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ON THE SEIDEL INTEGRAL GRAPHS WHICH BELONG TO THE CLASS $\alpha K_a \cup \beta K_{b,b}$

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We say that a simple graph G is Seidel integral if its Seidel spectrum consists entirely of integers. In this work we establish a characterization of Seidel integral graphs which belong to the class $\alpha K_a \cup \beta K_{b,b}$, where mG denotes the m-fold union of the graph G.

1. INTRODUCTION

Let G be a simple graph of order n and let $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ be the eigenvalues of its (0,1) adjacency matrix of G. The spectrum of G is the multiset of its eigenvalues and is denoted by $\sigma(G)$. A graph G is said to be integral if its spectrum $\sigma(G)$ consists only of integers [**1**]. The Seidel spectrum of G is the multiset of eigenvalues $\lambda_1^* \geq \lambda_2^* \geq \cdots \geq \lambda_n^*$ of its $(0, -1, 1)$ adjacency matrix $A^* = A^*(G)$ and is denoted by $\sigma^*(G)$. A graph G is said to be Seidel integral if its Seidel spectrum $\sigma^*(G)$ consists only of integers. We say that an eigenvalue μ is main if and only if $\langle \mathbf{j}, \mathbf{P}\mathbf{j}\rangle = n\cos^2\alpha > 0$, where $\mathbf{j}$ is the main vector (with coordinates equal to 1) and $\mathbf{P}$ is the orthogonal projection of the space $\mathbb{R}^n$ onto the eigenspace $\mathcal{E}_A(\mu)$. The quantity $\beta = |\cos\alpha|$ is called the main angle of μ. Similarly, we say that a Seidel eigenvalue μ^* is the Seidel main eigenvalue if and only if $\langle \mathbf{j}, \mathbf{P}^*\mathbf{j}\rangle = n\cos^2\alpha^* > 0$, where $\mathbf{P}^*$ is the orthogonal projection of the space $\mathbb{R}^n$ onto the eigenspace $\mathcal{E}_{A^*}(\mu^*)$. The quantity $\beta^* = |\cos\alpha^*|$ is called the Seidel main angle of μ^*. In [**1**] was proved that the graph G and its complement $\overline{G}$ have the

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same number of main eigenvalues. We also know that $|\mathcal{M}(G)| = |\mathcal{M}^*(G)|$, where $\mathcal{M}(G)$ and $\mathcal{M}^*(G)$ denote the sets of all main and the Seidel main eigenvalues of G, respectively. In particular, if G is a graph of order n with k main eigenvalues $\mu_1, \mu_2, \dots, \mu_k$ and Seidel main eigenvalues $\mu_1^*, \mu_2^*, \dots, \mu_k^*$ then $n_1+n_2+\cdots+n_k = n$ and $n_1^* + n_2^* + \cdots + n_k^* = n$, where $n_i = n\beta_i^2$ and $n_i^* = n(\beta_i^*)^2$ for $i = 1, 2, \dots, k$.

Let G be a graph of order n with exactly two main eigenvalues μ_1 and μ_2 and let $n_1 = n\beta_1^2$ and $n_2 = n\beta_2^2$.

Theorem 1 (Lepović [**3**]). *Let G be a graph of order n with two main eigenvalues μ_1 and μ_2. Then*

$$\mu_{1,2}^* = \frac{n-2-2\mu_1-2\mu_2}{2} \pm \frac{\sqrt{\big(2\mu_1-2\mu_2+n\big)^2 - 8n_1\big(\mu_1-\mu_2\big)}}{2}\,. \tag{1}$$

Besides, we have

$$n_{1,2}^* = \frac{n}{2} \pm \frac{n^2 + 2\big(n-2n_1\big)\big(\mu_1-\mu_2\big)}{2\sqrt{\big(2\mu_1-2\mu_2+n\big)^2 - 8n_1\big(\mu_1-\mu_2\big)}}\,, \tag{2}$$

where $n_1^* = n(\beta_1^*)^2$ and $n_2^* = n(\beta_2^*)^2$.

Further, let K_n and $K_{m,n}$ denote the complete graph and the complete bipartite graph, respectively. We know that $\sigma(K_n) = \{n-1, -1^{n-1}\}$ and $\sigma(K_{m,n}) = \{\sqrt{mn}\,, 0^{\,m+n-2}, -\sqrt{mn}\,\}$, where the multiplicity of a multiple eigenvalue is given in the form of an exponent. We note that $\alpha K_a \cup \beta K_{b,b}$ is an integral graph with the spectrum $\alpha\sigma(K_a) \cup \beta\sigma(K_{b,b})$ and with two main eigenvalues $\mu_a = a-1$ and $\mu_b = b$, for any $\alpha, \beta, a, b \in \mathbb{N}$ with $a \neq (b+1)$, where mG and $m\sigma(G)$ denote the m-fold union of the graph G and the m-fold union of the the spectrum $\sigma(G)$, respectively. As is pointed out [**3**], if G is an integral graph then G is Seidel integral if and only if the Seidel main spectrum of G contains integral values. Consequently, $\alpha K_a \cup \beta K_{b,b}$ is Seidel integral if and only if its largest Seidel main eigenvalue $\mu_1^* \in \mathbb{N}$.

Next, (i) we have established in [**4**] a characterization of integral graphs which belong to the class $\overline{\alpha K_a \cup \beta K_b}$; (ii) we have established in [**6**] a characterization of integral graphs which belong to the class $\overline{\alpha K_{a,a} \cup \beta K_{b,b}}$ and (iii) we have established in [**5**] a characterization of integral graphs which belong to the class $\overline{\alpha K_a \cup \beta K_{b,b}}$. Besides, we have established in [**7**] a characterization of Seidel integral graphs which belong to the class $\alpha K_a \cup \beta K_b$ and we have established in [**8**] a characterization of Seidel integral graphs which belong to the class $\alpha K_{a,a} \cup \beta K_{b,b}$. In this work we shall establish a characterization of Seidel integral graphs which belong to the class $\alpha K_a \cup \beta K_{b,b}$.

2. MAIN RESULTS

First, note that $o = \alpha a + 2\beta b$ is the order of $\alpha K_a \cup \beta K_{b,b}$. In the case that $a > (b+1)$ we find that $\mu_1 = a-1$ and $n_1 = \alpha a$. Then, according to (1), we get

$$\mu_1^* = \frac{\alpha a + 2\beta b - 2(a+b) + \delta}{2} \quad \text{and} \quad \mu_2^* = \frac{\alpha a + 2\beta b - 2(a+b) - \delta}{2}, \tag{3}$$

where $\delta = \sqrt{\big((\alpha+2)a + 2(\beta-1)b - 2\big)^2 - 8\alpha a\big(a-b-1\big)}$. It is not difficult to see that the same expression for δ is obtained if we assume $\mu_1 = b$. Using this fact, it follows that $\alpha K_a \cup \beta K_{b,b}$ is Seidel integral if and only if $(\alpha, \beta, a, b, \delta)$ represents a positive integral solution of the Diophantine equation

$$\big((\alpha+2)a + 2(\beta-1)b - 2\big)^2 - 8\alpha a\big(a-b-1\big) = \delta^2. \tag{4}$$

Therefore, the characterization of Seidel integral graphs which are related to the class $\alpha K_a \cup \beta K_{b,b}$ is reduced to the problem of finding the most general positive solution of the equation (4).

Next, $\mu_1^*\mu_2^* = 4\mu_1\mu_2 - 2(n_1-1)\mu_2 - 2(n_2-1)\mu_1 - (n-1)$ for any G with two main eigenvalues (see [**3**]). In the case that $G = \alpha K_a \cup \beta K_{b,b}$ this relation is transformed into

$$\big(\mu_1^* - 1\big)\big(\mu_2^* - 1\big) + 2(\alpha-2)a = 2ab(2-\alpha-2\beta). \tag{5}$$

In what follows the symbol (m, n) denotes the greatest common divisor of integers m, n while $m \mid n$ means that m divides n. Using this notation, we proceed to establish a characterization of Seidel integral graphs for the class $\alpha K_a \cup \beta K_{b,b}$ and give the list of all such graphs up to 25 vertices. The proof is based on the following statement [**2**].

Theorem 2. *The linear Diophantine equation $ax+by=c$ has at least one solution if and only if $d \mid c$ where $d = (a,b)$. In that case the most general solution of this equation is given in the form*

$$x = \frac{c}{d}\,x_0 - \frac{b}{d}\,z \quad \text{and} \quad y = \frac{c}{d}\,y_0 + \frac{a}{d}\,z \qquad (z \in \mathbb{Z}),$$

where (x_0, y_0) represents a particular solution[1] of the equation $ax+by=d$.

Theorem 3. *If $\alpha K_a \cup \beta K_{b,b}$ is Seidel integral then it belongs to one of the following classes of Seidel integral graphs*

$$\left[\pm\frac{(2t-1)k}{\tau}\,x_0 + \frac{2m(2t-1)}{\tau}\,z\right]K_a \cup \left[\pm\frac{(2t-1)k}{\tau}\,y_0 + \frac{a}{\tau}\,z\right](2n-1)K_{b,b}, \tag{6}$$

[1] A particular solution of the equation $ax+by=d$ may be obtained by using the EUCLID algorithm. In that case the coefficients a and b uniquely determine x_0 and y_0.

where (i) $a = \pm\big(t + 2\ell n - (\ell + n)\big)k + (2\ell - 1)m + 1$ *and* $b = (2\ell - 1)m$*; (ii)* $t, k, \ell, m, n \in \mathbb{N}$ *such that* $(m, 2n-1) = 1$, $(2n-1, 2t-1) = 1$ *and* $(2\ell-1, 2t-1) = 1$*; (iii)* $\tau = (a, 2m(2t-1))$ *such that* $\tau \mid (2t-1)k$*; (iv)* (x_0, y_0) *is a particular solution of the linear Diophantine equation* $ax - 2m(2t-1)y = \tau$ *and (v)* $z \geq z_0$ *where* z_0 *is the least integer such that* $\big(\pm\frac{(2t-1)k}{\tau}\, x_0 + \frac{2m(2t-1)}{\tau}\, z_0\big) \geq 1$ *and* $\big(\pm\frac{(2t-1)k}{\tau}\, y_0 + \frac{a}{\tau}\, z_0\big) \geq 1$*;*

$$\text{(7)} \qquad \Big[\pm\frac{2kt}{\tau}\, x_0 + \frac{2mt}{\tau}\, z\Big] K_a \cup \Big[\pm\frac{2kt}{\tau}\, y_0 + \frac{a}{\tau}\, z\Big](2n-1)K_{b,b}\,,$$

where (i) $a = \pm\big(t + \ell(2n-1)\big)k + \ell m + 1$ *and* $b = \ell m$*; (ii)* $t, k, \ell, m, n \in \mathbb{N}$ *such that* $(m, 2n-1) = 1$, $(2n-1, t) = 1$, $(\ell, t) = 1$ *and* $(t + \ell(2n-1), 2) = 1$*; (iii)* $\tau = (a, 2mt)$ *such that* $\tau \mid 2kt$*; (iv)* (x_0, y_0) *is a particular solution of the linear Diophantine equation* $ax - (2mt)y = \tau$ *and (v)* $z \geq z_0$ *where* z_0 *is the least integer such that* $\big(\pm\frac{2kt}{\tau}\, x_0 + \frac{2mt}{\tau}\, z_0\big) \geq 1$ *and* $\big(\pm\frac{2kt}{\tau}\, y_0 + \frac{a}{\tau}\, z_0\big) \geq 1$*;*

$$\text{(8)} \quad \Big[\pm\frac{2(2t-1)k}{\tau}\, x_0 + \frac{(2m-1)(2t-1)}{\tau}\, z\Big] K_a \cup \Big[\pm\frac{2(2t-1)k}{\tau}\, y_0 + \frac{a}{\tau}\, z\Big] nK_{b,b}\,,$$

where (i) $a = \pm\big(2t-1+2\ell n\big)k + (2m-1)\ell + 1$ *and* $b = (2m-1)\ell$*; (ii)* $t, k, \ell, m, n \in \mathbb{N}$ *such that* $(2m-1, 2n) = 1$, $(2n, 2t-1) = 1$ *and* $(\ell, 2t-1) = 1$*; (iii)* $\tau = (a, (2m-1)(2t-1))$ *such that* $\tau \mid 2(2t-1)k$*; (iv)* (x_0, y_0) *is a particular solution of the linear Diophantine equation* $ax - (2m-1)(2t-1)y = \tau$ *and (v)* $z \geq z_0$ *where* z_0 *is the least integer such that* $\big(\pm\frac{2(2t-1)k}{\tau}\, x_0 + \frac{(2m-1)(2t-1)}{\tau}\, z_0\big) \geq 1$ *and* $\big(\pm\frac{2(2t-1)k}{\tau}\, y_0 + \frac{a}{\tau}\, z_0\big) \geq 1$. *In these classes the symbol '*$\pm$*' is related to '*$+$*' if* $a > (b+1)$ *and '*$\pm$*' is related to '*$-$*' if* $a \leq b$.

Proof. Let us assume that $\mu_1^* \in \mathbb{N}$ and let $\theta = \frac{\rho}{\varphi}$ so that $\mu_1^* - 1 = \theta a$ and $(\rho, \varphi) = 1$. Using (3) and (5) we obtain

$$\mu_2^* = -\frac{2\big(\alpha-2\big)(b+1) + 4\beta b}{\theta} + 1 \quad \text{and} \quad \delta = \theta a + \frac{2\big(\alpha-2\big)(b+1) + 4\beta b}{\theta}\,.$$

Then by a straightforward calculation it is not difficult to see that (4) may be transformed in the form $\frac{\theta+2}{\theta} = \frac{\alpha(a-b-1)}{\theta a - ((\alpha-2)(b+1)+2\beta b)}$. Let c be a constant such that $(\overline{1})$ $\alpha(a-b-1) = c(\theta+2)$ and $(\overline{2})$ $\theta a - ((\alpha-2)(b+1) + 2\beta b) = c\theta$. Combining $(\overline{1})$ and $(\overline{2})$ we find that $2c = (\alpha-\theta)a + 2(\beta-1)b - 2$. Observe that $2c$ is an integer because $\theta a = (\mu_1^* - 1) \in \mathbb{N}$. Consequently, using $(\overline{1})$ or $(\overline{2})$ we arrive at $2\alpha(a-b-1) = \big((\alpha-\theta)a + 2(\beta-1)b - 2\big)(\theta+2)$. Hence,

$$\text{(9)} \qquad (a-b-1) = r\big((\alpha-\theta)a + 2(\beta-1)b - 2\big) \quad \text{and} \quad (\theta+2) = 2r\alpha\,,$$

where $r = \frac{s}{t}$ such that $(s,t) = 1$. Making use of (9), by an easy calculation we obtain $(\overline{3})$ $2r\beta b = (2r-1)\big(r\alpha a - (a-b-1)\big)$.

Using now the right-hand side of relation (9), note that $2r\alpha a = \mu_1^* + 2a - 1$ which shows that $(2r\alpha a)$ is integral and $2r - 1 = \frac{2s-t}{t} > 0$. Since $2\beta b = (2 - \frac{1}{r})\big(r\alpha a - (a-b-1)\big)$ (see $(\overline{3})$) it turns out that $r \mid |a-b-1|$. Consider first the case when $a > (b+1)$. With this condition we obtain $(\overline{4})$ $(a-b-1) = \gamma r$ where $(\overline{5})$ $\gamma = kt$. Then $(\overline{3})$ is reduced to the form $(\overline{6})$ $\beta = \frac{(2s-t)}{b}\frac{(\alpha a-kt)}{2t}$. We shall consider the following two cases:

Case 1. (t is odd). Let $t \to 2t-1$ where $p \to q$ means that 'p is replaced with q'. Let $(2s-(2t-1), b) = 2\ell - 1$ and let $m, n \in \mathbb{N}$ such that (1.1) $(2s-(2t-1)) = (2\ell-1)(2n-1)$ and (1.2) $b = (2\ell-1)m$, where $(m, 2n-1) = 1$. Since $(2s-(2t-1), 2t-1) = 1$ according to (1.1) we obtain $(2n-1, 2t-1) = 1$ and $(2\ell-1, 2t-1) = 1$. Therefore, using $(\overline{6})$ we have $\beta = \frac{(\alpha a-(2t-1)k)(2n-1)}{2m(2t-1)}$. Since $(2m(2t-1), 2n-1) = 1$ it follows that $2m(2t-1) \mid (\alpha a-(2t-1)k)$. Consequently, setting (1.3) $\alpha a-(2t-1)k = \eta(2m(2t-1))$ we get (1.4) $\beta = \eta(2n-1)$. We can see that (1.3) represents a linear Diophantine equation in variables α and η. Of course, if $(a, 2m(2t-1)) = \tau$ then (1.3) has at least one solution if and only if $\tau \mid (2t-1)k$. In this case, according to Proposition 2, we obtain that

$$\alpha = \frac{(2t-1)k}{\tau}\,x_0 + \frac{2m(2t-1)}{\tau}\,z \quad \text{and} \quad \eta = \frac{(2t-1)k}{\tau}\,y_0 + \frac{a}{\tau}\,z\,,$$

where $ax_0 - 2m(2t-1)y_0 = \tau$. Finally, making use of $(\overline{4})$, $(\overline{5})$, and according to (1.1), (1.2), (1.4) and the last relation, we get easily that $a = (t+2\ell n-(\ell+n))k + (2\ell-1)m+1$ and $\beta = \left[\frac{(2t-1)k}{\tau}\,y_0 + \frac{a}{\tau}\,z\right](2n-1)$, which provides the corresponding class of integral graphs represented in (6).

Case 2. (t is even). Since $(s,t) = 1$ it follows that s is an odd number. Let $s \to 2s-1$ and $t \to 2t$. Let $((2s-1)-t, b) = \ell$ and let $m, n \in \mathbb{N}$ such that (2.1) $(2s-1)-t = \ell n$ and (2.2) $b = \ell m$, where $(m,n) = 1$. Using (2.1) we obtain $2s-1 = t+\ell n$. Since $2s-1$ is an odd number it must be $(t+\ell n, 2) = 1$. Next, since $((2s-1)-t, t) = 1$ according to (2.1) we obtain $(n,t) = 1$ and $(\ell, t) = 1$. Therefore, using $(\overline{6})$ we have $\beta = \frac{(\alpha a-2kt)n}{2mt}$. Consider the case when n is an odd number. Setting $n \to 2n-1$ we have $(2mt, 2n-1) = 1$, which provides that $2mt \mid (\alpha a - 2kt)$. Consequently, setting (2.3) $\alpha a - 2kt = \eta(2mt)$ we get (2.4) $\beta = \eta(2n-1)$. We can see that (2.3) represents a linear Diophantine equation in variables α and η. Of course, if $(a, 2mt) = \tau$ then (2.3) has at least one solution if and only if $\tau \mid 2kt$. In this case, according to Proposition 2, we obtain that

$$\alpha = \frac{2kt}{\tau}\,x_0 + \frac{2mt}{\tau}\,z \quad \text{and} \quad \eta = \frac{2kt}{\tau}\,y_0 + \frac{a}{\tau}\,z\,,$$

where $ax_0 - (2mt)y_0 = \tau$. Finally, making use of $(\overline{4})$, $(\overline{5})$, and according to (2.1), (2.2), (2.4) and the last relation, we get easily that $a = (t+\ell(2n-1))k + \ell m + 1$ and $\beta = \left[\frac{2kt}{\tau}\,y_0 + \frac{a}{\tau}\,z\right](2n-1)$, which provides the corresponding class of integral graphs represented in (7). Consider the case when n is an even number. Setting $n \to 2n$ note that m and t must be two odd numbers because $(m, 2n) = 1$ and $2s-1 = t+\ell(2n)$ (see (2.1)). Let $m \to 2m-1$ and $t \to 2t-1$. In view of this, we obtain $\beta = \frac{(\alpha a-2(2t-1)k)n}{(2m-1)(2t-1)}$. Since $((2m-1)(2t-1), n) = 1$ it follows that $(2m-1)(2t-1) \mid (\alpha a - 2(2t-1)k)$. Consequently, setting (2.5) $\alpha a - 2(2t-1)k =$

$\eta(2m-1)(2t-1)$ we get (2.6) $\beta = \eta n$. We can see that (2.5) represents a linear Diophantine equation in variables α and η. Of course, if $(a,(2m-1)(2t-1)) = \tau$ then (2.5) has at least one solution if and only if $\tau \mid 2(2t-1)k$. In this case, according to Proposition 2, we obtain that

$$\alpha = \frac{2(2t-1)k}{\tau}\, x_0 + \frac{(2m-1)(2t-1)}{\tau}\, z \quad \text{and} \quad \eta = \frac{2(2t-1)k}{\tau}\, y_0 + \frac{a}{\tau}\, z\,,$$

where $ax_0 - (2m-1)(2t-1)y_0 = \tau$. Finally, making use of $(\overline{4})$, $(\overline{5})$, and according to (2.1), (2.2), (2.6) and the last relation, we get easily that $a = (2t-1+2\ell n)k + (2m-1)\ell + 1$ and $\beta = \left[\frac{2(2t-1)k}{\tau}\, y_0 + \frac{a}{\tau}\, z\right] n$, which provides the corresponding class of integral graphs represented in (8).

Consider the case when $\mu_1 = b$. Then we can see that $(\overline{4})$ and $(\overline{6})$ are reduced to $(b+1) - a = \gamma r$ and $\beta = \frac{(2s-t)}{b}\frac{(\alpha a + kt)}{2t}$, respectively. We also note that if (x_0, y_0) is a particular solution of the equation $ax - by = c$ then $(-x_0, -y_0)$ represents a particular solution of the equation $by - ax = c$. Keeping in mind this fact and by using the same procedure applied in Cases 1 and 2, we easily obtain the corresponding classes of integral graphs represented in (6), (7) and (8), respectively. □

Proposition 4. *If $\alpha K_a \cup \beta K_{b,b}$ is a Seidel integral graph then it uniquely determines the parameters τ, t, k, ℓ, m, n.*

Proof. Assume that $\tau_1, t_1, k_1, \ell_1, m_1, n_1$ and $\tau_2, t_2, k_2, \ell_2, m_2, n_2$ determine the same Seidel integral graph $\alpha K_a \cup \beta K_{b,b}$. Since the parameters α, β, a, b determine the graph $\alpha K_a \cup \beta K_{b,b}$ up to isomorphism, using the second equality of (9) we have $\mu_1^* + 2a - 1 = 2r\alpha a$, which shows that $s_1 = t_1$ and $s_2 = t_2$ because $(s,t) = 1$. According to this, we note that the classes represented by relations (6), (7) and (8) are mutually disjoint. Consequently, without loss of generality we can assume that the corresponding Seidel integral graph determined by the parameters $\tau_1, t_1, k_1, \ell_1, m_1, n_1$ and $\tau_2, t_2, k_2, \ell_2, m_2, n_2$ belongs to the class of integral graphs displayed in relation (6). Next, using $(\overline{4})$ and $(\overline{5})$ we get $k_1 = k_2$. Since $(2s - (2t-1), b) = 2\ell - 1$ (see Case 1), we also have $\ell_1 = \ell_2$. Using (1.1) and (1.2) we obtain $n_1 = n_2$ and $m_1 = m_2$. Finally, since $(a, 2m(2t-1)) = \tau$ it follows that $\tau_1 = \tau_2$. □

Remark 5. *If (x_0, y_0) is obtained using the Euclid algorithm then a fixed Seidel integral graph $\alpha K_a \cup \beta K_{b,b}$ also uniquely determines the parameters x_0, y_0, z_0, z.*

Proposition 6. *If $\alpha K_a \cup \beta K_{b,b}$ is Seidel integral with $\mu_1^* = 1$ then it belongs to the class of Seidel integral graphs $K_a \cup K_{1,1}$ for any $a \in \mathbb{N}$.*

Proof. Let us assume that $\alpha K_a \cup \beta K_{b,b}$ is Seidel integral with $\mu_1^* = 1$. Using (5) we obtain $b(2 - \alpha - 2\beta) = \alpha - 2$, which provides that $b = 1, \alpha = 1$ and $\beta = 1$. Using (3) we find that $\delta = a + 2$. Since $\delta = \mu_1^* - \mu_2^*$ we obtain $\mu_2^* = -(a+1)$. □

Theorem 7. *If $\alpha K_a \cup \beta K_{b,b}$ is Seidel integral with $\alpha = 1$, $\beta = 1$ and $b \geq 2$ then it belongs to one of the following classes of Seidel integral graphs:*

(1^0) *where* (*i*) $a = (2t-1)n$, $b = sm+1$ *and* $a \neq b+1$; (*ii*) $m = 2x_0+(2s+(2t-1))z$ *and* $n = 2y_0 + (2t-1)z$; (*iii*) $s, t \in \mathbb{N}$ *such that* $(2s, 2t-1) = 1$; (*iv*) (x_0, y_0) *is a particular solution of the linear Diophantine equation* $(2t-1)x-(2s+(2t-1))y = 1$ *and* (*v*) $z \geq z_0$ *where* z_0 *is the least integer such that* $\big(2x_0 + (2s + (2t-1))z_0\big) \geq 1$ *and* $\big(2y_0 + (2t-1)z_0\big) \geq 1$;

(2^0) *where* (*i*) $a = tn$, $b = (2s-1)m+1$ *and* $a \neq b+1$; (*ii*) $m = \frac{2}{\tau}x_0 + \frac{(2s-1)+t}{\tau}z$ *and* $n = \frac{2}{\tau}y_0 + \frac{2t}{\tau}z$; (*iii*) $s, t \in \mathbb{N}$ *such that* $(2s-1, 2t) = 1$; (*iv*) (x_0, y_0) *is a particular solution of the linear Diophantine equation* $(2t)x - ((2s-1)+t)y = \tau$, *where* $\tau = ((2s-1)+t, 2t)$ *and* (*v*) $z \geq z_0$ *where* z_0 *is the least integer such that* $\big(\frac{2}{\tau}x_0 + \frac{(2s-1)+t}{\tau}z_0\big) \geq 1$ *and* $\big(\frac{2}{\tau}y_0 + \frac{2t}{\tau}z_0\big) \geq 1$.

Proof. Let us assume that $\alpha K_a \cup \beta K_{b,b}$ is Seidel integral with $\alpha = 1$, $\beta = 1$ and $b \geq 2$. Using (5) we find that $(\mu_1^* - 1)(\mu_2^* - 1) = 2a(1-b)$. Let $\mu_1^* - 1 = (2a)r$ where $(s, t) = 1$. Then

$$\mu_2^* = -\frac{t}{s}\big(b-1\big) + 1 \quad \text{and} \quad \delta = \frac{(2a)s}{t} + \frac{t}{s}\big(b-1\big)\,, \tag{10}$$

from which we obtain the following two cases:

Case 1. (t is odd). Let $t \to 2t-1$ where $t \in \mathbb{N}$. Since $\mu_1^* - 1 = \frac{(2s)a}{2t-1}$, $\mu_2^* - 1 = -\frac{(2t-1)(b-1)}{s}$ and $(2s, 2t-1) = 1$ it follows that $(2t-1) \mid a$ and $s \mid (b-1)$. Setting (1.1) $a = (2t-1)n$ and (1.2) $b-1 = sm$ we obtain from (10) that $\mu_1^* = 2sn+1$, $\mu_2^* = -(2t-1)m+1$ and $\delta = 2sn+(2t-1)m$. Using (3) we arrive at (1.3) $(2t-1)m-(2s+(2t-1))n = 2$. We note that (1.3) represents a linear Diophantine equation in variables m and n. Of course, since $(2t-1, 2s+(2t-1)) = 1$ this equation has at least one solution. Consequently, according to Theorem 2 we obtain that $m = 2x_0+(2s+(2t-1))z$ and $n = 2y_0+(2t-1)z$, where $(2t-1)x_0-(2s+(2t-1))y_0 = 1$. So we obtain the corresponding class of Seidel integral graphs displayed in (1^0).

Case 2. (t is even). Since $(s, t) = 1$ it follows that s is an odd number. Let $s \to 2s-1$ and $t \to 2t$ where $s, t \in \mathbb{N}$. Since $\mu_1^* - 1 = \frac{(2s-1)a}{t}$, $\mu_2^* - 1 = -\frac{(2t)(b-1)}{2s-1}$ and $(2s-1, 2t) = 1$ it follows that $t \mid a$ and $(2s-1) \mid (b-1)$. Setting (2.1) $a = tn$ and (2.2) $b-1 = (2s-1)m$ we obtain from (10) that $\mu_1^* = (2s-1)n+1$, $\mu_2^* = -2tm+1$ and $\delta = (2s-1)n+2tm$. Using (3) we arrive at (2.3) $(2t)m - ((2s-1)+t)n = 2$. We note that (2.3) represents a linear Diophantine equation in variables m and n. Of course, since $(2t, (2s-1)+t) = \tau$ where $\tau = 1$ or $\tau = 2$, this equation has at least one solution. Consequently, according to Theorem 2 we obtain that $m = \frac{2}{\tau}x_0 + \frac{(2s-1)+t}{\tau}z$ and $n = \frac{2}{\tau}y_0 + \frac{2t}{\tau}z$, where $(2t)x_0 - ((2s-1)+t)y_0 = \tau$. So we obtain the corresponding class of Seidel integral graphs displayed in (2^0). □

Proposition 8. *If* $\alpha K_a \cup \beta K_{b,b}$ *is a Seidel integral graph with* $\alpha = 1$, $\beta = 1$ *and* $b \geq 2$ *then it uniquely determines the parameters* m, n, s, t.

Proof. Let us assume that m_1, n_1, s_1, t_1 and m_2, n_2, s_2, t_2 determine the same Seidel integral graph $\alpha K_a \cup \beta K_{b,b}$ with $\alpha = 1$, $\beta = 1$ and $b \geq 2$. Since the parameters

α, β, a, b determine the graph $\alpha K_a \cup \beta K_{b,b}$ up to isomorphism, using the first equality of (10) we have $(\mu_2^* - 1)r = -(b-1)$, which shows that $s_1 = s_2$ and $t_1 = t_2$ because $(s,t) = 1$. In view of this, we note that the classes represented by Theorem 7 (1^0) and (2^0) are mutually disjoint. Consequently, without loss of generality, we can assume that the corresponding Seidel integral graph determined by the parameters m_1, n_1, s_1, t_1 and m_2, n_2, s_2, t_2 belong to the class Theorem 7 (1^0). Hence, using (1.1) and (1.2) we have $(2t_1 - 1)n_1 = (2t_2 - 1)n_2$ and $s_1 m_1 = s_2 m_2$, which provides that $m_1 = m_2$ and $n_1 = n_2$. □

In order to demonstrate a procedure for obtaining the Seidel integral graphs which belong to the class $\alpha K_a \cup \beta K_{b,b}$ for a fixed Seidel main eigenvalue μ_1^*, we prove the following two results:

Proposition 9. *If $\alpha K_a \cup \beta K_{b,b}$ is Seidel integral with $\mu_1^* = 2$ then it is $K_1 \cup K_{3,3}$ for any $\alpha, \beta, a, b \in \mathbb{N}$.*

Proof. First, according to the proof of Theorem 3, we have $\mu_1^* - 1 = \theta a$. Using that $2s - t > 0$ and using the right-hand side of relation (9), we obtain

$$\frac{2a+1}{\alpha a} = \frac{2s}{t} > 1\,,$$

which provides that $\alpha = 1$ or $\alpha = 2$.

Case 1. ($\alpha = 1$). Since $\frac{2a+1}{2a} = \frac{s}{t}$ and $(2a+1, 2a) = 1$, $(s,t) = 1$, we obtain $s = 2a+1$ and $t = 2a$. Consider the case when $a > (b+1)$. Using ($\overline{6}$) we find that $a(1-2k) < 0$, a contradiction.

Consider the case when $a \le b$. Using (5) we find that $\mu_2^* - 1 = 2a(b+1-2\beta b)$. Since $\mu_1^* + \mu_2^* = \alpha a + 2\beta b - 2(a+b)$ (see (3)), we easily obtain $2\beta b(2a+1) - 2b(a+1) = 3(a+1)$, which provides that $\beta = 1$. So we arrive at $a(2b-3) = 3$ from which we obtain that $a = 1$ and $b = 3$, which provides the Seidel integral graph given in Proposition 9.

Case 2. ($\alpha = 2$). Since $\frac{2a+1}{4a} = \frac{s}{t}$ and $(2a+1, 4a) = 1$, $(s,t) = 1$, we obtain $s = 2a+1$ and $t = 4a$. Consider the case when $a > (b+1)$. Using ($\overline{6}$) we find that $2a(1-2k) < 0$, a contradiction.

Consider the case when $a \le b$. Using (5) we find that $\mu_2^* - 1 = -4ab\beta$. Using (3) we easily obtain $2b(2a\beta + \beta - 1) = 3$, a contradiction. □

Proposition 10. *If $\alpha K_a \cup \beta K_{b,b}$ is Seidel integral with $\mu_1^* = 3$ then it is: (1^0) $K_1 \cup K_{6,6}$ or (2^0) $K_2 \cup K_{4,4}$ or (3^0) $2K_1 \cup K_{2,2}$ for any $\alpha, \beta, a, b \in \mathbb{N}$.*

Proof. Using that $\mu_1^* - 1 = \theta a$ and using the right-hand side of relation (9), we find that $\frac{2a+2}{\alpha a} = \frac{2s}{t}$, which provides that $\alpha = 1$ or $\alpha = 2$ or $\alpha = 3$.

Case 1. ($\alpha = 1$). Since $\frac{a+1}{a} = \frac{s}{t}$ and $(a+1, a) = 1$, $(s,t) = 1$, we obtain $s = a+1$ and $t = a$. Consider the case when $a > (b+1)$. Using ($\overline{6}$) we find that $a(1-k) \le 0$, a contradiction.

Consider the case when $a \le b$. Using (5) we find that $\mu_2^* - 1 = a(b+1-2\beta b)$. Using (3) we easily obtain $2a+4 = 2\beta b(a+1) - b(a+2)$. In view of this, it follows that $\beta = 1$ or $\beta = 2$. Indeed, if we assume that $\beta \ge 3$ then

$$2a + 4 \ge 6b(a+1) - b(a+2) = 5ab + 4b\,,$$

a contradiction. Consider the case when when $\beta = 2$. Then $2a+4 = 3ab+2b$ which provides that $b = 1$. Then $2a+4 = 3a+2$ which yields $a = 2$, a contradiction because $a \le b$. Consider the case when $\beta = 1$. Then we get $a(b-2) = 4$ from which we obtain integral solutions $a = 1$, $b = 6$ or $a = 2$, $b = 4$, which provides the Seidel integral graphs given in Proposition 10 (1^0) and (2^0), respectively.

Case 2. ($\alpha = 2$). Consider the case when $a > (b+1)$. We note that $(a+1, 2a) = 1$ or $(a+1, 2a) = 2$. Consider the case when $(a+1, 2a) = 1$. Since $\frac{a+1}{2a} = \frac{s}{t}$ and $(s,t) = 1$, we obtain $s = a+1$ and $t = 2a$. Using ($\overline{6}$) we find that $2a(1-k) \le 0$, a contradiction. Consider the case when $(a+1, 2a) = 2$. In this situation a is an odd number. Let $a = 2\varepsilon + 1$ where $\varepsilon \in \mathbb{N}$. Since $\frac{\varepsilon+1}{2\varepsilon+1} = \frac{s}{t}$ and $(2\varepsilon+1, \varepsilon+1) = 1$, $(s,t) = 1$, we obtain $s = \varepsilon + 1$ and $t = 2\varepsilon + 1$. Then $(\alpha a - kt) = (2-k)(2\varepsilon+1)$, which provides that $k = 1$. Using ($\overline{6}$) we get

$$\beta = \frac{2(\varepsilon+1) - (2\varepsilon+1)}{2b} \cdot \frac{(2-1)(2\varepsilon+1)}{2\varepsilon+1}\,,$$

form which we obtain $2\beta b = 1$, a contradiction.

Consider the case when $a \le b$. Using (5) we find that $\mu_2^* - 1 = -2ab\beta$. Using (3) we easily obtain $(a\beta + \beta - 1)b = 2$, which provides that $\beta = 1$, $a = 1$ and $b = 2$. In view of this, we obtain the Seidel integral graph given in Proposition 10 (3^0).

Case 3. ($\alpha = 3$). In this situation we have $a = 1$ and $a \le b$. Using (5) we find that $\mu_2^* - 1 = -(2\beta b + b + 1)$. Using (3) we easily obtain $2 = (4\beta - 1)b$, a contradiction. ☐

Theorem 11. *If $\alpha K_a \cup \beta K_{b,b}$ is Seidel integral with $\mu_1^* = 2a+1$ then it belongs to one of the following classes of Seidel integral graphs:* (1^0) $K_a \cup (\beta+1)K_{b,b}$ *where* $a = (4\beta+1)m - 1$ *and* $b = 3m$ *or* (2^0) $K_a \cup 3\beta K_{b,b}$ *where* $a = (4\beta-1)m-1$ *and* $b = m$ *or* (3^0) $K_a \cup K_{b,b}$ *where* $a = m$ *and* $b = 3(m+1)$ *or* (4^0) $2K_a \cup (\beta+1)K_{b,b}$ *where* $a = (2\beta+1)m - 1$ *and* $b = m$ *or* (5^0) $2K_a \cup K_{b,b}$ *where* $a = m$ *and* $b = m+1$ *or* (6^0) $3K_a \cup \beta K_{b,b}$ *where* $a = (4\beta-1)m - 1$ *and* $b = m$ *for any* $\beta, m \in \mathbb{N}$.

Proof. Let us assume that $\alpha K_a \cup \beta K_{b,b}$ is Seidel integral with $\mu_1^* = 2a+1$. Using that $\mu_1^* - 1 = \theta a$ we obtain $\theta = 2$. Using the right-hand side of relation (9), we find that $2r\alpha = 4$. Since $2r > 1$ it follows that $\alpha = 1$ or $\alpha = 2$ or $\alpha = 3$.

Case 1. ($\alpha = 1$). In this situation $s = 2$ and $t = 1$. Consider the case when $a > (b+1)$. Using ($\overline{4}$) and ($\overline{5}$) we find that $a = 2k + b + 1$. Using ($\overline{6}$) we obtain $\beta = \frac{3(k+b+1)}{2b}$. Consider the case when $3 \mid b$. Setting $b = 3m$ we obtain $\beta = 1 + \frac{k+m+1}{2m}$, which provides that $2m \mid (k+m+1)$. Setting[2] $k+m+1 = 2m\ell$

[2]Setting $k+m+1 = 2m\ell$ note that $\alpha K_a \cup \beta K_{b,b}$ is reduced to the regular graph $K_4 \cup 2K_{3,3}$ for $\ell = 1$ and $m = 1$. Of course, since $a \ne b+1$ this case is excluded.

we obtain that $k=(2\ell-1)m-1$ and $\beta=\ell+1$. Replacing ℓ with β we obtain the corresponding class of Seidel integral graphs displayed in (1^0). Consider the case when $3\nmid b$. In this situation $2b\mid(k+b+1)$. Setting $k+b+1=2b\ell$ we obtain that $k=(2\ell-1)b-1$ and $\beta=3\ell$. Replacing ℓ with β and replacing b with m we obtain the corresponding class of Seidel integral graphs displayed in (2^0).

Consider the case when $a\le b$. Using (5) we find that $\mu_2^*-1=b(1-2\beta)+1$. Using (3) we easily obtain $3(a+1)=(4\beta-3)b$. Consider the case when $3\mid b$. Setting $b=3(m+1)$ we obtain $a=(4\beta-3)(m+1)-1$. Since $a\le b$ it follows that $\beta=1$. So we obtain the corresponding class of Seidel integral graphs displayed in (3^0). Consider the case when $3\nmid b$. Of course, in this case $3\mid(4\beta-3)$ which provides that $3\mid\beta$. Setting $\beta=3m$ we obtain $a+1=(4m-1)b$, a contradiction because $a\le b$.

Case 2. $(\alpha=2)$. In this situation $s=1$ and $t=1$. Consider the case when $a>(b+1)$. Using $(\overline{4})$ and $(\overline{5})$ we find that $a=k+b+1$. Using $(\overline{6})$ we obtain $\beta=\frac{2b+k+2}{2b}$. Then $2b\mid(k+2)$ which provides that $k+2=2b\ell$. In view of this, we find that $k=2b\ell-2$ and $\beta=\ell+1$. Replacing ℓ with β and replacing b with m we obtain the corresponding class of Seidel integral graphs displayed in (4^0).

Consider the case when $a\le b$. Using (5) we find that $\mu_2^*-1=-2\beta b$. Using (3) we easily obtain $a+1=(2\beta-1)b$. Since $a\le b$ it follows that $\beta=1$. Replacing b with $m+1$ we obtain the corresponding class of Seidel integral graphs displayed in (5^0).

Case 3. $(\alpha=3)$. In this situation $s=2$ and $t=3$. Consider the case when $a>(b+1)$. Using $(\overline{4})$ and $(\overline{5})$ we find that $a=2k+b+1$. Using $(\overline{6})$ we obtain $\beta=\frac{k+b+1}{2b}$. Setting $k+b+1=2b\ell$ we obtain that $k=(2\ell-1)b-1$ and $\beta=\ell$. Replacing ℓ with β and replacing b with m we obtain the corresponding class of Seidel integral graphs displayed in (6^0).

Finally, consider the case when $a\le b$. Using (5) we find that $\mu_2^*-1=-b(2\beta+1)-1$. Using (3) we easily obtain $a+1=(4\beta-1)b$, a contradiction. This completes the proof. □

Theorem 12. *If $\alpha K_a\cup\beta K_{b,b}$ is Seidel integral with $\mu_1^*=a+1$ then it belongs to one of the following classes of Seidel integral graphs: (1^0) $K_a\cup(\beta+1)K_{b,b}$ where $a=(3\beta+1)m-2$ and $b=m$ or (2^0) $K_a\cup K_{b,b}$ where $a=m$ and $b=m+2$ or (3^0) $2K_a\cup\beta K_{b,b}$ where $a=2(3\beta-1)m-2$ and $b=m$ for any $\beta,m\in\mathbb{N}$.*

Proof. Let us assume that $\alpha K_a\cup\beta K_{b,b}$ is Seidel integral with $\mu_1^*=a+1$. Using that $\mu_1^*-1=\theta a$ we obtain $\theta=1$. Using the right-hand side of relation (9), we find that $2r\alpha=3$. Since $2r>1$ it follows that $\alpha=1$ or $\alpha=2$.

Case 1. $(\alpha=1)$. In this situation $s=3$ and $t=2$. Consider the case when $a>(b+1)$. Using $(\overline{4})$ and $(\overline{5})$ we find that $a=3k+b+1$. Using $(\overline{6})$ we obtain $\beta=\frac{b+k+1}{b}$. Setting $k+1=\ell b$ we obtain that $k=\ell b-1$ and $\beta=\ell+1$. Replacing ℓ with β and replacing b with m we obtain the corresponding class of Seidel integral graphs displayed in (1^0).

Consider the case when $a\le b$. Using (5) we find that $\mu_2^*-1=2(b+1-2\beta)$.

Using (3) we easily obtain $a+2=(3\beta-2)b$, which provides that $\beta=1$. Replacing b with $m+2$ we obtain the corresponding class of Seidel integral graphs displayed in (2^0).

Case 2. ($\alpha=2$). In this situation $s=3$ and $t=4$. Consider the case when $a>(b+1)$. Using ($\overline{4}$) and ($\overline{5}$) we find that $a=3k+b+1$. Using ($\overline{6}$) we obtain $\beta=\frac{k+b+1}{2b}$. Setting $k+b+1=2b\ell$ we obtain that $k=(2\ell-1)b-1$ and $\beta=\ell$. Replacing ℓ with β and replacing b with m we obtain the corresponding class of Seidel integral graphs displayed in (3^0).

Consider the case when $a\le b$. Using (5) we find that $\mu_2^*-1=-4\beta b$. Using (3) we easily obtain $a+2=2(3\beta-1)b$, a contradiction. □

Theorem 13. *If* $(\alpha,\beta,a,b,\delta)$ *is a positive integral solution of the Diophantine equation* (4) *then it could be represented by one of the following forms:*

- $a=\pm\big(t+2\ell n-(\ell+n)\big)k+(2\ell-1)m+1$ *and* $b=(2\ell-1)m$;
- $\alpha=\pm\dfrac{(2t-1)k}{\tau}\,x_0+\dfrac{2m(2t-1)}{\tau}\,z$;
- $\beta=\left[\pm\dfrac{(2t-1)k}{\tau}\,y_0+\dfrac{a}{\tau}\,z\right](2n-1)$;
- $\delta=\pm(2\ell-1)(2n-1)k+\left[\pm\dfrac{(2t-1)k}{\tau}\,y_0+\dfrac{a}{\tau}\,z\right]4\big(t+2\ell n-(\ell+n)\big)m$,

with the same conditions (ii)–(v) which are related to (6);

- $a=\pm\big(t+\ell(2n-1)\big)k+\ell m+1$ *and* $b=\ell m$;
- $\alpha=\pm\dfrac{2kt}{\tau}\,x_0+\dfrac{2mt}{\tau}\,z$;
- $\beta=\left[\pm\dfrac{2kt}{\tau}\,y_0+\dfrac{a}{\tau}\,z\right](2n-1)$;
- $\delta=\pm 2k\ell(2n-1)+\left[\pm\dfrac{2kt}{\tau}\,y_0+\dfrac{a}{\tau}\,z\right]2\big(t+\ell(2n-1)\big)m$,

with the same conditions (ii)–(v) which are related to (7);

- $a=\pm\big(2t-1+2\ell n\big)k+(2m-1)\ell+1$ *and* $b=(2m-1)\ell$;
- $\alpha=\pm\dfrac{2(2t-1)k}{\tau}\,x_0+\dfrac{(2m-1)(2t-1)}{\tau}\,z$;
- $\beta=\left[\pm\dfrac{2(2t-1)k}{\tau}\,y_0+\dfrac{a}{\tau}\,z\right]n$;
- $\delta=\pm 4k\ell n+\left[\pm\dfrac{2(2t-1)k}{\tau}\,y_0+\dfrac{a}{\tau}\,z\right]\big(2t-1+2\ell n\big)(2m-1)$,

with the same conditions (ii)–(v) which are related to (8).

Proof. According to Theorem 3 it suffices to derive the expression for δ. First, from (3) we have (i) $\mu_1^* - \mu_2^* = \delta$ and (ii) $\mu_1^* + \mu_2^* = \alpha a + 2\beta b - 2(a+b)$. Using (i), (ii) and the equality $\mu_1^* = 2r\alpha a - (2a-1)$ (see (9)), by a straightforward calculation we obtain that $\delta = 4r\alpha a - (\alpha a + 2\beta b) - 2(a-b-1)$.

CASE 1. (t is odd). Using $\big((\overline{4})$ and $(\overline{5})\big)$, (1.1), (1.2), (1.3) and (1.4), we obtain that $a - b - 1 = \pm\, ks$, $s = t + 2\ell n - (\ell + n)$, $b = (2\ell - 1)m$, $\alpha a = \pm\, k(2t-1) + 2\eta m(2t-1)$, $r\alpha a = \pm\, ks + 2\eta ms$ and $\beta = \eta(2n-1)$. So we find that $\delta = \pm\,(2\ell-1)(2n-1)k + 4\eta m\big(t + 2\ell n - (\ell+n)\big)$, which provides the statement related to (6).

CASE 2. (t is even). Using $\big((\overline{4})$ and $(\overline{5})\big)$, (2.1), (2.2), (2.3) and (2.4), we obtain that $a-b-1 = \pm\, k(2s-1)$, $2s-1 = t+\ell n$, $b = \ell m$, $\alpha a = \pm\, k(2t)+\eta m(2t)$ and $r\alpha a = \pm\, k(2s-1)+\eta m(2s-1)$. Consider the case when n is an odd number. In this case we have $\beta = \eta(2n-1)$. So we find that $\delta = \pm\, 2k\ell(2n-1) + 2\eta m\big(t+\ell(2n-1)\big)$, which provides the statement related to (7). Finally, consider the case when n is an even number. In this case we have $2s-1 = (2t-1) + 2\ell n$, $b = (2m-1)\ell$, $\alpha a = \pm\, 2k(2t-1) + \eta(2m-1)(2t-1)$, $2r\alpha a = \pm\, 2k(2s-1) + \eta(2m-1)(2s-1)$ and $\beta = \eta n$. So we find that $\delta = \pm\, 2k\ell(2n-1) + 2\eta m\big(t + \ell(2n-1)\big)$, which provides the statement related to (8). □

3. APPENDIX

In this section we present the data given in Tables 1–6, which represent the set of all Seidel integral graphs from the class $\alpha K_a \cup \beta K_{b,b}$ whose order does not exceed 25. In these tables a Seidel integral graph is described by the parameters α, β, a, b and ones presented in the classes of Seidel integral graphs in Theorem 3. In Tables 1–6 the symbol 'i' is the identification number of an integral graph.

The Tables 1, 2 and 3 contains the Seidel integral graphs from the class $\alpha K_a \cup \beta K_{b,b}$ with $a > (b+1)$ whose order does not exceed 25, which belong to the classes Theorem 3 (6), (7) and (8), respectively. Therefore, there exits exactly $33 + 8 + 7 = 48$ non-isomorphic Seidel integral graphs which belong to the class $\alpha K_a \cup \beta K_{b,b}$ with $a > (b+1)$ whose order does not exceed 25.

The[3] Tables 4, 5 and 6 contains the Seidel integral graphs from the class $\alpha K_a \cup \beta K_{b,b}$ with $a \le b$ whose order does not exceed 25, which belong to the classes Theorem 3 (6), (7) and (8), respectively. Therefore, there exits exactly $30 + 12 + 10 = 52$ non-isomorphic Seidel integral graphs with $a \le b$ whose order does not exceed 25. In view of this, there exist exactly $48+52 = 100$ non-isomorphic Seidel integral graphs which belong to the class $\alpha K_a \cup \beta K_{b,b}$ whose order does not exceed 25.

[3]In Tables 4, 5 and 6 a particular solution (x_0, y_0) is related to the corresponding Diophantine equation $-(ax - by) = d$.

Table 1. Integral graphs with $o \leq 25$ and $a > (b+1)$ belonging to the class (6)

i	x_0	y_0	z	o	α	β	a	b	τ	t	k	ℓ	m	n	μ_1^*	μ_2^*
1	1	1	0	5	1	1	3	1	1	1	1	1	1	1	1	−4
2	0	−1	1	6	1	1	4	1	2	1	2	1	1	1	1	−5
3	1	2	−1	7	1	1	5	1	1	1	3	1	1	1	1	−6
4	0	−1	1	8	1	1	6	1	2	1	4	1	1	1	1	−7
5	1	3	−2	9	1	1	7	1	1	1	5	1	1	1	1	−8
6	0	−1	1	10	1	1	8	1	2	1	6	1	1	1	1	−9
7	1	4	−3	11	1	1	9	1	1	1	7	1	1	1	1	−10
8	0	−1	1	12	1	1	10	1	2	1	8	1	1	1	1	−11
9	1	5	−4	13	1	1	11	1	1	1	9	1	1	1	1	−12
10	0	−1	1	14	1	1	12	1	2	1	10	1	1	1	1	−13
11	0	−1	2	14	2	1	6	1	6	2	2	1	1	1	5	−5
12	0	−1	2	14	2	3	4	1	2	1	2	1	1	1	9	−5
13	1	6	−5	15	1	1	13	1	1	1	11	1	1	1	1	−14
14	0	−1	1	16	1	1	14	1	2	1	12	1	1	1	1	−15
15	1	7	−6	17	1	1	15	1	1	1	13	1	1	1	1	−16
16	−1	−1	2	17	3	1	5	1	5	3	1	1	1	1	9	−4
17	1	1	0	17	1	3	5	2	1	1	1	1	2	2	11	−8
18	1	1	1	17	3	4	3	1	1	1	1	1	1	1	13	−4
19	0	−1	1	18	1	1	16	1	2	1	14	1	1	1	1	−17
20	1	1	0	18	2	2	5	2	1	1	2	1	2	1	11	−7
21	−1	0	1	21	7	1	1	7	1	1	7	1	7	1	13	−8
22	−1	0	2	21	9	2	1	3	1	1	3	1	3	1	17	−4
23	5	2	−1	19	3	1	5	2	1	2	1	1	2	1	11	−6
24	0	−1	1	20	1	1	18	1	2	1	16	1	1	1	1	−19
25	0	−1	2	20	2	4	6	1	2	1	4	1	1	1	13	−7
26	1	9	−8	21	1	1	19	1	1	1	17	1	1	1	1	−20
27	0	−1	1	22	1	1	20	1	2	1	18	1	1	1	1	−21
28	0	−1	3	22	3	2	6	1	6	2	2	1	1	1	13	−5
29	0	−1	3	22	3	5	4	1	2	1	2	1	1	1	17	−5
30	1	10	−9	23	1	1	21	1	1	1	19	1	1	1	1	−22
31	0	−1	1	24	1	1	22	1	2	1	20	1	1	1	1	−23
32	1	11	−10	25	1	1	23	1	1	1	21	1	1	1	1	−24
33	1	2	0	25	1	10	5	1	1	1	1	1	1	3	21	−8

Table 2. Integral graphs with $o \leq 25$ and $a > (b+1)$ belonging to the class (7)

i	x_0	y_0	z	o	α	β	a	b	τ	t	k	ℓ	m	n	μ_1^*	μ_2^*
1	0	−1	1	14	1	2	6	2	2	1	1	2	1	1	7	−9
2	−1	−1	1	16	2	1	6	2	2	2	1	1	2	1	7	−7
3	0	−1	1	18	1	3	12	1	4	2	2	1	1	2	7	−15
4	0	−1	1	20	1	1	12	4	6	3	1	4	1	1	5	−17
5	0	−1	2	20	2	2	8	1	4	2	2	1	1	1	9	−7
6	1	1	0	24	1	1	12	6	4	2	1	3	2	1	7	−19
7	0	−1	2	24	2	1	10	2	10	5	1	2	1	1	9	−9
8	0	−1	1	24	1	3	12	2	2	1	3	2	1	1	13	−17

Table 3. Integral graphs with $o \leq 25$ and $a > (b+1)$ belonging to the class (8)

i	x_0	y_0	z	o	α	β	a	b	τ	t	k	ℓ	m	n	μ_1^*	μ_2^*
1	0	−1	1	11	1	3	5	1	1	1	1	1	1	1	6	−7
2	0	−1	1	13	1	2	9	1	3	2	1	1	1	2	4	−11
3	0	−1	1	16	1	4	8	1	1	1	2	1	1	1	9	−11
4	0	−1	1	21	1	1	15	3	5	3	1	3	1	1	4	−19
5	0	−1	1	21	1	5	11	1	1	1	3	1	1	1	12	−15
6	1	3	−1	22	1	2	10	3	1	1	2	1	2	1	11	−15
7	0	−1	1	23	1	4	15	1	5	3	1	1	1	4	10	−19

There exist exactly 3 non-isomorphic Seidel integral graphs with $\alpha = 1$, $\beta = 1$, $b \geq 2$ and $a > (b+1)$, which belong to the class $\alpha K_a \cup \beta K_{b,b}$ whose order does not exceed 25. They are represented in Table 2 under identification numbers $i = 4$, 6 and in Table 3 under identification number $i = 4$.

Graphs represented in Table 4 with identification numbers $i = 8$, 16, 20 are Seidel integral graphs with $\alpha = 1$, $\beta = 1$ and $b \geq 2$. Graphs represented in Table 5 with identification numbers $i = 1$, 4, 7, 10 are Seidel integral graphs with $\alpha = 1$, $\beta = 1$ and $b \geq 2$. Graphs represented in Table 6 with identification numbers $i = 1$, 2, 4, 6, 9 are Seidel integral graphs with $\alpha = 1$, $\beta = 1$ and $b \geq 2$. Therefore, there exits exactly $3 + 4 + 5 = 12$ non-isomorphic Seidel integral graphs with $\alpha = 1$, $\beta = 1$, $b \geq 2$ and $a \leq b$ whose order does not exceed 25. In view[4] of this, there exist exactly $3 + 12 = 15$ non-isomorphic Seidel integral graphs with $\alpha = 1$, $\beta = 1$ and $b \geq 2$, which belong to the class $\alpha K_a \cup \beta K_{b,b}$ whose order does not exceed 25.

[4]There exist exactly 3 non-isomorphic Seidel integral graphs which belong to the class Theorem 7 (1^0) whose order does not exceed 25. They are represented in Table 4 under identification numbers $i = 8$, 16, 20. There exist exactly 12 non-isomorphic Seidel integral graphs which belong to the class Theorem 7 (2^0) whose order does not exceed 25. In particular, they are represented (i) in Table 2 under identification numbers $i = 4$, 6; (ii) in Table 3 under identification number $i = 4$; (iii) in Table 5 under identification numbers $i = 1$, 4, 7, 10 and (iv) in Table 6 under identification numbers $i = 1$, 2, 4, 6, 9.

There exist exactly 7 non-isomorphic Seidel integral graphs with $\mu_1^* = 2a+1$ and $a > (b+1)$, which belong to the class $\alpha K_a \cup \beta K_{b,b}$ whose order does not exceed 25. They are represented in Table 1 under identification numbers $i = 12$, 17, 20, 21, 23, 25, 28.

There exist exactly 7 non-isomorphic Seidel integral graphs with $\mu_1^* = 2a+1$ and $a \leq b$, which belong to the class $\alpha K_a \cup \beta K_{b,b}$ whose order does not exceed 25. They are represented in Table 4 under identification numbers $i = 2$, 5, 8, 10, 17, 20, 23. In view[5] of this, there exist exactly $7 + 7 = 14$ non-isomorphic Seidel integral graphs with $\mu_1^* = 2a + 1$, which belong to the class $\alpha K_a \cup \beta K_{b,b}$ whose order does not exceed 25.

There exist exactly 8 non-isomorphic Seidel integral graphs with $\mu_1^* = a + 1$ and $a > (b+1)$, which belong to the class $\alpha K_a \cup \beta K_{b,b}$ whose order does not exceed 25. They are represented in Table 2 under identification numbers $i = 1$, 2, 5, 8 and in Table 3 under identification numbers $i = 1$, 3, 5, 6.

There exist exactly 7 non-isomorphic Seidel integral graphs with $\mu_1^* = a + 1$ and $a \leq b$, which belong to the class $\alpha K_a \cup \beta K_{b,b}$ whose order does not exceed 25. They are represented in Table 5 under identification numbers $i = 1$, 4, 7 and in Table 6 under identification numbers $i = 1$, 2, 4, 9. In view[6] of this, there exist exactly $8 + 7 = 15$ non-isomorphic Seidel integral graphs with $\mu_1^* = a + 1$, which belong to the class $\alpha K_a \cup \beta K_{b,b}$ whose order does not exceed 25.

[5] In particular, (i) there is no Seidel integral graph with $\mu_1^* = 2a + 1$ which belong to the class Theorem 11 (1^0) whose order does not exceed 25; (ii) there exist exactly 2 non-isomorphic Seidel integral graphs which belong to the class Theorem 11 (2^0) whose order does not exceed 25. They are represented in Table 1 under identification numbers $i = 17$, 21; (iii) there exist exactly 2 non-isomorphic Seidel integral graphs which belong to the class Theorem 11 (3^0) whose order does not exceed 25. They are represented in Table 4 under identification numbers $i = 8$, 20; (iv) there exist exactly 3 non-isomorphic Seidel integral graphs which belong to the class Theorem 11 (4^0) whose order does not exceed 25. They are represented in Table 1 under identification numbers $i = 12$, 20, 25; (v) there exist exactly 5 non-isomorphic Seidel integral graphs which belong to the class Theorem 11 (5^0) whose order does not exceed 25. They are represented in Table 4 under identification numbers $i = 2$, 5, 10, 17, 23 and (vi) there exist exactly 2 non-isomorphic Seidel integral graphs which belong to the class Theorem 11 (6^0) whose order does not exceed 25. They are represented in Table 1 under identification numbers $i = 23$, 28. In view of this, there exist exactly $0 + 2 + 2 + 3 + 5 + 2 = 14$ non-isomorphic Seidel integral graphs with $\mu_1^* = 2a + 1$, which belong to the class $\alpha K_a \cup \beta K_{b,b}$ whose order does not exceed 25.

[6] There exist exactly 6 non-isomorphic Seidel integral graphs which belong to the class Theorem 12 (1^0) whose order does not exceed 25. They are represented in Table 2 under identification numbers $i = 1$, 8 and in Table 3 under identification numbers $i = 1$, 3, 5, 6. There exist exactly 7 non-isomorphic Seidel integral graphs which belong to the class Theorem 12 (2^0) whose order does not exceed 25. They are represented in Table 5 under identification numbers $i = 1$, 4, 7 and in Table 6 under identification numbers $i = 1$, 2, 4, 9. There exist exactly 2 non-isomorphic Seidel integral graphs which belong to the class Theorem 12 (3^0) whose order does not exceed 25. They are represented in Table 2 under identification numbers $i = 2$, 5.

Table 4. Integral graphs with $o \leq 25$ and $a \leq b$ belonging to the class (6)

i	x_0	y_0	z	o	α	β	a	b	τ	t	k	ℓ	m	n	μ_1^*	μ_2^*
1	−1	0	1	3	1	1	1	1	1	1	1	1	1	1	1	−2
2	−1	0	1	6	2	1	1	2	1	1	2	1	2	1	3	−3
3	−1	0	2	7	3	2	1	1	1	1	1	1	1	1	5	−2
4	−1	0	1	9	3	1	1	3	1	1	3	1	3	1	5	−4
5	−1	0	1	10	2	1	2	3	2	1	2	1	3	1	5	−5
6	−1	0	3	11	5	3	1	1	1	1	1	1	1	1	9	−2
7	−1	0	1	12	4	1	1	4	1	1	4	1	4	1	7	−5
8	−1	0	1	13	1	1	1	6	1	1	3	2	2	1	3	−4
9	−1	0	1	13	9	1	1	2	1	2	1	1	2	1	11	−4
10	5	2	−1	14	2	1	3	4	1	1	2	1	4	1	7	−7
11	−1	0	2	14	6	2	1	2	1	1	2	1	2	1	11	−3
12	−1	0	1	15	5	1	1	5	1	1	5	1	5	1	9	−6
13	−1	0	1	15	3	3	1	2	1	1	1	1	2	2	11	−2
14	−1	0	4	15	7	4	1	1	1	1	1	1	1	1	13	−2
15	−1	0	1	16	3	1	2	5	2	1	4	1	5	1	9	−7
16	−1	0	1	17	1	1	3	7	3	2	1	4	1	1	5	−8
17	2	1	0	18	2	1	4	5	2	1	2	1	5	1	9	−9
18	−1	0	1	18	6	1	1	6	1	1	6	1	6	1	11	−7
19	−1	0	5	19	9	5	1	1	1	1	1	1	1	1	17	−2
20	−1	0	1	20	1	1	2	9	2	1	4	2	3	1	5	−7
21	−1	0	1	21	7	1	1	7	1	1	7	1	7	1	13	−8
22	−1	0	2	21	9	2	1	3	1	1	3	1	3	1	17	−4
23	7	3	−1	22	2	1	5	6	1	1	2	1	6	1	11	−11
24	−1	0	1	22	4	1	2	7	2	1	6	1	7	1	13	−9
25	−1	0	2	22	5	2	2	3	2	1	2	1	3	1	17	−5
26	−1	0	3	22	10	3	1	2	1	1	2	1	2	1	19	−3
27	−1	−1	1	23	1	2	3	5	1	1	1	3	1	1	13	−6
28	9	2	−3	23	3	1	3	7	1	1	5	1	7	1	13	−10
29	−1	0	6	23	11	6	1	1	1	1	1	1	1	1	21	−2
30	−1	0	1	24	8	1	1	8	1	1	8	1	8	1	15	−9

Table 5. Integral graphs with $o \leq 25$ and $a \leq b$ belonging to the class (7)

i	x_0	y_0	z	o	α	β	a	b	τ	t	k	ℓ	m	n	μ_1^*	μ_2^*
1	−1	0	1	10	1	1	2	4	2	1	1	2	2	1	3	−5
2	−1	0	1	14	2	1	1	6	1	1	2	2	3	1	5	−5
3	−1	0	1	14	8	1	1	3	1	2	1	1	3	1	11	−5
4	1	1	0	16	1	1	4	6	2	1	1	2	3	1	5	−9
5	−1	0	1	16	2	1	2	6	2	2	1	3	2	1	7	−7
6	−1	0	1	20	6	1	2	4	2	2	1	1	4	1	15	−7
7	1	1	0	22	1	1	6	8	2	1	1	2	4	1	7	−13
8	−1	0	1	22	3	1	2	8	2	3	1	4	2	1	11	−9
9	−1	0	2	22	3	2	2	4	2	1	1	2	2	1	15	−5
10	−1	0	1	24	1	1	4	10	4	2	1	5	2	1	7	−11
11	−1	0	1	24	2	1	2	10	2	1	3	2	5	1	9	−9
12	−1	0	1	24	6	1	2	6	2	3	1	2	3	1	17	−9

Table 6. Integral graphs with $o \leq 25$ and $a \leq b$ belonging to the class (8)

i	x_0	y_0	z	o	α	β	a	b	τ	t	k	ℓ	m	n	μ_1^*	μ_2^*
1	−1	0	1	7	1	1	1	3	1	1	1	1	2	1	2	−3
2	3	2	−1	13	1	1	3	5	1	1	1	1	3	1	4	−7
3	−1	0	2	16	4	2	1	3	1	1	1	1	2	1	11	−3
4	4	3	−1	19	1	1	5	7	1	1	1	1	4	1	6	−11
5	−1	0	1	19	9	1	1	5	1	2	1	1	3	1	14	−7
6	−1	0	1	21	1	1	1	10	1	1	2	2	3	1	4	−5
7	−1	0	1	21	3	1	1	9	1	1	3	1	5	1	8	−7
8	−1	0	1	23	3	2	1	5	1	1	1	1	3	2	14	−3
9	5	4	−1	25	1	1	7	9	1	1	1	1	5	1	8	−15
10	−1	0	3	25	7	3	1	3	1	1	1	1	2	1	20	−3

Table 7. Distribution of integral graphs with $o = 1, 2, \ldots, 400$

$003^{00,01}$	$005^{01,00}$	$006^{01,01}$	$007^{01,02}$	$008^{01,00}$	$009^{01,01}$	$010^{01,02}$
$011^{02,01}$	$012^{01,01}$	$013^{02,03}$	$014^{04,04}$	$015^{01,03}$	$016^{03,04}$	$017^{04,01}$
$018^{04,02}$	$019^{02,03}$	$020^{04,02}$	$021^{03,04}$	$022^{04,07}$	$023^{02,04}$	$024^{04,04}$
$025^{02,02}$	$026^{09,04}$	$027^{04,04}$	$028^{06,07}$	$029^{03,04}$	$030^{06,06}$	$031^{04,06}$
$032^{08,04}$	$033^{04,03}$	$034^{08,13}$	$035^{03,05}$	$036^{10,05}$	$037^{06,04}$	$038^{15,09}$
$039^{05,05}$	$040^{07,04}$	$041^{09,02}$	$042^{08,10}$	$043^{05,05}$	$044^{14,10}$	$045^{02,06}$
$046^{16,14}$	$047^{07,04}$	$048^{12,09}$	$049^{04,05}$	$050^{11,07}$	$051^{07,06}$	$052^{14,16}$
$053^{05,04}$	$054^{16,15}$	$055^{03,08}$	$056^{12,08}$	$057^{08,08}$	$058^{16,16}$	$059^{06,08}$
$060^{09,10}$	$061^{05,09}$	$062^{25,17}$	$063^{07,11}$	$064^{14,11}$	$065^{08,04}$	$066^{20,15}$
$067^{10,09}$	$068^{24,12}$	$069^{07,05}$	$070^{15,17}$	$071^{07,06}$	$072^{14,13}$	$073^{10,06}$
$074^{15,13}$	$075^{08,08}$	$076^{21,21}$	$077^{09,08}$	$078^{19,17}$	$079^{03,12}$	$080^{15,12}$
$081^{08,05}$	$082^{17,15}$	$083^{09,03}$	$084^{21,14}$	$085^{07,04}$	$086^{34,23}$	$087^{10,08}$
$088^{23,17}$	$089^{08,04}$	$090^{20,15}$	$091^{05,11}$	$092^{30,25}$	$093^{09,17}$	$094^{30,26}$
$095^{04,07}$	$096^{24,18}$	$097^{13,14}$	$098^{13,17}$	$099^{10,06}$	$100^{20,23}$	$101^{16,02}$
$102^{26,17}$	$103^{12,10}$	$104^{23,13}$	$105^{12,12}$	$106^{21,21}$	$107^{15,11}$	$108^{29,21}$
$109^{11,09}$	$110^{29,13}$	$111^{11,11}$	$112^{21,23}$	$113^{12,08}$	$114^{34,29}$	$115^{09,10}$
$116^{40,20}$	$117^{08,12}$	$118^{34,29}$	$119^{08,12}$	$120^{23,19}$	$121^{09,10}$	$122^{30,26}$
$123^{11,12}$	$124^{34,34}$	$125^{10,14}$	$126^{26,24}$	$127^{13,10}$	$128^{32,17}$	$129^{12,13}$
$130^{19,22}$	$131^{08,06}$	$132^{35,23}$	$133^{09,15}$	$134^{31,23}$	$135^{08,14}$	$136^{31,34}$
$137^{13,06}$	$138^{38,30}$	$139^{06,12}$	$140^{34,25}$	$141^{11,09}$	$142^{36,28}$	$143^{11,13}$
$144^{29,25}$	$145^{10,11}$	$146^{48,29}$	$147^{14,12}$	$148^{43,30}$	$149^{14,06}$	$150^{40,34}$
$151^{14,10}$	$152^{40,31}$	$153^{13,07}$	$154^{27,28}$	$155^{11,09}$	$156^{41,36}$	$157^{17,16}$
$158^{39,29}$	$159^{09,13}$	$160^{33,31}$	$161^{14,08}$	$162^{35,22}$	$163^{15,10}$	$164^{43,33}$
$165^{09,10}$	$166^{41,35}$	$167^{14,10}$	$168^{36,28}$	$169^{06,10}$	$170^{33,18}$	$171^{13,16}$
$172^{36,38}$	$173^{13,10}$	$174^{48,27}$	$175^{10,17}$	$176^{31,32}$	$177^{16,16}$	$178^{37,34}$
$179^{14,07}$	$180^{32,24}$	$181^{15,11}$	$182^{35,28}$	$183^{16,17}$	$184^{40,35}$	$185^{08,16}$
$186^{51,35}$	$187^{21,10}$	$188^{47,33}$	$189^{12,20}$	$190^{31,40}$	$191^{18,17}$	$192^{51,36}$
$193^{13,10}$	$194^{46,32}$	$195^{19,15}$	$196^{43,42}$	$197^{21,12}$	$198^{53,37}$	$199^{12,16}$
$200^{42,22}$	$201^{16,17}$	$202^{46,40}$	$203^{16,17}$	$204^{46,37}$	$205^{13,10}$	$206^{55,39}$
$207^{22,10}$	$208^{42,31}$	$209^{19,11}$	$210^{31,36}$	$211^{10,19}$	$212^{55,39}$	$213^{23,11}$
$214^{50,43}$	$215^{09,14}$	$216^{49,34}$	$217^{17,18}$	$218^{44,36}$	$219^{16,18}$	$220^{43,31}$
$221^{10,15}$	$222^{50,48}$	$223^{13,17}$	$224^{41,27}$	$225^{16,12}$	$226^{41,32}$	$227^{17,05}$
$228^{63,41}$	$229^{20,15}$	$230^{41,35}$	$231^{15,15}$	$232^{49,39}$	$233^{19,10}$	$234^{47,44}$
$235^{10,18}$	$236^{68,33}$	$237^{21,26}$	$238^{44,44}$	$239^{13,12}$	$240^{45,32}$	$241^{18,20}$
$242^{37,29}$	$243^{17,21}$	$244^{51,61}$	$245^{18,11}$	$246^{65,43}$	$247^{15,24}$	$248^{66,46}$
$249^{15,22}$	$250^{42,36}$	$251^{14,13}$	$252^{43,38}$	$253^{08,23}$	$254^{47,41}$	$255^{17,27}$

Table 7. (continued)

$256^{47,38}$	$257^{24,09}$	$258^{63,57}$	$259^{12,17}$	$260^{57,36}$	$261^{18,13}$	$262^{55,45}$
$263^{21,08}$	$264^{55,37}$	$265^{25,09}$	$266^{61,34}$	$267^{24,13}$	$268^{59,56}$	$269^{21,15}$
$270^{51,43}$	$271^{23,23}$	$272^{59,37}$	$273^{15,23}$	$274^{37,37}$	$275^{15,15}$	$276^{70,59}$
$277^{20,15}$	$278^{60,34}$	$279^{18,19}$	$280^{44,44}$	$281^{27,14}$	$282^{52,49}$	$283^{23,20}$
$284^{69,55}$	$285^{22,21}$	$286^{63,48}$	$287^{16,18}$	$288^{51,46}$	$289^{13,10}$	$290^{49,32}$
$291^{21,16}$	$292^{66,43}$	$293^{16,17}$	$294^{50,54}$	$295^{16,13}$	$296^{59,55}$	$297^{20,13}$
$298^{48,41}$	$299^{20,12}$	$300^{54,52}$	$301^{13,23}$	$302^{56,39}$	$303^{23,24}$	$304^{53,44}$
$305^{18,13}$	$306^{50,39}$	$307^{27,19}$	$308^{52,41}$	$309^{21,20}$	$310^{46,53}$	$311^{18,14}$
$312^{64,41}$	$313^{17,12}$	$314^{65,46}$	$315^{15,22}$	$316^{71,50}$	$317^{26,18}$	$318^{71,54}$
$319^{14,21}$	$320^{65,39}$	$321^{23,23}$	$322^{48,49}$	$323^{15,12}$	$324^{73,58}$	$325^{19,13}$
$326^{66,48}$	$327^{27,28}$	$328^{79,61}$	$329^{18,11}$	$330^{55,39}$	$331^{17,19}$	$332^{78,60}$
$333^{20,23}$	$334^{61,47}$	$335^{21,18}$	$336^{65,45}$	$337^{25,14}$	$338^{57,38}$	$339^{24,22}$
$340^{62,47}$	$341^{23,13}$	$342^{57,51}$	$343^{18,17}$	$344^{73,55}$	$345^{17,27}$	$346^{71,68}$
$347^{24,16}$	$348^{81,56}$	$349^{15,21}$	$350^{51,44}$	$351^{19,20}$	$352^{58,68}$	$353^{23,14}$
$354^{75,51}$	$355^{16,19}$	$356^{63,52}$	$357^{13,21}$	$358^{62,67}$	$359^{21,17}$	$360^{56,42}$
$361^{17,11}$	$362^{63,39}$	$363^{22,25}$	$364^{62,63}$	$365^{19,17}$	$366^{75,60}$	$367^{25,24}$
$368^{59,48}$	$369^{25,25}$	$370^{53,47}$	$371^{23,13}$	$372^{85,63}$	$373^{24,16}$	$374^{73,49}$
$375^{24,22}$	$376^{76,51}$	$377^{24,24}$	$378^{60,48}$	$379^{17,22}$	$380^{65,51}$	$381^{31,31}$
$382^{71,59}$	$383^{14,22}$	$384^{78,67}$	$385^{26,21}$	$386^{75,50}$	$387^{36,24}$	$388^{80,72}$
$389^{25,14}$	$390^{62,54}$	$391^{24,24}$	$392^{60,47}$	$393^{20,24}$	$394^{60,57}$	$395^{19,25}$
$396^{61,58}$	$397^{24,20}$	$398^{72,53}$	$399^{25,31}$	$400^{51,42}$		

There exist exactly 20202 non-isomorphic Seidel[7] integral graphs which belong to the class $\alpha K_a \cup \beta K_{b,b}$ whose order does not exceed 400. In particular, (i) there exist exactly 5121, 3490 and 2409 non-isomorphic Seidel integral graphs with $a > (b+1)$, which belong to the classes Theorem 3 (6), (7) and (8), respectively, whose order does not exceed 400. Therefore, there exits exactly $5121 + 3490 + 2409 = 11020$ non-isomorphic Seidel integral graphs with $a > (b+1)$ whose order does not exceed 400; (ii) there exist exactly 4178, 3470 and 1534 non-isomorphic Seidel integral graphs with $a \leq b$, which belong to the classes Theorem 3 (6), (7) and (8), respectively, whose order does not exceed 400. Therefore, there exits exactly $4178 + 3470 + 1534 = 9182$ non-isomorphic Seidel integral graphs with $a \leq b$ whose order does not exceed 400. In view[8] of this, there exist exactly

[7] In this work the data given in Tables 1–7 are obtained in two different ways: (i) they are generated by using relations $\big((6), (7), (8)\big)$ and (ii) by varying the parameters α, β, a, b in all possible ways in equation (4).

[8] In particular, (i) there exist exactly 213608 non-isomorphic Seidel integral graphs with $a > (b+1)$ which belong to the class $\alpha K_a \cup \beta K_{b,b}$ whose order does not exceed 2500; (ii) there exist exactly 175678 non-isomorphic Seidel integral graphs with $a \leq b$ which belong to the class $\alpha K_a \cup \beta K_{b,b}$ whose order does not exceed 2500. In view of this, there exist exactly $213608 + 175678 = 389286$ non-isomorphic Seidel integral graphs which belong to the class $\alpha K_a \cup \beta K_{b,b}$ whose order does not exceed 2500.

$11020 + 9182 = 20202$ non-isomorphic Seidel integral graphs which belong to the class $\alpha K_a \cup \beta K_{b,b}$ whose order does not exceed 400. Table 7 contains a distribution of those graphs in respect to their orders. In Table 7 the symbol $o^{m,n}$ denotes the number of integral graphs of the corresponding order $o = 1, 2, \ldots, 400$, where m and n denote the number of Seidel integral graphs with $a > (b+1)$ and $a \leq b$, respectively. In this table $o^{m,n}$ is omitted if the corresponding number $m = 0$ and $n = 0$.

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