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SYMMETRIC SUMS OF BINOMIAL COEFFICIENTS

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Two classes of symmetric sums on products of partial sums of binomial coefficients are examined. Explicit formulae are established that give rise to several interesting binomial identities, including a very particular case discovered by Chang and Shan in 1983.

1. INTRODUCTION AND MOTIVATION

Denote by $\mathbb{N}$ the set of natural numbers with $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. For $n \in \mathbb{N}$, Chang and Shan [**1**] discovered in 1983 the following beautiful identity

$$\frac{n}{2}\binom{2n}{n} = \sum_{\ell=0}^{n-1}\left\{\binom{n}{0}+\binom{n}{1}+\cdots+\binom{n}{\ell}\right\} \times\left\{\binom{n}{\ell+1}+\binom{n}{\ell+2}+\cdots+\binom{n}{n}\right\}.$$

Writing this compactly in a triple sum

$$\frac{n}{2}\binom{2n}{n} = \sum_{\ell=0}^{n-1}\sum_{i=0}^{\ell}\binom{n}{i}\sum_{j=\ell+1}^{n}\binom{n}{j} = \sum_{0\le i<j\le n}\binom{n}{i}\binom{n}{j}\sum_{\ell=i}^{j-1}1,$$

we get the following equivalent formula

$$\frac{n}{2}\binom{2n}{n} = \sum_{0\le i<j\le n}\binom{n}{i}\binom{n}{j}(j-i).$$

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This suggests us to examine, for $\lambda \in \mathbb{N}$, the following two general binomial sums

(1) $$\Phi_\lambda(n) = \sum_{0 \le i < j \le n} \binom{n}{i}\binom{n}{j}(j-i)^\lambda,$$

(2) $$\Psi_\lambda(n) = \sum_{0 \le i < j \le n} \binom{n}{i}\binom{n}{j}(j-i)_\lambda.$$

When $\boxed{\lambda = 0}$, we have $\Phi_0(n) = \Psi_0(n)$, which can be reformulated, under $j - i = \ell$, as

$$\Phi_0(n) = \sum_{0 \le i < j \le n} \binom{n}{i}\binom{n}{j} = \sum_{\ell=1}^{n}\sum_{i=0}^{n-\ell} \binom{n}{i}\binom{n}{n-\ell-i}$$
$$= \sum_{\ell=1}^{n} \binom{2n}{n+\ell} = \frac{1}{2}\sum_{\ell=1}^{n}\left\{\binom{2n}{n+\ell} + \binom{2n}{n-\ell}\right\}.$$

This results in the following closed expression

$$\Phi_0(n) = \Psi_0(n) = 2^{2n-1} - \frac{1}{2}\binom{2n}{n}.$$

For brevity, we shall utilize the following notations throughout the paper. For $\lambda \in \mathbb{N}_0$ and an indeterminate x, denote the usual monomial by x^λ. The rising and falling factorials will be defined by $(x)_0 = \langle x \rangle_0 = 1$ and

$$\left.\begin{aligned} (x)_\lambda &= x(x+1)\cdots(x+\lambda-1) \\ \langle x \rangle_\lambda &= x(x-1)\cdots(x-\lambda+1) \end{aligned}\right\} \quad \text{for} \quad \lambda \in \mathbb{N}.$$

2. THE FIRST BINOMIAL SUM $\Phi_\lambda(N)$

Recall that (cf. Chen and Chu [**2**, Equations 9 & 10])

(3) $$x^{2m} = \sum_{k=0}^{m} (-1)^k \langle y+x \rangle_k \langle y-x \rangle_k \sigma_{k,m-k}(y),$$

where the connection coefficient $\sigma_{k,m}(y)$ is the mth complete symmetric function

(4) $$\begin{aligned} \sigma_{k,m}(y) &= h_m\big(y^2, (y-1)^2, \cdots, (y-k)^2\big) \\ &= \frac{2(-1)^k}{\langle 2y \rangle_{2k+1}} \sum_{i=0}^{k} \binom{2y}{i}\binom{2k-2y}{k-i}(y-i)^{2k+2m+1} \\ &= \frac{2}{k!}\sum_{i=0}^{k}(-1)^i\binom{k}{i}\frac{(y-i)^{2k+2m+1}}{\langle 2y-i \rangle_{k+1}}. \end{aligned}$$

Letting $\ell = j - i$, we can reformulate the sum $\Phi_\lambda(n)$ as

$$\Phi_\lambda(n) = \sum_{0\le i<j\le n} \binom{n}{i}\binom{n}{j}(j-i)^\lambda = \sum_{\ell=1}^{n} \ell^\lambda \sum_{i=0}^{n-\ell} \binom{n}{i}\binom{n}{n-\ell-i}.$$

Evaluating the inner sum by the Chu–Vandermonde convolution, we find that

$$\Phi_\lambda(n) = \sum_{0\le i<j\le n} \binom{n}{i}\binom{n}{j}(j-i)^\lambda = \sum_{\ell=1}^{n} \binom{2n}{n+\ell}\ell^\lambda.$$

This will be our starting point for establishing alternative formulae for reducing computations of the binomial sums $\Phi_\lambda(n)$.

2.1 The sums of even order

According to (3), we can express $\Phi_{2\lambda}(n)$ as the double sum

$$\begin{aligned}\Phi_{2\lambda}(n) &= \sum_{\ell=1}^{n} \binom{2n}{n+\ell}\ell^{2\lambda} = \frac{1}{2}\sum_{\ell=-n}^{n} \binom{2n}{n+\ell} \\ &\quad\times \sum_{k=0}^{\lambda}(-1)^k \langle n+\ell\rangle_k \langle n-\ell\rangle_k \sigma_{k,\lambda-k}(n) \\ &= \frac{1}{2}\sum_{k=0}^{\lambda}(-1)^k \langle 2n\rangle_{2k}\sigma_{k,\lambda-k}(n) \sum_{\ell=-n}^{n} \binom{2n-2k}{n+\ell-k}.\end{aligned}$$

Since the last inner sum equals 2^{2n-2k}, we find the expression below in the theorem.

Theorem 1 ($\lambda \in \mathbb{N}$)**.**

$$\Phi_{2\lambda}(n) = \sum_{k=0}^{\lambda}(-1)^k 2^{2n-2k-1}\langle 2n\rangle_{2k}\sigma_{k,\lambda-k}(n).$$

This formula significantly reduces the computation of $\Phi_{2\lambda}(n)$ to a few terms (precisely $\lambda+1$ terms). In fact, if we write

$$\Phi_{2\lambda}(n) = 2^{2n-\lambda-1}P_\lambda(n),$$

then we have the following explicit formulae.

Corollary 2.

$$\begin{aligned}P_1(n) &= n,\\ P_2(n) &= 3n^2 - n,\\ P_3(n) &= 15n^3 - 15n^2 + 4n,\\ P_4(n) &= 105n^4 - 210n^3 + 147n^2 - 34n,\\ P_5(n) &= 945n^5 - 3150n^4 + 4095n^3 - 2370n^2 + 496n.\end{aligned}$$

According to Theorem 1, the polynomial $P_\lambda(n)$ should be of degree "2λ" in n. Instead, the above examples suggest that the degree of $P_\lambda(n)$ is equal to "λ". Despite further experimental results by *Mathematica* that confirm this fact, the authors have unfortunately failed to prove it.

2.2 The sums of odd order

Analogously, we can treat the sum $\Phi_{2\lambda+1}(n)$ by

$$\begin{aligned}\Phi_{2\lambda+1}(n) &= \sum_{\ell=1}^{n}\binom{2n}{n+\ell}\ell^{2\lambda+1} = \sum_{\ell=1}^{n}\binom{2n}{n+\ell}\ell \\ &\times \sum_{k=0}^{\lambda}(-1)^k\langle n+\ell\rangle_k\langle n-\ell\rangle_k\sigma_{k,\lambda-k}(n) \\ &= \sum_{k=0}^{\lambda}(-1)^k\langle 2n\rangle_{2k}\sigma_{k,\lambda-k}(n)\sum_{\ell=1}^{n}\binom{2n-2k}{n+\ell-k}\ell.\end{aligned}$$

By writing

$$\ell = \frac{n+\ell-k}{2} - \frac{n-\ell-k}{2},$$

we may evaluate the inner sum with respect to ℓ as follows:

$$\begin{aligned}\sum_{\ell=1}^{n}\binom{2n-2k}{n+\ell-k}\ell &= \sum_{\ell=1}^{n}\binom{2n-2k}{n+\ell-k}\frac{(n+\ell-k)-(n-\ell-k)}{2} \\ &= (n-k)\sum_{\ell=1}^{n-k}\binom{2n-2k-1}{n-\ell-k} - (n-k)\sum_{\ell=1}^{n-k-1}\binom{2n-2k-1}{n+\ell-k} \\ &= (n-k)\binom{2n-2k-1}{n-k} = \frac{n-k}{2}\binom{2n-2k}{n-k}.\end{aligned}$$

Hence, we get an analogous formula in the following theorem.

Theorem 3 ($\lambda \in \mathbb{N}_0$)**.**

$$\Phi_{2\lambda+1}(n) = \frac{n}{2}\sum_{k=0}^{\lambda}(-1)^k\langle 2n-1\rangle_{2k}\binom{2n-2k}{n-k}\sigma_{k,\lambda-k}(n).$$

Again, this formula reduces, for a small λ, the computation of $\Phi_{2\lambda+1}(n)$ to a few terms. Writing further

$$\Phi_{2\lambda-1}(n) = \frac{1}{2}\binom{2n}{n}Q_\lambda(n),$$

we highlight the first five polynomials in the next corollary.

Corollary 4.

$$\begin{aligned}
Q_1(n) &= n, \\
Q_2(n) &= n^2, \\
Q_3(n) &= 2n^3 - n^2, \\
Q_4(n) &= 6n^4 - 8n^3 + 3n^2, \\
Q_5(n) &= 24n^5 - 60n^4 + 54n^3 - 17n^2.
\end{aligned}$$

These examples suggest that the polynomial $Q_\lambda(n)$ is of degree "λ" instead of "$2\lambda - 1$", which is implied by Theorem 3. An intriguing problem is how to show that the degree of $Q_\lambda(n)$ is exactly "λ"?

3. THE SECOND BINOMIAL SUM $\Psi_\lambda(N)$

By letting $\ell = j - i$ first and then carrying out the same procedure as that for $\Phi_\lambda(n)$, we can manipulate the double sum $\Psi_\lambda(n)$ as

$$\Psi_\lambda(n) = \sum_{0 \le i < j \le n} \binom{n}{i}\binom{n}{j}(j-i)_\lambda = \sum_{\ell=1}^{n} (\ell)_\lambda \sum_{i=0}^{n-\ell} \binom{n}{i}\binom{n}{n-\ell-i},$$

which leads us to the following expression

$$\Psi_\lambda(n) = \sum_{0 \le i < j \le n} \binom{n}{i}\binom{n}{j}(j-i)_\lambda = \sum_{\ell=1}^{n} \binom{2n}{n+\ell}(\ell)_\lambda.$$

By means of the Chu–Vandermonde convolution formula, it is not hard to show the linear relation

$$(\ell)_\lambda = \sum_{k=0}^{\lambda} \binom{\lambda}{k} \frac{(-n)_\lambda}{(-n)_k} \langle n+\ell \rangle_k.$$

Then we can manipulate the double sum

$$\begin{aligned}
\Psi_\lambda(n) &= \sum_{\ell=1}^{n} \binom{2n}{n+\ell} \sum_{k=0}^{\lambda} \binom{\lambda}{k} \frac{(-n)_\lambda}{(-n)_k} \langle n+\ell \rangle_k \\
&= \sum_{k=0}^{\lambda} \binom{\lambda}{k} \frac{(-n)_\lambda}{(-n)_k} \langle 2n \rangle_k \sum_{\ell=1}^{n} \binom{2n-k}{n-\ell}.
\end{aligned}$$

The inner sum with respect to ℓ can be evaluated by

$$\sum_{\ell=1}^{n} \binom{2n-k}{n-\ell} = \frac{1}{2}\left\{ 2^{2n-k} - \binom{2n}{n}\chi(k=0) + \sum_{i=1}^{k-1} \binom{2n-k}{n-i} \right\},$$

where χ stands for the logical function with $\chi(\text{true}) = 1$ and $\chi(\text{false}) = 0$. This formula reduces remarkably the computation of the sum on the left because the number of terms involved is independent of n and at most $k+1 \le \lambda+2$. Observe further that

$$\sum_{k=0}^{\lambda}\binom{\lambda}{k}\frac{(-n)_\lambda}{(-n)_k}\langle 2n\rangle_k\binom{2n-k}{n-k} = \binom{2n}{n}(-n)_\lambda\sum_{k=0}^{\lambda}\binom{\lambda}{k}(-1)^k = 0 \text{ for } \lambda \neq 0.$$

By substitution and simplification, we establish the following reduction formula.

Theorem 5 ($\lambda \in \mathbb{N}$)**.**

$$\Psi_\lambda(n) = \frac{(-n)_\lambda}{2}\sum_{k=0}^{\lambda}\binom{\lambda}{k}\frac{\langle 2n\rangle_k}{(-n)_k}\Big\{2^{2n-k} + \sum_{i=1}^{k}\binom{2n-k}{n-i}\Big\}.$$

According to this formula, we can see that $\Psi_\lambda(n)$ results in a linear combination of 2^{2n} and $\binom{2n}{n}$ with the polynomial coefficients in n, whose degrees are determined, by making use of *Mathematica* commands, to be $\lfloor\frac{\lambda}{2}\rfloor$ and $\lceil\frac{\lambda}{2}\rceil$, respectively. According to the parity of λ, the first ten examples are recorded in the following two corollaries.

Corollary 6 ($\lambda \in \mathbb{N}$)**.** *Write formally*

$$\Psi_{2\lambda}(n) = 2^{2n-2\lambda-1}(\lambda+1)_\lambda\mathcal{P}_\lambda(n) + \lambda!(\lambda-\tfrac{1}{2})\binom{2n}{n}\mathcal{Q}_\lambda(n).$$

Then $\mathcal{P}_\lambda(n)$ and $\mathcal{Q}_\lambda(n)$ are monic polynomials given by

$$\begin{aligned}
\mathcal{P}_1(n) &= n,\\
\mathcal{P}_2(n) &= n^2 + 7n, ,\\
\mathcal{P}_3(n) &= n^3 + 33n^2 + 62n,\\
\mathcal{P}_4(n) &= n^4 + 90n^3 + 683n^2 + 762n,\\
\mathcal{P}_5(n) &= n^5 + 190n^4 + 3635n^3 + 14630n^2 + 12264n;
\end{aligned}$$

$$\begin{aligned}
\mathcal{Q}_1(n) &= n,\\
\mathcal{Q}_2(n) &= n^2 + n,\\
\mathcal{Q}_3(n) &= n^3 + 7n^2 + 4n,\\
\mathcal{Q}_4(n) &= n^4 + 22n^3 + 67n^2 + 30n,\\
\mathcal{Q}_5(n) &= n^5 + 50n^4 + 431n^3 + 862n^2 + 336n.
\end{aligned}$$

Corollary 7 ($\lambda \in \mathbb{N}$)**.** *Write formally*

$$\Psi_{2\lambda-1}(n) = 2^{2n-2\lambda+1}(\lambda-1)_{\lambda+1}\mathrm{P}_\lambda(n) + \frac{(\lambda-1)!}{2}\binom{2n}{n}\mathrm{Q}_\lambda(n).$$

Then $\mathrm{P}_\lambda(n)$ *and* $\mathrm{Q}_\lambda(n)$ *are monic polynomials given by*

$$\begin{aligned}
\mathrm{P}_1(n) &= 0,\\
\mathrm{P}_2(n) &= n,\\
\mathrm{P}_3(n) &= n^2+3n,\\
\mathrm{P}_4(n) &= n^3+13n^2+18n,\\
\mathrm{P}_5(n) &= n^4+34n^3+179n^2+170n;
\end{aligned}$$

$$\begin{aligned}
\mathrm{Q}_1(n) &= n,\\
\mathrm{Q}_2(n) &= n^2+2n,\\
\mathrm{Q}_3(n) &= n^3+17n^2+12n,\\
\mathrm{Q}_4(n) &= n^4+57n^3+242n^2+120n,\\
\mathrm{Q}_5(n) &= n^5+134n^4+1691n^3+4054n^2+1680n.
\end{aligned}$$

Furthermore, by appealing to the Stirling inversions (see Comtet [**3**, Chapter 5])

$$(j-i)^\lambda = \sum_{k=1}^{\lambda}(-1)^{\lambda-k}S(\lambda,k)(j-i)_k,$$
$$(j-i)_\lambda = \sum_{k=1}^{\lambda}(-1)^{\lambda-k}s(\lambda,k)(j-i)^k;$$

where $s(\lambda,k)$ and $S(\lambda,k)$ are the Stirling numbers of the first and the second kind, respectively. We immediately get, from (1) and (2) the following reciprocal relations

$$\Phi_\lambda(n) = \sum_{k=1}^{\lambda}(-1)^{\lambda-k}S(\lambda,k)\Psi_k(n),$$
$$\Psi_\lambda(n) = \sum_{k=1}^{\lambda}(-1)^{\lambda-k}S(\lambda,k)\Phi_k(n).$$

This last relation explains why the expression of $\Psi_\lambda(n)$ displayed in Theorem 5 consists of both 2^{2n} and $\binom{2n}{n}$, that correspond to $\Phi_k(n)$ with k being even and odd, respectively.

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