-1}du\right] \\
&\geq \lim_{n\to\infty}\frac{1}{2n}I_1\geq M>0.
\end{aligned}$$

This completes the proof of the lemma. □

3. MAIN RESULTS AND SOME COROLLARIES

Theorem 1. *If there exists a constant $\delta_0 > 0$, such that $K(\sigma \pm \delta_0) < \infty$, then the following conditions are equivalent:*

(i) There exists a constant $M > 0$, such that for any $f(x) \geq 0$, satisfying

$$0 < \int_{-\infty}^{\infty} |x|^{p(1-\sigma)-1} f^p(x)dx < \infty,$$

we have the following inequality:

$$\begin{aligned} J &:= \left[\int_{-\infty}^{\infty} |y|^{p\sigma_1 - 1} \left(\int_{-\infty}^{\infty} h(xy)f(x)dx\right)^p dy\right]^{\frac{1}{p}} \\ &> M\left[\int_{-\infty}^{\infty} |x|^{p(1-\sigma)-1} f^p(x)dx\right]^{\frac{1}{p}}. \end{aligned} \tag{13}$$

(ii) There exists a constant $M > 0$, such that for any $f(x), g(y) \geq 0$, satisfying

$$0 < \int_{-\infty}^{\infty} |x|^{p(1-\sigma)-1} f^p(x)dx < \infty$$

and

$$0 < \int_{=\infty}^{\infty} |y|^{q(1-\sigma_1)-1} g^q(y)dy < \infty,$$

we have the following inequality:

$$\begin{aligned} I &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(xy)f(x)g(y)dxdy \\ &> M\left[\int_{-\infty}^{\infty} |x|^{p(1-\sigma)-1} f^p(x)dx\right]^{\frac{1}{p}} \left[\int_{-\infty}^{\infty} |y|^{q(1-\sigma_1)-1} g^q(y)dy\right]^{\frac{1}{q}}. \end{aligned} \tag{14}$$

(iii) $\sigma_1 = \sigma$, and $K(\sigma) > 0$.

Proof. $(i) \Rightarrow (ii)$. By Hölder's inequality (cf. [**6**]), we have

$$\begin{aligned} I &= \int_{-\infty}^{\infty} \left(|y|^{\sigma_1 - \frac{1}{p}} \int_{-\infty}^{\infty} h(xy)f(x)dx\right)\left(|y|^{\frac{1}{p}-\sigma_1} g(y)\right) dy \\ &\geq J\left[\int_{-\infty}^{\infty} |y|^{q(1-\sigma_1)-1} g^q(y)dy\right]^{\frac{1}{q}}. \end{aligned} \tag{15}$$

Then by (13), we have (14).

(ii) $\Rightarrow$ *(iii)*. Since $K(\sigma \pm \delta_0) < \infty$, by Lemma 1, we have $\sigma_1 = \sigma$. Then by Lemma 2, we have $K(\sigma) \geq M > 0$.

(iii) $\Rightarrow$ *(i)*. Setting $u = xy$, we define the following weight function: For $y \in (-\infty, 0) \cup (0, \infty)$,

$$\begin{aligned} \omega(\sigma, y) &:= |y|^{\sigma} \int_{-\infty}^{\infty} h(xy)|x|^{\sigma-1} dx \\ &= \int_{0}^{\infty} (h(-u) + h(u)) u^{\sigma-1} du = K(\sigma). \end{aligned} \tag{16}$$

By Hölder's inequality with weight and (16), we have

$$\begin{aligned} &\left(\int_{-\infty}^{\infty} h(xy) f(x) dx\right)^p \\ = &\left\{\int_{-\infty}^{\infty} h(xy) \left[\frac{|y|^{(\sigma-1)/p}}{|x|^{(\sigma-1)/q}} f(x)\right] \left[\frac{|x|^{(\sigma-1)/q}}{|y|^{(\sigma-1)/p}}\right] dx\right\}^p \\ \geq &\int_{-\infty}^{\infty} h(xy) \frac{|y|^{\sigma-1}}{|x|^{(\sigma-1)p/q}} f^p(x) dx \left[\int_{-\infty}^{\infty} h(xy) \frac{|x|^{\sigma-1}}{|y|^{(\sigma-1)q/p}} dx\right]^{p/q} \\ = &\left[\omega(\sigma, y)|y|^{q(1-\sigma)-1}\right]^{p-1} \int_{-\infty}^{\infty} h(xy) \frac{|y|^{\sigma-1}}{|x|^{(\sigma-1)p/q}} f^p(x) dx \\ = &(K(\sigma))^{p-1} |y|^{-p\sigma+1} \int_{-\infty}^{\infty} h(xy) \frac{|y|^{\sigma-1}}{|x|^{(\sigma-1)p/q}} f^p(x) dx. \end{aligned} \tag{17}$$

If (17) assumes the form of equality for some $y \in (-\infty, 0) \cup (0, \infty)$, then (cf. [**6**]) there exist constants A and B, such that they are not all zero, and

$$A \frac{|y|^{\sigma-1}}{|x|^{(\sigma-1)p/q}} f^p(x) = B \frac{|x|^{\sigma-1}}{|y|^{(\sigma-1)q/p}} \text{ } a.e. \text{ } in \text{ } \mathbf{R}.$$

Let us suppose that $A \neq 0$ (otherwise $B = A = 0$). Then it follows that

$$|x|^{p(1-\sigma)-1} f^p(x) = |y|^{q(1-\sigma)} \frac{B}{A|x|} \text{ } a.e. \text{ } in \text{ } \mathbf{R},$$

which contradicts the fact that

$$0 < \int_{-\infty}^{\infty} |x|^{p(1-\sigma)-1} f^p(x) dx < \infty.$$

Hence, (17) assumes the form of strict inequality.

For $\sigma_1 = \sigma$, by Fubini's theorem (cf. [**7**]) and (17), we have

$$\begin{aligned} J &> (K(\sigma))^{\frac{1}{q}} \left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(xy)\frac{|y|^{\sigma-1}}{|x|^{(\sigma-1)p/q}} f^p(x)dxdy\right]^{\frac{1}{p}} \\ &= (K(\sigma))^{\frac{1}{q}} \left\{\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} h(xy)\frac{|y|^{\sigma-1}}{|x|^{(\sigma-1)(p-1)}}dy\right] f^p(x)dx\right\}^{\frac{1}{p}} \\ &= (K(\sigma))^{\frac{1}{q}} \left[\int_{-\infty}^{\infty} \omega(\sigma,x)|x|^{p(1-\sigma)-1} f^p(x)dx\right]^{\frac{1}{p}} \\ &= K(\sigma)\left[\int_{-\infty}^{\infty} |x|^{p(1-\sigma)-1} f^p(x)dx\right]^{\frac{1}{p}}. \end{aligned}$$

Since $K(\sigma) \in \mathbf{R}_+$, setting $0 < M \le K(\sigma)$, (13) follows.

Therefore, the conditions (i), (ii) and (iii) are equivalent.

This completes the proof of the theorem. □

For $\sigma_1 = \sigma$, we have the following theorem:

Theorem 2. *If there exists a constant $\delta_0 > 0$, such that $K(\sigma \pm \delta_0) < \infty$, then the following conditions are equivalent:*

(i) There exists a constant $M > 0$, such that for any $f(x) \ge 0$ satisfying

$$0 < \int_{-\infty}^{\infty} |x|^{p(1-\sigma)-1} f^p(x)dx < \infty,$$

we have the following inequality:

$$\begin{aligned} &\left[\int_{-\infty}^{\infty} |y|^{p\sigma-1}\left(\int_{-\infty}^{\infty} h(xy)f(x)dx\right)^p dy\right]^{\frac{1}{p}} \\ &> M\left[\int_{-\infty}^{\infty} |x|^{p(1-\sigma)-1} f^p(x)dx\right]^{\frac{1}{p}}. \end{aligned} \tag{18}$$

(ii) There exists a constant $M > 0$, such that for any $f(x), g(y) \ge 0$,

$$0 < \int_{-\infty}^{\infty} |x|^{p(1-\sigma)-1} f^p(x)dx < \infty$$

and

$$0 < \int_{-\infty}^{\infty} |y|^{q(1-\sigma)-1} g^q(y)dy < \infty,$$

we have the following inequality:

$$\begin{aligned} &\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(xy)f(x)g(y)dxdy \\ &> M\left[\int_{-\infty}^{\infty} |x|^{p(1-\sigma)-1} f^p(x)dx\right]^{\frac{1}{p}} \left[\int_{-\infty}^{\infty} |y|^{q(1-\sigma)-1} g^q(y)dy\right]^{\frac{1}{q}}. \end{aligned} \tag{19}$$

(iii) $K(\sigma) > 0$.

Moreover, if (iii) follows, then the constant factor $M = K(\sigma) \in \mathbf{R}_+$ in (18) and (19) is the best possible.

Proof. For $\sigma_1 = \sigma$ in Theorem 1, we can still conclude that the conditions (i), (ii) and (iii) in Theorem 2 are equivalent.

When condition (iii) is satisfied, if there exists a constant $M \geq K(\sigma)$, such that (19) is valid, then by Lemma 2, we have $K(\sigma) \geq M$. Hence, the constant factor $M = K(\sigma) \in \mathbf{R}_+$ in (19) is the best possible.

The constant factor $M = K(\sigma)$ in (18) is still the best possible. Otherwise, by (15) (for $\sigma_1 = \sigma$), we would reach a contradiction that the constant factor $M = K(\sigma)$ in (19) is not the best possible.

This completes the proof of the theorem. □

In particular, for $\sigma = \frac{1}{p}$ in Theorem 2, we have the following corollary.

Corollary 1. *If there exists a constant $\delta_0 > 0$, such that $K(\frac{1}{p} \pm \delta_0) < \infty$, then the following conditions are equivalent:*

(i) There exists a constant $M > 0$, such that for any $f(x) \geq 0$ satisfying

$$0 < \int_{-\infty}^{\infty} |x|^{p-2} f^p(x) dx < \infty,$$

we have the following inequality:

$$\left[\int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} h(xy) f(x) dx\right)^p dy\right]^{\frac{1}{p}} > M \left(\int_{-\infty}^{\infty} |x|^{p-2} f^p(x) dx\right)^{\frac{1}{p}}. \tag{20}$$

(ii) There exists a constant $M > 0$, such that for any $f(x), g(y) \geq 0$ satisfying

$$0 < \int_{-\infty}^{\infty} |x|^{p-2} f^p(x) dx < \infty$$

and

$$0 < \int_{-\infty}^{\infty} g^q(y) dy < \infty,$$

we have the following inequality:

$$\begin{aligned} &\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(xy) f(x) g(y) dx dy \\ > \ & M \left(\int_{-\infty}^{\infty} |x|^{p-2} f^p(x) dx\right)^{\frac{1}{p}} \left(\int_{-\infty}^{\infty} g^q(y) dy\right)^{\frac{1}{q}}. \end{aligned} \tag{21}$$

(iii) $K(\frac{1}{p}) > 0$.

If Condition (iii) follows, then the constant $M = K(\frac{1}{p})\,(\in \mathbf{R}_+)$ in (20) and (21) is the best possible.

Setting

$$y = \frac{1}{Y}, \quad G(Y) = g\left(\frac{1}{Y}\right)\frac{1}{Y^2}$$

in Theorem 1-2, then replacing Y by y, we obtain the following corollary.

Corollary 2. *If there exists a constant $\delta_0 > 0$, such that $K(\sigma \pm \delta_0) < \infty$, then the following conditions are equivalent:*

(i) There exists a constant $M > 0$, such that for any $f(x) \geq 0$ satisfying

$$0 < \int_0^\infty |x|^{p(1-\sigma)-1} f^p(x)dx < \infty,$$

we have the following inequality:

$$\begin{aligned} &\left[\int_{-\infty}^\infty |y|^{-p\sigma_1-1}\left(\int_{-\infty}^\infty h\left(\frac{x}{y}\right)f(x)dx\right)^p dy\right]^{\frac{1}{p}} \\ &> M\left[\int_{-\infty}^\infty |x|^{p(1-\sigma)-1} f^p(x)dx\right]. \end{aligned} \tag{22}$$

(ii) There exists a constant $M > 0$, such that for any $f(x), G(y) \geq 0$ satisfying

$$0 < \int_{-\infty}^\infty |x|^{p(1-\sigma)-1} f^p(x)dx < \infty$$

and

$$0 < \int_{-\infty}^\infty |y|^{q(1+\sigma_1)-1} G^q(y)dy < \infty,$$

we have the following inequality:

$$\begin{aligned} &\int_{-\infty}^\infty \int_{-\infty}^\infty h\left(\frac{x}{y}\right) f(x)G(y)dxdy \\ &> M\left[\int_{-\infty}^\infty |x|^{p(1-\sigma)-1} f^p(x)dx\right]^{\frac{1}{p}}\left[\int_{-\infty}^\infty y^{q(1+\sigma_1)-1} G^q(y)dy\right]^{\frac{1}{q}}. \end{aligned} \tag{23}$$

(iii) $\sigma_1 = \sigma$, and $K(\sigma) > 0$.

If condition (iii) is satisfied, then the constant $M = K(\sigma)\,(\in \mathbf{R}_+)$ in (22) and (23) (for $\sigma_1 = \sigma$) is the best possible.

Note. $h(\frac{x}{y})$ is a homogeneous function of degree 0, namely, $h(\frac{x}{y}) = k_0(x, y)$.

Setting $h(u) = k_\lambda(u, 1)$, where $k_\lambda(x, y)$ $(x, y \in \mathbf{R})$ is the homogeneous function of degree $-\lambda \in \mathbf{R}$, with

$$K_\lambda(\sigma) := \int_{-\infty}^\infty k_\lambda(u, 1)|u|^{\sigma-1}du = \int_0^\infty (k_\lambda(-u, 1) + k_\lambda(u, 1))u^{\sigma-1}du,$$

then for

$$g(y) = |y|^\lambda G(y) \quad and \quad \mu = \lambda - \sigma_1$$

in Corollary 2, we derive the following theorem.

Theorem 3. *If there exists a constant $\delta_0 > 0$, such that $K_\lambda(\sigma \pm \delta_0) < \infty$, then the following conditions are equivalent:*

(i) There exists a constant $M > 0$, such that for any $f(x) \geq 0$ satisfying

$$0 < \int_{-\infty}^{\infty} |x|^{p(1-\sigma)-1} f^p(x)dx < \infty,$$

we have the following inequality:

$$\left[\int_{-\infty}^{\infty} |y|^{p\mu-1} \left(\int_{-\infty}^{\infty} k_\lambda(x,y)f(x)dx\right)^p dy\right]^{\frac{1}{p}} > M\left[\int_{-\infty}^{\infty} |x|^{p(1-\sigma)-1} f^p(x)dx\right]^{\frac{1}{p}}. \tag{24}$$

(ii) There exists a constant $M > 0$, such that for any $f(x), g(y) \geq 0$ satisfying

$$0 < \int_{-\infty}^{\infty} |x|^{p(1-\sigma)-1} f^p(x)dx < \infty$$

and

$$0 < \int_{-\infty}^{\infty} |y|^{q(1-\mu)-1} g^q(y)dy < \infty,$$

we have the following inequality:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} k_\lambda(x,y)f(x)g(y)dxdy > M\left[\int_{-\infty}^{\infty} |x|^{p(1-\sigma)-1} f^p(x)dx\right]^{\frac{1}{p}} \left[\int_{-\infty}^{\infty} |y|^{q(1-\mu)-1} g^q(y)dy\right]^{\frac{1}{q}}. \tag{25}$$

(iii) $\mu + \sigma = \lambda$, and $K_\lambda(\sigma) > 0$.

If condition (iii) is satisfied, then the constant $M = K_\lambda(\sigma)\,(\in \mathbf{R}_+)$ in (24) and (25) is the best possible.

Remark 1. *If $\lambda = 0, \mu = -\sigma_1,\ k_0(x,y) = h(\frac{x}{y})$, then Theorem 3 reduces to Corollary 2.*

In particular, for $\lambda = 1, \sigma = \frac{1}{q}$, and $\mu = \frac{1}{p}$ in Theorem 3, we derive the corollary below.

Corollary 3. *If there exists a constant $\delta_0 > 0$, such that $K_1(\frac{1}{q} \pm \delta_0) < \infty$, then the following conditions are equivalent:*

(i) There exists a constant $M > 0$, such that for any $f(x) \geq 0$ satisfying

$$0 < \int_{-\infty}^{\infty} f^p(x)dx < \infty,$$

we have the following inequality:

$$\left[\int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} k_1(x,y)f(x)dx\right)^p dy\right]^{\frac{1}{p}} > M \left(\int_{-\infty}^{\infty} f^p(x)dx\right). \tag{26}$$

(ii) There exists a constant $M > 0$, such that for any $f(x), g(y) \geq 0$ satisfying

$$0 < \int_{-\infty}^{\infty} f^p(x)dx < \infty,$$

and

$$0 < \int_{-\infty}^{\infty} g^q(y)dy < \infty,$$

we have the following inequality:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} k_1(x,y)f(x)g(y)dxdy > M \left(\int_{-\infty}^{\infty} f^p(x)dx\right)^{\frac{1}{p}} \left(\int_{-\infty}^{\infty} g^q(y)dy\right)^{\frac{1}{q}}. \tag{27}$$

(iii) $K_1(\frac{1}{q}) > 0$.

If condition (iii) is satisfied, then the constant $M = K_1(\frac{1}{q})(\in \mathbf{R}_+)$ in (26) and (27) is the best possible.

For $\lambda = 1, \sigma = \frac{1}{p}$, and $\mu = \frac{1}{q}$ in Theorem 3, we have

Corollary 4. *If there exists a constant $\delta_0 > 0$, such that $K_1(\frac{1}{p} \pm \delta_0) < \infty$, then the following conditions are equivalent:*

(i) There exists a constant $M > 0$, such that for any $f(x) \geq 0$ satisfying

$$0 < \int_{-\infty}^{\infty} |x|^{p-2} f^p(x)dx < \infty,$$

we have the following inequality:

$$\left[\int_{-\infty}^{\infty} |y|^{p-2} \left(\int_{-\infty}^{\infty} k_1(x,y)f(x)dx\right)^p dy\right]^{\frac{1}{p}} > M \left(\int_{-\infty}^{\infty} |x|^{p-2} f^p(x)dx\right)^{\frac{1}{p}}. \tag{28}$$

(ii) There exists a constant $M > 0$, such that for any $f(x), g(y) \geq 0$ satisfying

$$0 < \int_{-\infty}^{\infty} |x|^{p-2} f^p(x)dx < \infty,$$

and

$$0<\int_{-\infty}^{\infty}|y|^{q-2}g^{q}(y)dy<\infty,$$

we have the following inequality:

$$\begin{aligned}&\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}k_1(x,y)f(x)g(y)dxdy\\ &> M\left(\int_{-\infty}^{\infty}|x|^{p-2}f^{p}(x)dx\right)^{\frac{1}{p}}\left(\int_{-\infty}^{\infty}|y|^{q-2}g^{q}(y)dy\right)^{\frac{1}{q}}.\end{aligned}\tag{29}$$

(iii) $K_1(\frac{1}{p})>0$.

If condition (iii) is satisfied, then the constant $M=K_1(\frac{1}{p})\,(\in\mathbf{R}_+)$ in (28) and (29) is the best possible.

Example 1. *Setting*

$$h(u)=\frac{|\ln|u||^{\beta}}{(\max\{|u|,1\})^{\lambda-1}|u-1|},$$

$$h(xy)=\frac{|\ln|xy||^{\beta}}{(\max\{|xy|,1\})^{\lambda-1}|xy-1|},$$

and

$$k_{\lambda}(x,y)=\frac{|\ln|x/y||^{\beta}}{(\max\{|x|,|y|\})^{\lambda-1}|x-y|}\quad(u,x,y\in\mathbf{R}),$$

for $\beta>0,\sigma,\mu>0,\sigma+\mu=\lambda$, *it follows that*

$$\begin{aligned}K(\sigma) &= K_{\lambda}(\sigma)=\int_{0}^{\infty}\frac{|\ln u|^{\beta}u^{\sigma-1}}{(\max\{u,1\})^{\lambda-1}}\left(\frac{1}{u+1}+\frac{1}{|u-1|}\right)du\\ &= \int_0^1(-\ln u)^{\beta}\left(\frac{1}{u+1}+\frac{1}{1-u}\right)(u^{\sigma-1}+u^{\mu-1})du\\ &= 2\int_0^1(-\ln u)^{\beta}\frac{1}{1-u^2}(u^{\sigma-1}+u^{\mu-1})du\\ &= 2\int_0^1(-\ln u)^{\beta}\sum_{k=0}^{\infty}u^{2k}(u^{\sigma-1}+u^{\mu-1})du.\end{aligned}$$

By the Lebesgue term by term integration theorem (cf. [7]), we have

$$\begin{aligned}K(\sigma) &= K_{\lambda}(\sigma)=2\sum_{k=0}^{\infty}\int_0^1(-\ln u)^{\beta}(u^{2k+\sigma-1}+u^{2k+\mu-1}du\\ &= 2\sum_{k=0}^{\infty}\left[\frac{1}{(2k+\sigma)^{\beta+1}}+\frac{1}{(2k+\mu)^{\beta+1}}\right]\int_0^{\infty}v^{\beta}e^{-v}dv\\ &= \frac{\Gamma(\beta+1)}{2^{\beta}}\left(\zeta\left(\beta+1,\frac{\sigma}{2}\right)+\zeta\left(\beta+1,\frac{\mu}{2}\right)\right)\in\mathbf{R}_+,\end{aligned}$$

where

$$\zeta(s,a)=\sum_{k=0}^{\infty}\frac{1}{(k+a)^s}\quad (Re\, s>1; a>0)$$

stands for the extended Riemann zeta function. Note that $\zeta(s,1)=\sum_{k=1}^{\infty}\frac{1}{k^s}$ $(Re\, s>1)$ *is the Riemann zeta function) (cf. [**12**]).*

Example 2. *Setting*

$$h(u)=\frac{1}{|u-1|^{\lambda}},\quad h(xy)=\frac{1}{|xy-1|^{\lambda}},$$

$$k_{\lambda}(x,y)=\frac{1}{|x-y|^{\lambda}}\,(u,x,y\in\mathbf{R}),$$

for $\sigma,\mu>0,\sigma+\mu=\lambda<1$, *it follows that*

$$\begin{aligned} K(\sigma) &= K_{\lambda}(\sigma)=\int_0^{\infty}\left(\frac{1}{(u+1)^{\lambda}}+\frac{1}{|u-1|^{\lambda}}\right)u^{\sigma-1}du \\ &= \int_0^{\infty}\frac{u^{\sigma-1}}{(u+1)^{\lambda}}du+\int_0^1\frac{1}{(1-u)^{\lambda}}(u^{\sigma-1}+u^{\mu-1})du \\ &= B(\sigma,\mu)+B(1-\lambda,\sigma)+B(1-\lambda,\mu)\in\mathbf{R}_+. \end{aligned}$$

We can use the above examples in Theorem 1 and Theorem 3 with the particular kernels.

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