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ON APPROXIMATING PROPERTIES OF BLENDED VARIANT OF KANTOROVICH-STANCU TYPE LUPAŞ OPERATORS

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We introduce blended variant of Kantorovich-Stancu type Lupaş operators and study convergence and $\mathfrak{q}$-statistical convergence properties using Korovkin theorem. We investigate rate of convergence interms of modulus of continuity, Peetre's K-functional and Lipschitz functions. Some Direct results and Voronovskaja type theorem are established. Moreover, graphical analysis and error estimations are presented using MATLAB.

1. INTRODUCTION

Weierstrass approximation theorem is the most celebrated theorem in approximation theory, which states that a function defined and continuous on a closed and bounded interval is uniformly approximable by certain polynomials. In 1912, Bernstein gave most clear, concise and elegant proof of this theorem by constructing Bernstein polynomials. Since then many researchers have explored approximation properties of operators in different function spaces. Some of the much studied operators are Kantorovich [**11**], Mirakjan [**17**], Szász [**26**], Stancu [**25**]. These operators exhibit the enhancement of rate of approximation by producing sharper estimates. Romanian mathematician Lupaş[**16**] obtained the following operators. which is defined for $u \geq 0,\ m \in \mathbb{N}$, as

$$L_{\mathfrak{m}}(\mathfrak{h};\mathfrak{u}) = (1-a)^{\mathfrak{m}\mathfrak{u}} \sum_{j=0}^{\infty} \frac{(\mathfrak{m}\mathfrak{u})_j}{j!} h\left(\frac{j}{\mathfrak{m}}\right) a^j, \quad \mathfrak{u} \geq 0,$$

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where $(\kappa)_j = \kappa(\kappa+1)(\kappa+2)\dots(\kappa+j-1), j \geq 1$ and $(\kappa)_0 = 1$, $h : [0,\infty) \to \mathbb{R}$. Agratini [**1**] obtained from the Lupas operators the following operators

$$\mathfrak{L}_{\mathfrak{m}}(\mathfrak{h};\mathfrak{u}) = \frac{1}{2^{\mathfrak{m}\mathfrak{u}}}\sum_{j=0}^{\infty}\frac{(\mathfrak{m}\mathfrak{u})_j}{2^j j!}h\left(\frac{j}{\mathfrak{m}}\right), \quad \mathfrak{u} \geq 0. \tag{1}$$

To approximate the Lebesgue integrable functions and study their approximation properties, Agritini gave the following Kantorovich type operators

$$\mathcal{K}_{\mathfrak{m}}(\mathfrak{h};\mathfrak{u}) = \frac{\mathfrak{m}}{2^{\mathfrak{m}\mathfrak{u}}}\sum_{j=0}^{\infty}\frac{(\mathfrak{m}\mathfrak{u})_j}{2^j j!}\int_{\frac{j}{\mathfrak{m}}}^{\frac{j+1}{\mathfrak{m}}}\mathfrak{h}\left(\frac{j}{\mathfrak{m}}\right), \quad \mathfrak{u} \geq 0. \tag{2}$$

More results on Kantorovich type operators can be found in ([**12**],[**18**], [**21**], [**23**], [**24**]) and blended form of operators in ([**4**], [**8**], [**19**]). In literature Stancu type generalizations of a number of useful operators have been constructed and studied, *e.g.*, in ([**2**], [**10**],[**13**], [**15**], [**20**]). Taqseer et al [**14**] obtained the following Stancu type generalization of Lupaş operators(1):

$$\mathfrak{L}_{m}^{\gamma,\eta}(\mathfrak{h};\mathfrak{u}) = \frac{1}{2^{\mathfrak{m}\mathfrak{u}}}\sum_{j=0}^{\infty}\frac{(\mathfrak{m}\mathfrak{u})_j}{2^j j!}\mathfrak{h}\left(\frac{j+\gamma}{\mathfrak{m}+\eta}\right), \tag{3}$$

where $h : [0,\infty) \to \mathbb{R}$, $0 \leq \gamma \leq \eta$ and investigated approximating properties for these operators. The aim of this paper is to introduce the blended variant of Kantorovich -Stancu type Lupaş operators defined in (3) and establish approximation results for them.

$$\mathfrak{K}_{\mathfrak{m},\delta}^{*\gamma,\eta}(\mathfrak{h};\mathfrak{u}) = \frac{1}{2^{\mathfrak{m}\mathfrak{u}}}\sum_{j=0}^{\infty}\frac{(\mathfrak{m}\mathfrak{u})_j}{2^j j!}\int_0^1\mathfrak{h}\left(\frac{j+\gamma+s^{\delta}}{\mathfrak{m}+\eta}\right)ds, \quad u \geq 0, \tag{4}$$

where

$$\mathfrak{h} : [0,\infty) \to \mathbb{R}, \ 0 \leq \gamma \leq \eta.$$

The rest of the paper is structured as follows. In Section 2, the proposed generalization is constructed and certain estimates are obtained. The first, second and third moments are derived. Convergence behaviour in weighted space is studied in Section 3. Main results are embodied in Section 4. In Section 5, $\mathfrak{q}$-statistical convergence is explored by using Korovkin theorem. Graphical estimation is presented in Section 6.

2. AUXILIARY RESULTS

To establish main results of this paper, the following lemma is essential.

Lemma 1. *For* $\xi_i(\mathfrak{t}) = \mathfrak{t}^{\mathfrak{i}}$, $(\mathfrak{i} = 0,1,2,3)$, *the following estimates are obtained*

$$\begin{aligned}
(i)\ \mathfrak{K}^{*\gamma,\eta}_{\mathfrak{m},\delta}(\xi_0;\mathfrak{u}) &= 1\\
(ii)\ \mathfrak{K}^{*\gamma,\eta}_{\mathfrak{m},\delta}(\xi_1;\mathfrak{u}) &= \mathfrak{u} + \frac{1}{(\delta+1)(\mathfrak{m}+\eta)}\\
(iii)\ \mathfrak{K}^{*\gamma,\eta}_{\mathfrak{m},\delta}(\xi_2;\mathfrak{u}) &= \frac{\mathfrak{m}^2}{(\mathfrak{m}+\eta)^2}\mathfrak{u}^2 + \frac{2[(1+\delta)(1+\gamma)\mathfrak{m} + (\mathfrak{m}+\eta)]}{(\delta+1)(\mathfrak{m}+\eta)^2}\mathfrak{u}\\
&+ \frac{1+\gamma(2\delta+1)}{(2\delta+1)(\mathfrak{m}+\eta)^2}
\end{aligned}$$

Proof. Simple calculation yields (i).
(ii) We have

$$\begin{aligned}
\mathfrak{K}^{*\gamma,\eta}_{\mathfrak{m},\delta}(\mathfrak{t};\mathfrak{u}) &= \frac{1}{2^{\mathfrak{m}\mathfrak{u}}}\sum_{j=0}^{\infty}\frac{(\mathfrak{m}\mathfrak{u})_j}{2^j j!}\int_0^1 \frac{j+\gamma+s^{\delta}}{\mathfrak{m}+\eta}ds\\
&= \frac{1}{2^{\mathfrak{m}\mathfrak{u}}}\sum_{j=0}^{\infty}\frac{(\mathfrak{m}\mathfrak{u})_j}{2^j j!}\frac{(j+\gamma)}{(\mathfrak{m}+\eta)} + \frac{1}{2^{\mathfrak{m}\mathfrak{u}}(\mathfrak{m}+\eta)(\delta+1)}\sum_{j=0}^{\infty}\frac{(\mathfrak{m}\mathfrak{u})_j}{2^j j!}\\
&= \mathfrak{u} + \frac{1}{(\mathfrak{m}+\eta)(\delta+1)}
\end{aligned}$$

(iii) One obtains

$$\begin{aligned}
\mathfrak{K}^{*\gamma,\eta}_{\mathfrak{m},\delta}(\mathfrak{t}^2;\mathfrak{u}) &= \frac{1}{2^{\mathfrak{m}\mathfrak{u}}}\sum_{j=0}^{\infty}\frac{(\mathfrak{m}\mathfrak{u})_j}{2^j j!}\int_0^1 \left(\frac{j+\gamma+s^{\delta}}{\mathfrak{m}+\eta}\right)^2 ds\\
&= \frac{1}{2^{\mathfrak{m}\mathfrak{u}}}\sum_{j=0}^{\infty}\frac{(\mathfrak{m}\mathfrak{u})_j}{2^j j!}\left(\frac{j+\gamma}{\mathfrak{m}+\eta}\right)^2 + \frac{1}{2^{\mathfrak{m}\mathfrak{u}}}\sum_{j=0}^{\infty}\frac{(\mathfrak{m}\mathfrak{u})_j}{2^j j!}\frac{2(j+\gamma)}{(\delta+1)(\mathfrak{m}+\eta)^2}\\
&+ \frac{1}{2^{\mathfrak{m}\mathfrak{u}}}\sum_{j=0}^{\infty}\frac{(\mathfrak{m}\mathfrak{u})_j}{2^j j!}\frac{1}{(2\delta+1)(\mathfrak{m}+\eta)^2}\\
&= \frac{1}{2^{\mathfrak{m}\mathfrak{u}}(\mathfrak{m}+\eta)^2}\sum_{j=0}^{\infty}\frac{(\mathfrak{m}\mathfrak{u})_j}{2^j j!}\left(\frac{j+\gamma}{\mathfrak{m}+\eta}\right)^2\\
&+ \frac{1}{2^{\mathfrak{m}\mathfrak{u}-1}(\mathfrak{m}+\eta)(\delta+1)}\sum_{j=0}^{\infty}\frac{(\mathfrak{m}\mathfrak{u})_j}{2^j j!}\left(\frac{j+\gamma}{\mathfrak{m}+\eta}\right) + \frac{1}{(\mathfrak{m}+\eta)^2(\delta+1)}\\
&= \frac{\mathfrak{m}^2}{(\mathfrak{m}+\eta)^2}\mathfrak{u}^2 + \frac{2((1+\delta)(1+\gamma)\mathfrak{m} + \mathfrak{m}+\eta)}{(\delta+1)(\mathfrak{m}+\eta)}\mathfrak{u}\\
&+ \frac{1+\gamma(2\delta+1)}{(2\delta+1)(\mathfrak{m}+\eta)^2}
\end{aligned}$$

And hence the lemma is proved completely. □

In the following lemma, we obtain moments for the operators defined in (4).

Lemma 2. *The first and the second moments for the operators* (4) *are derived as*

$$(i)\ \mathfrak{K}^{*\gamma,\eta}_{\mathfrak{m},\delta}((\mathfrak{t}-\mathfrak{u});\mathfrak{u}) = \frac{1}{(\delta+1)(\mathfrak{m}+\eta)}$$

$$(ii)\ \mathfrak{K}^{*\gamma,\eta}_{\mathfrak{m},\delta}((\mathfrak{t}-\mathfrak{u})^2;\mathfrak{u}) = \left(\left(\frac{\mathfrak{m}}{\mathfrak{m}+\eta}\right)^2-1\right)\mathfrak{u}^2+\frac{2\mathfrak{m}(1+\gamma)}{(\mathfrak{m}+\eta)^2}\mathfrak{u}+\frac{1+\gamma(2\delta+1)}{(2\delta+1)(\mathfrak{m}+\eta)^2}$$

Proof. (i) is straightly followed by linearity of operators(4).

Using Lemma 1 and linearity, one obtains

$$\begin{aligned}\mathfrak{K}^{*\gamma,\eta}_{\mathfrak{m},\delta}((\mathfrak{t}-\mathfrak{u})^2;\mathfrak{u}) &= \mathfrak{K}^{*\gamma,\eta}_{\mathfrak{m},\delta}(\mathfrak{t}^2;\mathfrak{u})-2\mathfrak{u}\mathfrak{K}^{*\gamma,\eta}_{\mathfrak{m},\delta}(\mathfrak{t};\mathfrak{u})+\mathfrak{u}^2\mathfrak{K}^{*\gamma,\eta}_{\mathfrak{m},\delta}(1;\mathfrak{u})\\ &=\left(\frac{\mathfrak{m}}{\mathfrak{m}+\eta}\right)^2\mathfrak{u}^2+\frac{2[(1+\delta)(1+\gamma)\mathfrak{m}+(\mathfrak{m}+\eta)]}{(\delta+1)(\mathfrak{m}+\eta)^2}\mathfrak{u}\\ &+\frac{1+\gamma(2\delta+1)}{(2\delta+1)(\mathfrak{m}+\eta)^2}-\frac{2\mathfrak{u}}{(1+\delta)(\mathfrak{m}+\eta)}-\mathfrak{u}^2\\ &=\left(\left(\frac{\mathfrak{m}}{\mathfrak{m}+\eta}\right)^2-1\right)\mathfrak{u}^2\\ &+\frac{2[(1+\delta)(1+\gamma)\mathfrak{m}+(\mathfrak{m}+\eta)-(\mathfrak{m}+\eta)]}{(\delta+1)(\mathfrak{m}+\eta)^2}\mathfrak{u}+\frac{1+\gamma(2\delta+1)}{(2\delta+1)(\mathfrak{m}+\eta)^2}\\ &=\left(\left(\frac{\mathfrak{m}}{\mathfrak{m}+\eta}\right)^2-1\right)\mathfrak{u}^2+\frac{2\mathfrak{m}(1+\gamma)}{(\mathfrak{m}+\eta)^2}\mathfrak{u}+\frac{1+\gamma(2\delta+1)}{(2\delta+1)(\mathfrak{m}+\eta)^2},\end{aligned}$$

proving (ii) and, hence, the lemma. □

Lemma 3. *For any* $\mathfrak{h}\in C[0,\infty)$*, we obtain the following bound*

$$|\mathfrak{K}^{*\gamma,\eta}_{\mathfrak{m},\delta}(\mathfrak{h};\mathfrak{u})|\leq\|\mathfrak{h}\|.$$

Proof. From the definition of $\mathfrak{K}^{*\gamma,\eta}_{\mathfrak{m},\delta}$ given in (4), we have

$$\begin{aligned}||\mathfrak{K}^{*\gamma,\eta}_{\mathfrak{m},\delta}(\mathfrak{h};\mathfrak{u})| &\leq \frac{1}{2^{\mathfrak{m}\mathfrak{u}}}\sum_{j=0}^{\infty}\frac{(mu)_j}{2^j j!}\int_0^1\left|\mathfrak{h}\left(\frac{\mathfrak{j}+\gamma+\mathfrak{s}^{\delta}}{\mathfrak{m}+\eta}\right)\right|d\mathfrak{s}\\ &\leq\|\mathfrak{h}\|\frac{1}{2^{\mathfrak{m}\mathfrak{u}}}\sum_{j=0}^{\infty}\frac{(mu)_j}{2^j j!}\int_0^1 d\mathfrak{s}=\|\mathfrak{h}\|\end{aligned}$$

□

3. CONVERGENCE IN WEIGHTED SPACE

In this section, we consider the space $\mathfrak{B}_{\varrho}[0,\infty)=\{\mathfrak{h}\mid \mathfrak{h}:[0,\infty)\to\mathbb{R}\}$ such that $|\mathfrak{h}(\mathfrak{u})|\le \mathcal{L}_{\mathfrak{h}}\varrho(\mathfrak{u})$, where $\mathcal{L}_{\mathfrak{h}}$ is a constant associated with the function $\mathfrak{h}$ and $\varrho(\mathfrak{u})=1+\mathfrak{u}^2$ is a weight function, endowed with the norm

$$\|\mathfrak{h}\|_{\varrho}=\sup_{\mathfrak{u}\in[0,\infty)}\frac{|\mathfrak{h}(\mathfrak{u})|}{\varrho(\mathfrak{u})}$$

and study convergence of operators $\mathfrak{K}^{*\gamma,\eta}_{\mathfrak{m},\delta}(.;\mathfrak{u})$.
Let $\mathcal{C}_{\varrho}[0,\infty)=\{\mathfrak{h}\in\mathcal{B}_{\varrho}[0,\infty):\mathfrak{h}\text{ is continuous on}[0,\infty)\}$, and
$\mathcal{C}^0_{\varrho}[0,\infty)=\left\{\mathfrak{h}\in\mathcal{C}_{\varrho}[0,\infty):\lim_{\mathfrak{u}\to\infty}\frac{|\mathfrak{h}(\mathfrak{u})|}{\varrho(\mathfrak{u})}\text{ is finite}\right\}$.
We prove the following lemma

Lemma 4. *Let $\mathfrak{K}^{*\gamma,\eta}_{\mathfrak{m},\delta}(.;\mathfrak{u})$ be operators defined by (4). Then, for the weight function $\varrho(\mathfrak{u})=1+\mathfrak{u}^2$,*

$$\|\mathfrak{K}^{*\gamma,\eta}_{\mathfrak{m},\delta}(\varrho;\mathfrak{u})\|_{\varrho}\le G,$$

where G is a positive constant greater than 1.

Proof. Using linearity and Lemma 1, we obtain

$$\begin{aligned}
\mathfrak{K}^{*\gamma,\eta}_{\mathfrak{m},\delta}(\varrho;\mathfrak{u}) &= \mathfrak{K}^{*\gamma,\eta}_{\mathfrak{m},\delta}(1+\mathfrak{u}^2;\mathfrak{u})\\
&= \mathfrak{K}^{*\gamma,\eta}_{\mathfrak{m},\delta}(1;\mathfrak{u})+\mathfrak{K}^{*\gamma,\eta}_{\mathfrak{m},\delta}(\mathfrak{u}^2;\mathfrak{u})\\
&= 1+\frac{\mathfrak{m}^2}{(\mathfrak{m}+\eta)^2}\mathfrak{u}^2+\frac{2[(1+\delta)(1+\gamma)\mathfrak{m}+(\mathfrak{m}+\eta)]}{(1+\delta)(\mathfrak{m}+\eta)^2}\mathfrak{u}\\
&+ \frac{1+\gamma(2\delta+1)}{(2\delta+1)(\mathfrak{m}+\eta)^2}.
\end{aligned}$$

Then, $\|\mathfrak{K}^{*\gamma,\eta}_{\mathfrak{m},\delta}(\varrho;\mathfrak{u})\|_{\varrho}$

$$\begin{aligned}
&= \sup_{\mathfrak{u}\ge 0}\left\{\frac{1}{1+\mathfrak{u}^2}+\frac{\mathfrak{m}^2}{(\mathfrak{m}+\eta)^2}\frac{\mathfrak{u}^2}{1+\mathfrak{u}^2}+\frac{2[(1+\delta)(1+\gamma)\mathfrak{m}+(\mathfrak{m}+\eta)]}{(1+\delta)(\mathfrak{m}+\eta)^2}\frac{\mathfrak{u}}{1+\mathfrak{u}^2}\right.\\
&\qquad\left.+\frac{1+\gamma(2\delta+1)}{(2\delta+1)(\mathfrak{m}+\eta)^2}\frac{1}{1+\mathfrak{u}^2}\right\}\\
&< 1+\frac{\mathfrak{m}^2}{(\mathfrak{m}+\eta)^2}+\frac{2[(1+\delta)(1+\gamma)\mathfrak{m}+(\mathfrak{m}+\eta)]}{(1+\delta)(\mathfrak{m}+\eta)^2}+\frac{1+\gamma(2\delta+1)}{(2\delta+1)(\mathfrak{m}+\eta)^2}.
\end{aligned}$$

Noting that

$$\lim_{m\to\infty}\frac{\mathfrak{m}^2}{(\mathfrak{m}+\eta)^2}=1,$$

and
$$\lim_{m\to\infty}\frac{2[(1+\delta)(1+\gamma)\mathfrak{m}+(\mathfrak{m}+\eta)]}{(1+\delta)(\mathfrak{m}+\eta)^2}=0=\lim_{m\to\infty}\frac{1+\gamma(2\delta+1)}{(2\delta+1)(\mathfrak{m}+\eta)^2},$$

we find that there exists a constant $G>1$ such that

$$\|\mathfrak{K}^{*\gamma,\eta}_{\mathfrak{m},\delta}(\varrho;\mathfrak{u})\|_\varrho\le G,$$

which proves the lemma. □

The following theorem is proved.

Theorem 5. *For the operators* $\mathfrak{K}^{*\gamma,\eta}_{\mathfrak{m},\delta}(.;\mathfrak{u})$ *defined in* (2.1), *and for each* $\mathfrak{f}\in C^0_\varrho[0,\infty)$, *we have*
$$\lim_{\mathfrak{m}\to\infty}\|\mathfrak{K}^{*\gamma,\eta}_{\mathfrak{m},\delta}(\mathfrak{f};\mathfrak{u})-\mathfrak{f}(\mathfrak{u})\|_\varrho=0$$
where $\varrho(\mathfrak{u})=1+\mathfrak{u}^2$ *is the weight function.*

Proof. It is sufficient to establish that, for k=0, 1, 2,
$$\lim_{\mathfrak{m}\to\infty}\|\mathfrak{K}^{*\gamma,\eta}_{\mathfrak{m},\delta}(\mathfrak{s}^k;\mathfrak{u})-\mathfrak{u}^k\|_\varrho=0,$$
Using $\mathfrak{K}^{*\gamma,\eta}_{\mathfrak{m},\delta}(1;\mathfrak{u})=1$ from Lemma 1 (i), we get
$$\|\mathfrak{K}^{*\gamma,\eta}_{\mathfrak{m},\delta}(1;\mathfrak{u})-1\|_\varrho=0.$$
From Lemma 1 (ii), we have $\|\mathfrak{K}^{*\gamma,\eta}_{\mathfrak{m},\delta}(\mathfrak{s};\mathfrak{u})-\mathfrak{u}\|_\varrho$
$$=\sup_{\mathfrak{u}\ge0}\left|\frac{1}{(\delta+1)(\mathfrak{m}+\eta)}\right|$$
Hence, we have
$$\lim_{\mathfrak{m}\to\infty}\|\mathfrak{K}^{*\gamma,\eta}_{\mathfrak{m},\delta}(\mathfrak{s};\mathfrak{u})-\mathfrak{u}\|_\varrho=0.$$
Given Lemma 1 (iii), it follows that
$\|\mathfrak{K}^{*\gamma,\eta}_{\mathfrak{m},\delta}(\mathfrak{s}^2;\mathfrak{u})-\mathfrak{s}^2\|_\varrho$
$$\begin{aligned}
&= \sup_{\mathfrak{u}\ge0}\left|\left(\left(\frac{\mathfrak{m}}{\mathfrak{m}+\eta}\right)^2-1\right)\frac{\mathfrak{u}^2}{1+\mathfrak{u}^2}+\frac{2\mathfrak{m}(1+\gamma)}{(\mathfrak{m}+\eta)^2}\frac{\mathfrak{u}}{1+\mathfrak{u}^2}\right.\\
&+\left.\frac{1+\gamma(2\delta+1)}{(2\delta+1)(\mathfrak{m}+\eta)^2}\frac{1}{1+\mathfrak{u}^2}\right|\\
&\le \left(\left(\frac{\mathfrak{m}}{\mathfrak{m}+\eta}\right)^2-1\right)+\frac{2\mathfrak{m}(1+\gamma)}{(\mathfrak{m}+\eta)^2}+\frac{1+\gamma(2\delta+1)}{(2\delta+1)(\mathfrak{m}+\eta)^2}\to0\ \ as\ \ \mathfrak{m}\to\infty,
\end{aligned}$$
and this results in
$$\lim_{m\to\infty}\|\mathfrak{K}^{*\gamma,\eta}_{\mathfrak{m},\delta}(\mathfrak{s}^2;\mathfrak{u})-\mathfrak{s}^2\|_\varrho=0,$$
and this concludes the proof in context of [**7**]. □

4. MAIN RESULTS

In the section, rate of convergence is computed for the operators $\mathfrak{K}^{*\gamma,\eta}_{\mathfrak{m},\delta}(,;\mathfrak{u})$ defined in (4) to the function $\mathfrak{h}$ in respect of Lipschitz class, modulus of continuity and Peetre's $\mathcal{K}$-functional.
Let $\lambda > 0$, $\mathfrak{C}^2_{\mathfrak{B}}[0,\infty) = \{\mathfrak{f} \in \mathfrak{C}_{\mathfrak{B}}[0,\infty);\ \mathfrak{f}',\ \mathfrak{f}'' \in \mathfrak{C}_{\mathfrak{B}}[0,\infty)\}$.
We determine the order of approximating operators $\mathfrak{K}^{*\gamma,\eta}_{\mathfrak{m},\delta}(.;\mathfrak{u})$ for the Lipschitz class. Let $\mathfrak{g} \in \mathfrak{C}[0,\infty)$ and $0 < \varsigma \leq 1$. We know that $\mathfrak{f}$ belongs to $Lip_{\mathfrak{M}}(\varsigma)$, when

$$|\mathfrak{g}(\mathfrak{u}) - \mathfrak{g}(\mathfrak{v})| \leq \mathfrak{M}|\mathfrak{u} - \mathfrak{v}|^{\varsigma}; \quad for\ all\ \mathfrak{u}, \mathfrak{v} \in [0,\infty). \tag{5}$$

We prove the following theorem

Theorem 6. *For all $\mathfrak{g} \in Lip_{\mathfrak{M}}(\varsigma)$, we have*

$$\|\mathfrak{K}^{*\gamma,\eta}_{\mathfrak{m},\delta}(\mathfrak{g};\mathfrak{u}) - \mathfrak{g}(\mathfrak{u})\|_{\mathfrak{C}[0,\infty)} \leq \mathfrak{M}\lambda^{\varsigma}_{\mathfrak{m}}$$

where

$$\lambda_{\mathfrak{m}} = \left(\left(\left(\frac{\mathfrak{m}}{\mathfrak{m}+\eta} \right)^2 - 1 \right) + \frac{2\mathfrak{m}(1+\gamma)}{(\mathfrak{m}+\eta)^2} + \frac{1+\gamma(2\delta+1)}{(2\delta+1)(\mathfrak{m}+\eta)^2} \right)^{\frac{1}{2}}$$

and $\mathfrak{M} > 0$ is a constant.

Proof. Let $\mathfrak{g} \in Lip_{\mathfrak{M}}(\varsigma)$ and $0 < \varsigma \leq 1$. By utilizing the linearity and monotonicity of the operators $\mathfrak{K}^{*\gamma,\eta}_{\mathfrak{m},\delta}(.;\mathfrak{u})$ and (5), we have

$$\begin{aligned} |\mathfrak{K}^{*\gamma,\eta}_{\mathfrak{m},\delta}(\mathfrak{g};\mathfrak{u}) - \mathfrak{g}(\mathfrak{u})| &\leq \mathfrak{K}^{*\gamma,\eta}_{\mathfrak{m},\delta}(|\mathfrak{g}(t) - \mathfrak{g}(\mathfrak{u})|;\mathfrak{u}) \\ &\leq \mathfrak{M}\mathfrak{K}^{*\gamma,\eta}_{\mathfrak{m},\delta}(|\mathfrak{t} - \mathfrak{u}|^{\varsigma};\mathfrak{u}) \end{aligned}$$

On applying Hölder's inequality for the parameters $\mathfrak{p} = \frac{2}{\varsigma}$ and $\mathfrak{q} = \frac{2}{2-\varsigma}$, we get

$$|\mathfrak{K}^{*\gamma,\eta}_{\mathfrak{m},\delta}(\mathfrak{g};\mathfrak{u}) - \mathfrak{g}(\mathfrak{u}| \leq \mathfrak{M}(\mathfrak{K}^{*\gamma,\eta}_{\mathfrak{m},\delta}(|\mathfrak{t} - \mathfrak{u}|^2;\mathfrak{u}))^{\frac{\varsigma}{2}}$$

On taking the maximum on both the sides for $\mathfrak{u} \in [0,\infty)$, we obtain

$$\|\mathfrak{K}^{*\gamma,\eta}_{\mathfrak{m},\delta}\mathfrak{g};\mathfrak{u}) - \mathfrak{g}(\mathfrak{u}\| \leq \mathfrak{M}(\max_{\mathfrak{u}} \mathfrak{K}^{*\gamma,\eta}_{\mathfrak{m},\delta}(|\mathfrak{t} - \mathfrak{u}|^2;\mathfrak{u}))^{\frac{\varsigma}{2}}$$

if

$$\lambda_{\mathfrak{m}} = \left(\left(\left(\frac{\mathfrak{m}}{\mathfrak{m}+\eta} \right)^2 - 1 \right) + \frac{2\mathfrak{m}(1+\gamma)}{(\mathfrak{m}+\eta)^2} + \frac{1+\gamma(2\delta+1)}{(2\delta+1)(\mathfrak{m}+\eta)^2} \right)^{\frac{1}{2}},$$

Taking into account the Lemma 2 (ii), the proof is accomplished. ∎

Next, the rate of convergence of our positive and linear operators $\mathfrak{K}^{*\gamma,\eta}_{\mathfrak{m},\delta}(.;\mathfrak{u})$ will be studied using the following Peetre's $\mathcal{K}$-functional for $\mathfrak{g} \in \mathfrak{C}_{\mathfrak{B}}[0,\infty]$ is defined as

$$\mathcal{L}_2(\mathfrak{g},\lambda) = inf\{\|\mathfrak{g}-\mathfrak{f}\| \; + \; \lambda\|\mathfrak{f}''\| : \mathfrak{f} \in \mathfrak{C}^2_{\mathfrak{B}}[0,\infty)\}. \tag{6}$$

Then there exists a positive constant $\mathfrak{D}$ such that

$$\mathcal{L}_2(\mathfrak{g},\lambda) \leq \mathfrak{D}\Omega_2(h,\sqrt{\lambda}), \tag{7}$$

and the second order modulus of continuity is described as

$$\Omega_2(\mathfrak{g};\sqrt{\lambda}) = \sup_{0\leq p\leq\sqrt{\lambda}} \sup_{\mathfrak{u}\in[0,\infty)} \{|\mathfrak{g}(\mathfrak{u}+2p) - 2\mathfrak{g}(\mathfrak{u}+p) + \mathfrak{g}(\mathfrak{u})|\}. \tag{8}$$

For $\mathfrak{g} \in \mathfrak{C}^2_{\mathfrak{B}}[0,\infty)$, the modulus of continuity in usual form is defined as

$$\Omega(\mathfrak{g};\lambda) = \sup_{0\leq p\leq\sqrt{\lambda}} \sup_{\mathfrak{u}\in[0,\infty)} \{|\mathfrak{g}(\mathfrak{u}+p) - \mathfrak{g}(\mathfrak{u})|\}.$$

$\Omega(\mathfrak{g};\delta)$ also satisfies the following properties
(1) $\lim_{\lambda\to 0} \Omega(\mathfrak{g};\lambda) \; = \; 0,$
(2) $|\mathfrak{g}(\mathfrak{u}) - \mathfrak{g}(\mathfrak{v})| \leq \Omega(\mathfrak{g};\lambda)\left(1 + \frac{(\mathfrak{u}-\mathfrak{v})^2}{\lambda^2}\right)$
Next, we have

Theorem 7. *Let $\mathfrak{g} \in \mathcal{C}[0,\infty)$ and $\mathfrak{v} \in [0,\infty)$, then the following bound holds*

$$|\mathfrak{K}^{*\gamma,\eta}_{\mathfrak{m},\delta}(\mathfrak{g};\mathfrak{u}) - \mathfrak{g}(\mathfrak{v})| \leq 2\Omega(\mathfrak{g};\sqrt{\lambda^{*\gamma,\eta}_{\mathfrak{m},\delta}(\mathfrak{v})}).$$

Proof. Given the linearity and positivity of these operators, along with the characteristic (2) of $\Omega(h;\delta)$, we get

$$\begin{aligned}|\mathfrak{K}^{*\gamma,\eta}_{\mathfrak{m},\delta}(\mathfrak{g};\mathfrak{u}) - \mathfrak{g}(\mathfrak{v})| &\leq \mathfrak{K}^{*\gamma,\eta}_{\mathfrak{m},\delta}(|\mathfrak{g}(\mathfrak{u}) - \mathfrak{g}(\mathfrak{v})|;\mathfrak{v}) \\ &\leq \Omega(\mathfrak{g};\lambda)\left(1 + \frac{1}{\lambda^2}\mathfrak{K}^{*\gamma,\eta}_{\mathfrak{m},\delta}((\mathfrak{u}-\mathfrak{v})^2;\mathfrak{v})\right).\end{aligned}$$

The desired outcome is simply achieved by choosing
$\lambda^2 \; = \; \lambda^{*\gamma,\eta}_{\mathfrak{m},\delta} \; = \; \mathfrak{K}^{*\gamma,\eta}_{\mathfrak{m},\delta}((\mathfrak{u}-\mathfrak{v})^2;\mathfrak{v}).$ □

Theorem 8. *Let $\mathfrak{g} \in \mathfrak{C}[0,\infty)$. Then, there exists a positive constant $\mathfrak{D}$ such that*

$$|\mathfrak{K}^{*\gamma,\eta}_{\mathfrak{m},\delta}(\mathfrak{g};\mathfrak{u}) - \mathfrak{g}(\mathfrak{u})| \leq \mathfrak{D}\Omega_2(\mathfrak{g},\sqrt{\varphi^{*(\gamma,\eta)}_{\mathfrak{m},\delta}(\mathfrak{u})}) + \Omega\left(\mathfrak{g};\frac{\gamma-\eta\mathfrak{u}}{\mathfrak{m}+\eta} + \frac{1}{2(\mathfrak{m}+\eta)}\right)$$

where

$$\varphi^{*(\gamma,\eta)}_{\mathfrak{m},\delta}(u) = \mathfrak{K}^{*\gamma,\eta}_{\mathfrak{m},\delta}((\mathfrak{t}-\mathfrak{u})^2;\mathfrak{u}) + \left(\frac{1}{(\delta+1)(\mathfrak{m}+\eta)}\right)^2$$

Proof. For $u \in [0,\infty)$, consider the auxiliary operators $\overline{\mathfrak{K}}_{\mathfrak{m},\delta}^{*\gamma,\eta}$ defined as

$$\overline{\mathfrak{K}}_{\mathfrak{m},\delta}^{*\gamma,\eta}(\mathfrak{g};\mathfrak{u}) = \mathfrak{K}_{\mathfrak{m},\delta}^{*\gamma,\eta}(\mathfrak{g};\mathfrak{u}) + \mathfrak{g}(\mathfrak{u}) - \mathfrak{g}(\mathfrak{K}_{\mathfrak{m},\delta}^{*\gamma,\eta}(\mathfrak{t};\mathfrak{u}))$$

Using the construction and linearity of $\overline{\mathfrak{K}}_{\mathfrak{m},\delta}^{*\gamma,\eta}$, we have $\overline{\mathfrak{K}}_{\mathfrak{m},\delta}^{*\gamma,\eta}(1;\mathfrak{u}) = 1$ and $\overline{\mathfrak{K}}_{m,\delta}^{*\gamma,\eta}(\mathfrak{t};\mathfrak{u}) = \mathfrak{u}$.
Let $\mathfrak{l} \in \mathfrak{C}_{\mathfrak{B}}^2[0,\infty)$ and using Taylor's expansion of $\mathfrak{l}$ at $\mathfrak{s} = \mathfrak{u}$, we have

$$\mathfrak{l}(\mathfrak{s}) = \mathfrak{l}(\mathfrak{u}) + (\mathfrak{s}-\mathfrak{u})\mathfrak{l}''(\mathfrak{u}) + \int_{\mathfrak{u}}^{\mathfrak{s}} \frac{(\mathfrak{s}-\mathfrak{u})^2}{2}\mathfrak{l}''(\mathfrak{u})d\mathfrak{u}$$

On applying $\overline{\mathfrak{K}}_{\mathfrak{m},\delta}^{*\gamma,\eta}(,;\mathfrak{u})$ both the sides, which following

$$\begin{aligned}
|\overline{\mathfrak{K}}_{\mathfrak{m},\delta}^{*\gamma,\eta}(\mathfrak{l};\mathfrak{u}) - \mathfrak{l}(\mathfrak{u})| &\le \mathfrak{K}_{\mathfrak{m},\delta}^{*\gamma,\eta}\left(\left|\int_{\mathfrak{u}}^{\mathfrak{s}}(\mathfrak{s}-\mathfrak{v})\mathfrak{f}''(\mathfrak{v})d\mathfrak{v}\right|;\mathfrak{u}\right)\\
&+\left|\int_{\mathfrak{u}}^{\frac{1}{(\delta+1)(\mathfrak{m}+\eta)}}\left|\left(\frac{1}{(\delta+1)(\mathfrak{m}+\eta)}-\mathfrak{v}\right)\right||\mathfrak{l}''(\mathfrak{v})|d\mathfrak{v}\right|\\
&\le \mathfrak{K}_{\mathfrak{m},\delta}^{*\gamma,\eta}((\mathfrak{s}-\mathfrak{u})^2;\mathfrak{u})\|\mathfrak{l}''\| + \left(\frac{1}{(\delta+1)(\mathfrak{m}+\eta)}\right)^2\|\mathfrak{l}''\|\\
&= \varphi_m^{*(\gamma,\eta)}(\mathfrak{u})\|\mathfrak{l}''\|
\end{aligned}$$

From (6), we have

$$|\overline{\mathcal{K}}_m^{*\gamma,\eta}(\mathfrak{g};\mathfrak{u})| \le 3\|\mathfrak{g}\|$$

On using (8) and (7) in (6, we obtain for $\mathfrak{f} \in \mathfrak{C}_{\mathfrak{B}}^2[0,\infty)$

$$\begin{aligned}
|\mathfrak{K}_{\mathfrak{m},\delta}^{*\gamma,\eta}(\mathfrak{g};\mathfrak{u}) - \mathfrak{g}(\mathfrak{u})| &\le |\overline{\mathfrak{K}}_{m,\delta}^{*\gamma,\eta}(\mathfrak{g}-\mathfrak{f};\mathfrak{u})| + |\overline{\mathfrak{K}}_{m,\delta}^{*\gamma,\eta}(\mathfrak{f};\mathfrak{u}) - \mathfrak{f}(\mathfrak{u})|\\
&+ |\mathfrak{f}(\mathfrak{u}) - \mathfrak{g}(\mathfrak{u})| + \left|\mathfrak{g}\left(\frac{1}{(\delta+1)(\mathfrak{m}+\eta)}\right) - \mathfrak{g}(\mathfrak{u})\right|\\
&\le 4\|\mathfrak{g}-\mathfrak{f}\| + |\overline{\mathfrak{K}}_{\mathfrak{m},\delta}^{*\gamma,\eta}(\mathfrak{f};\mathfrak{u}) - \mathfrak{f}(\mathfrak{u})|\\
&+ \left|\mathfrak{g}\left(\frac{1}{(\delta+1)(\mathfrak{m}+\eta)}\right) - \mathfrak{g}(\mathfrak{u})\right|\\
&\le 4\|\mathfrak{g}-\mathfrak{f}\| + \varphi_{\mathfrak{m},\delta}^{*(\gamma,\eta)}\|\mathfrak{f}''\| + \Omega\left(\mathfrak{g};\frac{u}{(\delta+1)(\mathfrak{m}+\eta)}\right)
\end{aligned}$$

By taking the infimum of the right side over all $f \in \mathcal{C}_B^2[0,\infty)$, we get

$$|\mathfrak{K}_{\mathfrak{m},\delta}^{*\gamma,\eta}(\mathfrak{g};\mathfrak{u}) - \mathfrak{g}(\mathfrak{u})| \le 4\Omega_2(h;\varphi_{\mathfrak{m},\delta}^{*(\gamma,\eta)}(\mathfrak{u})) + \Omega\left(\mathfrak{g};\frac{\mathfrak{u}}{(\delta+1)(\mathfrak{m}+\eta)}\right)$$

Thus by using (7), we get

$$|\mathfrak{K}_{\mathfrak{m},\delta}^{*\gamma,\eta}(\mathfrak{g};\mathfrak{u}) - \mathfrak{g}(\mathfrak{u})| \le \mathfrak{D}(\mathfrak{g};\sqrt{\varphi_{\mathfrak{m},\delta}^{*(\gamma,\eta)}(\mathfrak{u})}) + \Omega\left(\mathfrak{g};\frac{\mathfrak{u}}{(\delta+1)(\mathfrak{m}+\eta)}\right)$$

This concludes the proof. ∎

Lemma 9. *For the operators defined in (4), we also have*

(a) $\lim_{\mathfrak{m}\to\infty} \mathfrak{m}(\mathfrak{K}^{*\gamma,\eta}_{\mathfrak{m},\delta}((\mathfrak{t}-\mathfrak{u});\mathfrak{u})) = \frac{1}{\delta+1}$

(b) $\lim_{\mathfrak{m}\to\infty} \mathfrak{m}(\mathfrak{K}^{*\gamma,\eta}_{\mathfrak{m},\delta}((\mathfrak{t}-\mathfrak{u})^2;\mathfrak{u})) = 1+\gamma.$

Theorem 10. *Let* $\mathfrak{g}''$ *exists at a point* $\mathfrak{u}\in[0,\infty)$ *for* $\mathfrak{g}\in\mathfrak{C}[0,\infty)$. *Then the following holds for the operators* $\mathfrak{K}^{*\gamma,\eta}_{\mathfrak{m},\delta}(.;\mathfrak{u})$

$$\lim_{\mathfrak{m}\to\infty} \mathfrak{m}(\mathfrak{K}^{*\gamma,\eta}_{\mathfrak{m},\delta}(\mathfrak{g};\mathfrak{u})-\mathfrak{g}(\mathfrak{u})) = \frac{1}{1+\delta}\mathfrak{g}'(\mathfrak{u})+\mathfrak{g}''(\mathfrak{u}).$$

Proof. The Taylor's expansion for the $\mathfrak{g}$ is

$$\mathfrak{g}(\mathfrak{s}) = \mathfrak{g}(\mathfrak{u})+(\mathfrak{s}-\mathfrak{u})\mathfrak{g}'(\mathfrak{u})+\frac{1}{2}(\mathfrak{s}-\mathfrak{u})^2\ \mathfrak{g}''(\mathfrak{u})+\varphi(\mathfrak{s},\mathfrak{u})(\mathfrak{s}-\mathfrak{u})^2, \tag{9}$$

where $\varphi(\mathfrak{s},\mathfrak{u})$ represents the remainder and $\varphi(\mathfrak{s},\mathfrak{u})\to 0$ as $\mathfrak{s}\to\mathfrak{u}$.
On applying the operators $\mathfrak{K}^{*\gamma,\eta}_{\mathfrak{m},\delta}(.;\mathfrak{u})$ on both sides of (9), we get

$$\begin{aligned}\mathfrak{K}^{*\gamma,\eta}_{\mathfrak{m},\delta}(\mathfrak{g};\mathfrak{u})-\mathfrak{g}(\mathfrak{u}) &= \mathfrak{K}^{*\gamma,\eta}_{\mathfrak{m},\delta}((\mathfrak{s}-\mathfrak{u});\mathfrak{u})\mathfrak{g}'(\mathfrak{u})\\ &+ \frac{1}{2}\,\mathfrak{K}^{*\gamma,\eta}_{\mathfrak{m},\delta}((\mathfrak{s}-\mathfrak{u})^2;\mathfrak{u})\mathfrak{g}''(\mathfrak{u})+\mathfrak{K}^{*\gamma,\eta}_{\mathfrak{m},\delta}(\varphi(\mathfrak{s},\mathfrak{u})(\mathfrak{s}-\mathfrak{u})^2;\mathfrak{u}).\end{aligned}$$

By applying the Cauchy-Schwarz inequality and Lemma 2 (ii), we obtain

$$\mathfrak{K}^{*\gamma,\eta}_{\mathfrak{m},\delta}\big(\varphi(\mathfrak{s},\mathfrak{u})(\mathfrak{s}-\mathfrak{u})^2;\mathfrak{u}\big) \le \big(\mathfrak{K}^{*\gamma,\eta}_{\mathfrak{m},\delta}(\varphi^2(\mathfrak{s},\mathfrak{u});\mathfrak{u})\big)^{\frac{1}{2}}\big(\mathfrak{K}^{*\gamma,\eta}_{\mathfrak{m},\delta}((\mathfrak{s}-\mathfrak{u})^4;\mathfrak{u})\big)^{\frac{1}{2}}.$$

Since $\mathfrak{K}^{*\gamma,\eta}_{\mathfrak{m},\delta}(\mathfrak{g};\mathfrak{u})\to\mathfrak{g}(\mathfrak{u})$, we get

$$\begin{aligned}\lim_{\mathfrak{m}\to\infty}\ \mathfrak{K}^{*\gamma,\eta}_{\mathfrak{m},\delta}(\varphi^2(\mathfrak{s},\mathfrak{u});\mathfrak{u}) &= \varphi^2(\mathfrak{u},\mathfrak{u})=0,\\ \lim_{\mathfrak{m}\to\infty} \mathfrak{m}\mathfrak{K}^{*\gamma,\eta}_{\mathfrak{m},\delta}(\varphi^2(\mathfrak{s},\mathfrak{u});\mathfrak{u}) &= 0.\end{aligned}$$

Combining the above limits using the Lemma 9, it follows that

$$\begin{aligned}\lim_{\mathfrak{m}\to\infty} \mathfrak{m}(\mathfrak{K}^{*\gamma,\eta}_{\mathfrak{m},\delta}(\mathfrak{g};\mathfrak{u})-\mathfrak{g}(\mathfrak{u})) &= \lim_{\mathfrak{m}\to\infty} \mathfrak{m}(\mathfrak{K}^{*\gamma,\eta}_{\mathfrak{m},\delta}((\mathfrak{s}-\mathfrak{u});\mathfrak{u}))\\ &+ \lim_{\mathfrak{m}\to\infty} \mathfrak{m}(\mathfrak{K}^{*\gamma,\eta}_{\mathfrak{m},\delta}((\mathfrak{s}-\mathfrak{u})^2;\mathfrak{u}))\\ &+ \lim_{\mathfrak{m}\to\infty} \mathfrak{m}(\mathfrak{K}^{*\gamma,\eta}_{\mathfrak{m},\delta}(\varphi(\mathfrak{s},\mathfrak{u})(\mathfrak{s}-\mathfrak{u})^2;\mathfrak{u}))\\ &= \frac{1}{\delta+1}\mathfrak{g}'(u)+(1+\gamma)\mathfrak{g}''(u)\end{aligned}$$

This proves the theorem. □

5. $\mathfrak{Q}$-STATISTICAL CONVERGENCE

The $\mathfrak{q}$-statistical convergence of our operators $\mathfrak{K}^{*\gamma,\eta}_{\mathfrak{m},\delta}(.;\mathfrak{u})$ has been proved in this section. The $\mathfrak{q}$-statistical convergence was presented in [**6**] and extensivly used since then e.g. ([**5**], [**9**], [**15**], and [**22**]). First, let us go over few terminology from $\mathfrak{q}$-calculus. Let $\mathfrak{r}$ any non-negative integer, then the $\mathfrak{q}$-analog of the number $\mathfrak{r}$ is defined as

$$[\mathfrak{r}]_{\mathfrak{q}} = \begin{cases} \frac{1-\mathfrak{q}^{\mathfrak{r}}}{1-\mathfrak{q}} & \text{if } \mathfrak{q} \neq 1 \\ \mathfrak{r} & \text{if } \mathfrak{q} = 1, \end{cases}$$

where $\mathfrak{q} > 0$ is a positive real number.
The $\mathfrak{q}$-analog of factorial is characterized as:

$$[\mathfrak{r}]_{\mathfrak{q}}! = \begin{cases} [1]_{\mathfrak{q}}[2]_{\mathfrak{q}} \cdots [\mathfrak{r}]_{\mathfrak{q}} & \text{if } \mathfrak{r} = 1, 2, \dots \\ 1 & \text{if } \mathfrak{r} = 0. \end{cases}$$

and

$$(1+\mathfrak{v})^{\mathfrak{r}}_{\mathfrak{q}} := \begin{cases} \prod_{l=0}^{\mathfrak{s}-1}\left(1+\mathfrak{q}^{l}\mathfrak{v}\right) & \mathfrak{r} = 1, 2, \dots \\ 1 & \mathfrak{r} = 0. \end{cases}$$

If l is a non-negative interger such that $l \leq \mathfrak{r}$, then the $\mathfrak{q}$-analog of the binomial coefficient is defined as

$$\begin{bmatrix} \mathfrak{r} \\ l \end{bmatrix}_{\mathfrak{q}} = \frac{[\mathfrak{r}]!}{[l]![\mathfrak{r}-l]!}.$$

The $\mathfrak{q}$-integration in the interval $[0, c]$ was defined by Jackson in [7] as

$$\int_0^c \mathfrak{h}(\mathfrak{u}) d_{\mathfrak{q}}\mathfrak{u} := c(1-\mathfrak{q}) \sum_{m=0}^{\infty} \mathfrak{h}\left(c\mathfrak{q}^m\right)\mathfrak{q}^m, 0 < \mathfrak{q} < 1 \text{ and } c > 0.$$

The $\mathfrak{q}$-analog of *Cesàro* matrix $\mathfrak{A}_1$ is defined by

$$\mathfrak{A}_1(\mathfrak{q}) = (\mathfrak{a}^1_{\mathfrak{m}p}(\mathfrak{q}^p))^{\infty}_{\mathfrak{m},p=0},$$

where $\mathfrak{a}^1_{\mathfrak{m}p}(\mathfrak{q}^p)$ is regular for $\mathfrak{q} \geq 1$ and is defined as

$$\mathfrak{a}^1_{\mathfrak{m}p}(\mathfrak{q}^p) = \left\{ \begin{array}{cc} \frac{\mathfrak{q}^p}{[\mathfrak{m}+1]_{\mathfrak{q}}}, & \text{if } p \leq \mathfrak{m} \\ 0, & \text{otherwise} \end{array} \right\}.$$

For a subset $\mathfrak{M}$ of $\mathbb{N}$, the asymptotic density is defined by

$$\vartheta(\mathfrak{M}) = \lim_{\mathfrak{m}} \frac{1}{\mathfrak{m}} \#\{p \leq \mathfrak{m} : p \in \mathfrak{M}\},$$

where the cardinality of the set is represented by $\#$.
Also, the $\mathfrak{q}$-density is described as

$$\begin{aligned} \upsilon_{\mathfrak{q}}(\mathfrak{M}) = \upsilon_{\mathfrak{A}^{\mathfrak{q}}_1}(\mathfrak{M}) &= \lim\inf_{\mathfrak{m}\to\infty} (\mathfrak{A}^{\mathfrak{q}}_{1}\chi_{\mathfrak{M}})_{\mathfrak{m}} \\ &= \lim\inf_{\mathfrak{m}\to\infty} \sum_{j\in\mathfrak{M}} \frac{\mathfrak{q}^{j-1}}{[\mathfrak{m}]}, \mathfrak{q} \geq 1 \end{aligned}$$

The $\mathfrak{q}$-statistical convergence is defined as follows.
A sequence $\zeta = (\zeta_n)$ is said to be $\mathfrak{q}$-statistical convergent to $\mathfrak{r}$, if $\zeta_q(\mathfrak{M}_\varepsilon) = 0$, for every $\varepsilon > 0$, and $\mathfrak{M}_\varepsilon$ is defined as $\mathfrak{M}_\varepsilon = \{n \leq k : |\zeta_n - \mathfrak{r}| > \varepsilon\}$. Moreover, it is denoted as $st_{\mathfrak{q}} - \lim \zeta_n = \mathfrak{r}$. For each $\varepsilon > 0$,

$$\lim_k \frac{1}{[k]} \#\left\{n \leq k : \mathfrak{q}^{n-1}|\xi_n - \mathfrak{r}| \geq \varepsilon\right\} = 0.$$

For an infinite set $\mathfrak{M}$, $v(\mathfrak{M}) = 0$ implies $v_q(\mathfrak{M}) = 0$. Hence, statistical convergence always implies $\mathfrak{q}$-statistical convergence but the converse is not true. (see [**3**], Example 15). We have

Theorem 11. *For all* $\mathfrak{g} \in \mathfrak{C}^0_\varrho[0,\infty]$, *we have*

$$st_{\mathfrak{q}} - \lim_{\mathfrak{m}} \|\mathfrak{K}^{*\gamma,\eta}_{\mathfrak{m},\delta}(\mathfrak{g};\mathfrak{u}) - \mathfrak{g}(\mathfrak{u})\|_\varrho = 0, \ \mathfrak{u} \in [0,\infty).$$

Proof. To establish the theorem, it is sufficient to demonstrate that

$$st_q - \lim_m \|\mathfrak{K}^{*\gamma,\eta}_{\mathfrak{m},\delta}(\mathfrak{e}_j; u) - \mathfrak{e}_j\|_\varrho = 0, \quad for\ j = 0, 1\ and\ 2$$

Since $\mathfrak{K}^{*\gamma,\eta}_{\mathfrak{m},\delta}(1;\mathfrak{u}) = 1$ then $st_{\mathfrak{q}} - \lim_{\mathfrak{m}} \|\mathfrak{K}^{*\gamma,\eta}_{\mathfrak{m},\delta}(\mathfrak{e}_0; u) - \mathfrak{e}_0\|_\varrho = 0$.
By Lemma 1, we have

$$\begin{aligned}
\|\mathfrak{K}^{*\gamma,\eta}_{\mathfrak{m},\delta}(\mathfrak{e}_1;\mathfrak{u}) - \mathfrak{u}\|_\varrho &= \sup_{\mathfrak{u}\in[0,\infty)} \frac{1}{1+\mathfrak{u}^2}\left|\frac{1}{(\delta+1)(\mathfrak{m}+\eta)}\right| \\
&\leq \frac{1}{(\delta+1)(\mathfrak{m}+\eta)} \sup_{u\in[0,\infty)} \frac{u}{1+u^2} \\
&\leq \frac{1}{(\delta+1)(\mathfrak{m}+\eta)}.
\end{aligned}$$

We now consider the subsequent sets for a given $\varepsilon > 0$:

$$\begin{aligned}
\mathfrak{B}_1 : &= \left\{\mathfrak{m} : \|\mathfrak{K}^{*\gamma,\eta}_{\mathfrak{m},\delta}(\mathfrak{e}_1;\mathfrak{u}) - \mathfrak{u}\| \geq \varepsilon\right\}, \\
\mathfrak{B}_2 : &= \left\{\mathfrak{m} : \left|\frac{1}{(\delta+1)(\mathfrak{m}+\eta)}\right| \geq \varepsilon\right\}.
\end{aligned}$$

This follows that $\mathfrak{B}_1 \subseteq \mathfrak{B}_2$, which clearly shows that $v_{\mathfrak{q}}(\mathfrak{B}_1) \leq v_{\mathfrak{q}}(\mathfrak{B}_2)$.
Letting the limit as $\mathfrak{m} \to \infty$, we establish

$$st_{\mathfrak{q}} - \lim_{\mathfrak{m}} \|\mathfrak{K}^{*\gamma,\eta}_{\mathfrak{m},\delta}(\mathfrak{e}_1;\mathfrak{u}) - \mathfrak{u}\|_\varrho = 0.$$

By Lemma 1, we have

$$\begin{aligned}
\|\mathfrak{K}^{*\gamma,\eta}_{\mathfrak{m},\delta}(\mathfrak{e}_2;u)-\mathfrak{u}^2\|_{\varrho} &= \sup_{\mathfrak{u}\in[0,\infty)}\frac{1}{1+\mathfrak{u}^2}\left|\left(\left(\frac{\mathfrak{m}}{(\mathfrak{m}+\eta)}\right)^2-1\right)\mathfrak{u}^2\right.\\
&+ \left.\frac{2\mathfrak{m}(1+\gamma)}{(\mathfrak{m}+\eta)^2}\mathfrak{u}+\frac{1+\gamma(2\delta+1)}{(2\delta+1)(\mathfrak{m}+\eta)^2}\right|\\
&\leq \left|\left(\frac{\mathfrak{m}}{(\mathfrak{m}+\eta)}\right)^2-1\right|\sup_{\mathfrak{u}\in[0,\infty)}\frac{\mathfrak{u}^2}{1+\mathfrak{u}^2}\\
&+ \frac{2\mathfrak{m}(1+\gamma)}{(\mathfrak{m}+\eta)^2}\sup_{\mathfrak{u}\in[0,\infty)}\frac{\mathfrak{u}}{1+\mathfrak{u}^2}\\
&+ \frac{1+\gamma(2\delta+1)}{(2\delta+1)(\mathfrak{m}+\eta)^2}\frac{1}{1+\mathfrak{u}^2}.
\end{aligned}$$

We consider the sets $\mathcal{S}_1$, $\mathcal{S}_2$ and $\mathcal{S}_3$ for a given $\varepsilon>0$ as follows,

$$\begin{aligned}
\mathcal{S}_1: &= \left\{\mathfrak{m}:\|\mathfrak{K}^{*\gamma,\eta}_{\mathfrak{m},\delta}(\mathfrak{e}_2;\mathfrak{u})-\mathfrak{u}^2\|\geq\varepsilon\right\},\\
\mathcal{S}_2: &= \left\{\mathfrak{m}:\left|\left(\frac{\mathfrak{m}}{(\mathfrak{m}+\eta)}\right)^2-1\right|\geq\frac{\varepsilon}{3}\right\},\\
\mathcal{S}_3: &= \left\{m:\frac{2\mathfrak{m}(1+\gamma)}{(\mathfrak{m}+\eta)^2}\geq\frac{\varepsilon}{3}\right\},\\
\mathcal{S}_4: &= \left\{\mathfrak{m}:\frac{1+\gamma(2\delta+1)}{(2\delta+1)(\mathfrak{m}+\eta)^2}\geq\frac{\varepsilon}{3}\right\}.
\end{aligned}$$

This follows that $\mathcal{S}_1\subseteq\mathcal{S}_2\cup\mathcal{S}_3\cup\mathcal{S}_4$, which implies that
$\upsilon_{\mathfrak{q}}(\mathcal{S}_1)\leq\upsilon_{\mathfrak{q}}(\mathcal{S}_2)+\upsilon_{\mathfrak{q}}(\mathcal{S}_3)+\upsilon_{\mathfrak{q}}(\mathcal{S}_4)$.
By taking the limit as $\mathfrak{m}\to\infty$, we have

$$st_{\mathfrak{q}}-\lim_{\mathfrak{m}}\|\mathfrak{K}^{*\gamma,\eta}_{\mathfrak{m},\delta}(\mathfrak{e}_2;\mathfrak{u})-\mathfrak{u}^2\|_{\varrho}=0.$$

This concludes the proof of the theorem. □

6. GRAPHICAL ANALYSIS AND ERROR ESTIMATION

To validate theoretical conclusions using various parameter values, we compare error estimations using MATLAB and illustrate approximation for our operators (4) through extensive numerical examples with graphical representations in the following section. The advantage of these operators is that they offer approximation methods in domains of integrable functions on $[0,\infty)$ as well. It also gives approximation operators more flexibility.

Example 12. *We illustrate the graph of the operators* $\mathfrak{h}(\mathfrak{u}) = 2\mathfrak{u}^4 - 9\mathfrak{u}^3 + 5\mathfrak{u}^2 - 12\mathfrak{u}$ *for the values of the parameters* $\gamma = 2,\ \delta = 2$ *and* $\eta = 3$ *and* $\mathfrak{m} \in \{5,\ 10,\ 15,\ 25,\ 35\}$ *in Figure* 1*.The absolute error* $E^{*\gamma,\eta}_{\mathfrak{m},\delta}(\mathfrak{h};\mathfrak{u}) = |\mathfrak{K}^{*\gamma,\eta}_{\mathfrak{m},\delta}(\mathfrak{h};\mathfrak{u}) - \mathfrak{h}(\mathfrak{u})|$ *of the operators with the function* $\mathfrak{h}(\mathfrak{u})$ *is depicted in Figure* 2*. The absolute errors at different points is shown in Table* 1*.*

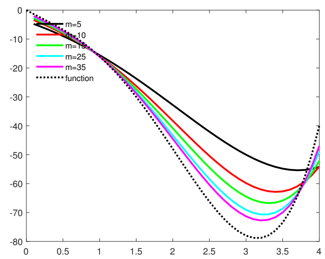

Figure 1: Convergence of $\mathfrak{K}^{*\gamma,\eta}_{\mathfrak{m}.\delta}(h;\mathfrak{u})$ to $h(\mathfrak{u}) = 2\mathfrak{u}^4 - 9\mathfrak{u}^3 + 5\mathfrak{u}^2 - 12\mathfrak{u}$

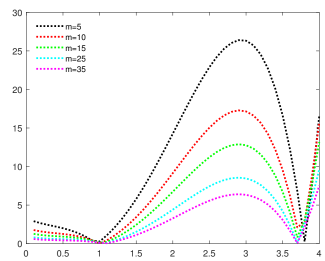

Figure 2: Absolute Error of approximation of $\mathfrak{K}^{*\gamma,\eta}_{\mathfrak{m},\delta}(h;\mathfrak{u})$ with $h(\mathfrak{u}) = 2\mathfrak{u}^4 - 9\mathfrak{u}^3 + 23\mathfrak{u}^2 - 15\mathfrak{u}$.

Table 1: Absolute Error of approximation by $\mathcal{K}^{*\gamma,\eta}_{m,\delta}(h;u)$ for $h(u) = \mathfrak{u}^4 - 9\mathfrak{u}^3 + 23\mathfrak{u}^2 - 15\mathfrak{u}$ for $\gamma = 2,\ \eta = 2$ *and* $\delta = 2$

u	m=5	m=10	m=15	m=20	m=25
0.4	2.2160	1.3513	0.9655	0.6118	0.4470
0.8	0.9550	0.7462	0.5909	0.4112	0.3141
1.2	2.0472	1.0315	0.6623	0.3725	0.2548
1.6	7.2993	4.4421	3.1861	2.0322	1.4910
2.0	14.2693	9.1473	6.7366	4.4129	3.2814
2.4	21.3833	14.0099	10.4337	6.9118	5.1684
2.8	26.0267	17.0946	12.7610	8.4781	6.3499
3.2	24.5431	15.6673	11.5658	7.6133	5.6795
3.6	12.2351	6.1953	4.0594	2.3707	1.6663
4.0	16.6362	15.6525	13.1834	9.6438	7.5253

Example 13. *Next, we consider the graph of the function* $\mathfrak{h}(\mathfrak{u}) = -3sin(0.8\mathfrak{u})$ *for the parameters* $\gamma = 2,\ \delta = 3$ *and* $\eta = 3$ *and keeping the values of* $\mathfrak{m} = 4, 7, 10$ *and* 14*. Then the graph of the approximation by the operators(4) and the absolute error estimates are plotted in Figures* 3 *and* 4 *respectively. Table 2 incorporates absolute errors at uniform values for the function.*

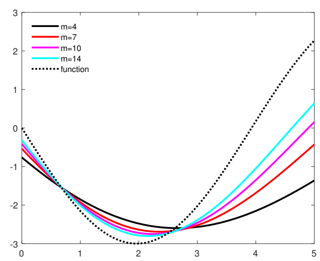

Figure 3: Convergence of $\mathfrak{K}^{*\gamma,\eta}_{\mathfrak{m},\delta}(\mathfrak{h};\mathfrak{u})$ to $\mathfrak{h}(\mathfrak{u}) = -3sin(0.8\mathfrak{u})$.

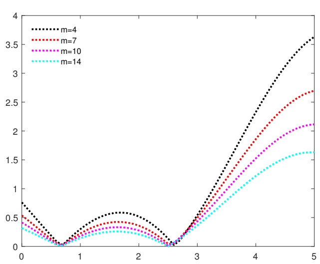

Figure 4: Absolute Error of approximation of $\mathfrak{K}^{*\gamma,\eta}_{\mathfrak{m},\delta}(h;\mathfrak{u})$ with $h(\mathfrak{u}) = -3sin(0.8\mathfrak{u})$.

Table 2: Absolute Error of approximation of $\mathfrak{K}^{*\gamma,\eta}_{\mathfrak{m},\delta}(h;\mathfrak{u})$ with $h(\mathfrak{u}) = -3sin(0.8\mathfrak{u})$ for $\gamma = 2,\ \eta = 3$ *and* $\delta = 3$.

u	m=4	m=7	m=10	m=14
0.5	0.1883	0.0401	0.0279	0.0190
1.0	0.2992	0.1486	0.0967	0.0634
1.5	0.5646	0.0596	0.0373	0.0239
2.0	0.5193	0.3980	0.2538	0.1644
2.5	0.1370	0.6800	0.4325	0.2797
3.0	0.5406	0.7191	0.4531	0.2914
3.5	1.4108	0.3289	0.1954	0.1211
4.0	2.3264	0.6772	0.4609	0.3096
4.5	3.1200	2.4858	1.6362	1.0792
5.0	3.6322	5.2832	3.4507	2.2662

Example 14. *Figure 5 depicts the approximation action of the operators (4) for* $\mathfrak{h}(\mathfrak{u}) = \mathfrak{u}^3 - 6\mathfrak{u}^2 + 9\mathfrak{u}$, $\gamma = 2$, $\delta = 3$ *and* $\eta = 3$ *for the* $\mathfrak{m} = 5, 10, 15, 25$ *and 35. The graph of the errors is diplayed in Figure 6. In Table 3, absolute errors at numerous points is presented.*

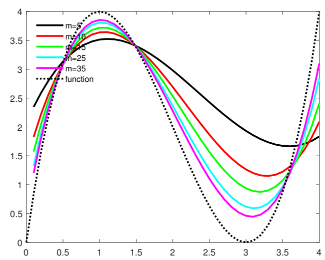

Figure 5: Convergence of $\mathfrak{K}^{*\gamma,\eta}_{\mathfrak{m},\delta}(\mathfrak{h};\mathfrak{u})$ to $\mathfrak{h}(\mathfrak{u}) = \mathfrak{u}^3 - 6\mathfrak{u}^2 + 9\mathfrak{u}$.

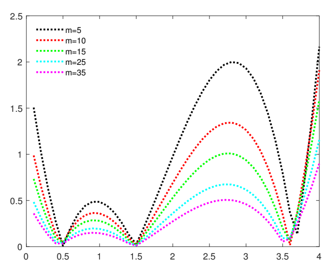

Figure 6: Absolute Error of approximation of $\mathfrak{K}^{*\gamma,\eta}_{\mathfrak{m},\delta}(h;\mathfrak{u})$ with $\mathfrak{h}(\mathfrak{u}) = \mathfrak{u}^3 - 6\mathfrak{u}^2 + 9\mathfrak{u}$.

Table 3: Absolute Error of approximation of $\mathfrak{K}^{*\gamma,\eta}_{\mathfrak{m},\delta}(\mathfrak{h};\mathfrak{u})$ with $\mathfrak{h}(\mathfrak{u}) = \mathfrak{u}^3 - 6\mathfrak{u}^2 + 9\mathfrak{u}$ for $\gamma = 2,\ \eta = 3$ *and* $\delta = 3$.

u	m=5	m=10	m=15	m=20	m=25
0.5	0.2764	0.1402	0.0906	0.0515	0.0355
1.0	0.4407	0.3397	0.2683	0.1866	0.1425
1.5	0.3691	0.2684	0.2067	0.1404	0.1061
2.0	0.2010	0.1448	0.1136	0.0792	0.0608
2.5	0.9793	0.6908	0.5309	0.3618	0.2740
3.0	1.6755	1.1603	0.8834	0.5964	0.4498
3.5	1.9995	1.3441	1.0093	0.6726	0.5041
4.0	1.6610	1.0330	0.7468	0.4796	0.3530
4.5	0.3697	0.0178	0.0658	0.0933	0.0875
5.0	2.1647	1.9107	1.5903	1.1567	0.9013

Example 15. *In this example, a comparison of the approximation of our operators and the existing operators are exibits for the function* $\mathfrak{h}(\mathfrak{u}) = \mathfrak{u}^5 - 10\mathfrak{u}^4 + 35\mathfrak{u}^3 - 50\mathfrak{u}^2 + 24\mathfrak{u}$, *and* $m = 20$ *with* $\gamma = 2, \delta = 1$ *and* $\eta = 3$. *Figure* 7 *demonstrates convergence equivalence of our operators* $\mathfrak{K}^{*\gamma,\eta}_{\mathfrak{m},\delta}(\mathfrak{h};\mathfrak{u})$ *(4) with the operators* $\mathcal{K}_{\mathfrak{m}}(\mathfrak{h};\mathfrak{u})$ *in (2) and the operators* $\mathcal{L}_m(\mathfrak{h};\mathfrak{u})$ *in (1) for* $\mathfrak{h}(\mathfrak{u})$. *In Table* 4, *the absolute errors are also reckoned and represented graphically in the Figure* 8. *The above figures clearly indicate that operators in (4) provide finer approximation than* $\mathcal{L}_m(,;\mathfrak{u})$ *and* $\mathcal{K}_{\mathfrak{m}}(,;\mathfrak{u})$.

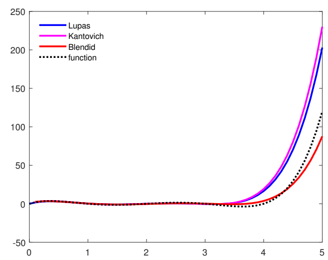

Figure 7: Comparison of convergence of approximation by operators $\mathfrak{K}^{*\gamma,\eta}_{\mathfrak{m},\delta}(\mathfrak{h};\mathfrak{u})$ with $\mathcal{L}_{\mathfrak{m}}(,;\mathfrak{u})$ and $\mathcal{K}_{\mathfrak{m}}(,;\mathfrak{u})$.

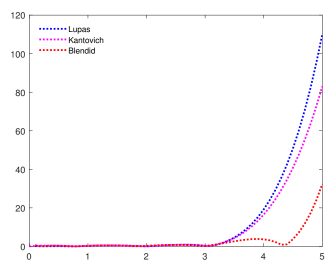

Figure 8: Comparison of Absolute Error of approximation by operators $\mathfrak{K}^{*\gamma,\eta}_{\mathfrak{m},\delta}(\mathfrak{h};\mathfrak{u})$ with $\mathcal{L}_{\mathfrak{m}}(,;\mathfrak{u})$ and $\mathcal{K}_{\mathfrak{m}}(,;\mathfrak{u})$.

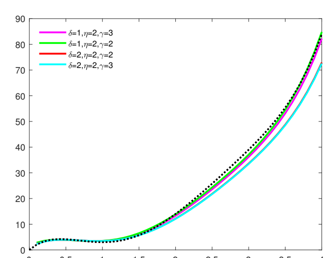

Figure 9: Convergence of $\mathfrak{K}^{*\gamma,\eta}_{\mathfrak{m},\delta}(h;\mathfrak{u})$ to $h(\mathfrak{u}) = \mathfrak{u}^5 - 10\mathfrak{u}^4 + 36\mathfrak{u}^3 - 49\mathfrak{u}^2 + 25\mathfrak{u}$.

Table 4: Comparison of Absolute Error of approximation by operators $\mathfrak{K}^{*\gamma,\eta}_{\mathfrak{m},\delta}(h;\mathfrak{u})$ with $\mathcal{L}_m(,;\mathfrak{u})$ and $\mathcal{K}_{\mathfrak{m}}(,;\mathfrak{u})$ for $h(\mathfrak{u}) = \mathfrak{u}^5 - 10\mathfrak{u}^4 + 35\mathfrak{u}^3 - 50\mathfrak{u}^2 + 24\mathfrak{u}$.

u	m=20		
	$\mathcal{L}_m$	$\mathcal{K}_{\mathfrak{m}}$	$\mathfrak{K}^{*\gamma,\eta}_{\mathfrak{m},\delta}$
0.5	0.1061	0.3441	0.3892
1.0	0.2383	0.2998	0.3347
1.5	0.4438	0.5317	0.4714
2.0	0.0116	0.0485	0.3019
2.5	0.7873	0.8407	0.8086
3.0	0.3053	0.2450	0.2056
3.5	4.6250	4.3387	2.6507
4.0	19.1816	16.5104	3.6716
4.5	50.9051	40.8700	3.7740
5.0	110.0736	83.0176	32.3369

Additionally, the convergence of the operators $\mathfrak{K}^{*\gamma,\eta}_{\mathfrak{m},\delta}(h;\mathfrak{u})$ to $h(\mathfrak{u}) = \mathfrak{u}^5 - 10\mathfrak{u}^4 + 36\mathfrak{u}^3 - 49\mathfrak{u}^2 + 25\mathfrak{u}$ for various combination of parameters η, γ and δ are depicted graphically in Figure 9. and the absolute errors are shown in Figure 10. Also, the absolute errors are organized the absulte errors at various points in Table 5.

Table 5: Absolute Error of approximation of $\mathfrak{K}^{*\gamma,\eta}_{\mathfrak{m},\delta}(h;\mathfrak{u})$ with $h(\mathfrak{u}) = \mathfrak{u}^5 - 10\mathfrak{u}^4 + 36\mathfrak{u}^3 - 49\mathfrak{u}^2 + 25\mathfrak{u}$ for different values of δ, η and γ.

u	$\delta = 1, \eta = 2$ $\gamma = 3$	$\delta = 1, \eta = 2$ $\gamma = 2$	$\delta = 2, \eta = 2$ $\gamma = 1$	$\delta = 2, \eta = 2$ $\gamma = 2$
0.4	0.2368	0.2442	0.2477	0.2366
0.8	0.1308	0.1632	0.1694	0.1357
1.2	0.9373	0.7803	0.7561	0.9098
1.6	1.1021	0.6998	0.6347	1.0337
2.0	0.4824	0.1224	0.2218	0.3806
2.4	0.4664	1.2061	1.3285	0.5903
2.8	1.0292	1.8857	2.0273	1.1727
3.2	0.5703	1.6490	1.8263	0.7518
3.6	1.1278	0.4765	0.7379	0.8564
4.0	3.5252	0.8204	0.3813	3.0665

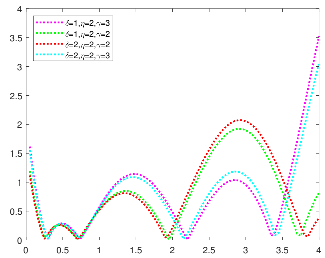

Figure 10: Absolute Error of approximation of $\mathfrak{K}^{*\gamma,\eta}_{\mathfrak{m},\delta}(h;\mathfrak{u})$ with $h(\mathfrak{u}) = \mathfrak{u}^5 - 10\mathfrak{u}^4 + 36\mathfrak{u}^3 - 49\mathfrak{u}^2 + 25\mathfrak{u}$.

7. CONCLUSIONS

A blended type generalization of Kantorovich-Stancu type Lupaş operators are constructed and their approximation properties have been studied. Convergence theorems are proved using Korovkin theorem. Convergence is explored in weighted space. Sharper results have been established.

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