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VORONOVSKAJA TYPE RESULTS FOR THE ALDAZ-KOUNCHEV-RENDER VERSIONS OF GENERALIZED BASKAKOV OPERATORS

Ana-Maria Acu, Margareta Heilmann, Ioan Raşa and Ancuţa Emilia Şteopoaie*

The modification of Bernstein operators given by Aldaz, Kounchev and Render has been intensively studied in the recent years. In this paper we define a corresponding modification for the general class of Baskakov-type operators. All these operators preserve the constants and j-th monomial for a given natural number j. We prove a general result of Voronovskaja type (Theorem 2.1) for positive linear operators acting on the infinite interval $[0,\infty)$ and use this theorem to prove a Voronovskaja-type theorem for the whole class of the modified Baskakov-type operators.

1. INTRODUCTION

In 1957 Baskakov [**5**] introduced a general method for the construction of positive linear operators depending on a real parameter c. It is well-known that all these operators preserve the linear functions.

Let $c \in \mathbb{R}$, $n \in \mathbb{R}$, $n > c$ for $c \geq 0$ and $-n/c \in \mathbb{N}$ for $c < 0$. Furthermore let $I_c = [0,\infty)$ for $c \geq 0$ and $I_c = [0, -1/c]$ for $c < 0$. Take $f : I_c \longrightarrow \mathbb{R}$ given in such a way that the corresponding integrals and series are convergent.

The Baskakov-type operators are defined by

$$B_n^{[c]}(f;x) = \sum_{k=0}^{\infty} p_{n,k}^{[c]}(x) f\left(\frac{k}{n}\right), \tag{1}$$

*Corresponding author. Ana-Maria Acu
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with the corresponding basis functions

$$p_{n,k}^{[c]}(x) = \begin{cases} \dfrac{n^k}{k!} x^k e^{-nx} & , c = 0, \\ \dfrac{n^{c,\overline{k}}}{k!} x^k (1+cx)^{-\left(\frac{n}{c}+k\right)} & , c \neq 0, \end{cases} \tag{2}$$

and $a^{c,\overline{k}} := \prod_{l=0}^{k-1}(a+cl), \quad a^{c,\overline{0}} := 1$.

Note that the sum in (1) is finite in case $c < 0$. The most significant cases are for $c = -1$ the Bernstein operators, the Szász-Mirakjan operators for $c = 0$ and the Baskakov operators for $c = 1$.

Starting from the classical Bernstein operators $B_n^{[-1]}$ defined on $C[0,1]$, several modifications have been considered with the aim to obtain operators preserving some prescribed functions. J.P. King [**13**] constructed linear positive operators which preserve the functions e_0 and e_2. Here and in what follows we use the notation $e_j(t) = t^j$, $j = 0, 1 \ldots$, $t \in I_c$.

A slight extension of King operators was considered by Cárdenas-Morales et al. [**7**], where a sequence of operators $B_{n,\alpha}$ that preserve e_0 and $e_2 + \alpha e_1$, $\alpha \in [0, +\infty)$ was introduced. Cárdenas-Morales et al. (see [**8**], [**12**]) introduced a modification of the Bernstein operator which preserves the functions e_0 and τ, where τ is a continuous strictly increasing function defined on $[0,1]$ with $\tau(0) = 0$ and $\tau(1) = 1$, $\tau'(x) > 0, x \in [0,1]$.

For a fixed $j \in \mathbb{N}$ and $n \geq j$, Aldaz, Kounchev and Render [**2**] introduced a polynomial operator $B_{n,j}^{[-1]} : C[0,1] \to C[0,1]$ preserving e_0 and e_j. The operator is a linear combination of the classical Berstein basis functions $p_{n,k}^{[-1]}$ but using

$$t_{n,k,j} = \left(\prod_{l=0}^{j-1} \frac{k-l}{n-l} \right)^{1/j}$$

for the point evaluations of the function f, i.e.,

$$B_{n,j}^{[-1]}(f;x) = \sum_{k=0}^{n} p_{n,k}^{[-1]}(x) f\left(t_{n,k,j}\right). \tag{3}$$

Several papers were subsequently devoted to their study. In particular, the Voronovskaja formula conjectured in [**9**] was proved in [**6**] (see also [**11**], [**10**], [**1**], where other properties of the operators can be found).

In the following we introduce the corresponding modification for the class of Baskakov-type operators, i.e.,

$$B_{n,j}^{[c]}(f;x) = \sum_{k=0}^{\infty} p_{n,k}^{[c]}(x) f\left(t_{n,k,j}^{[c]}\right), \; j \in \mathbb{N},$$

where

$$t_{n,k,j}^{[c]} = \left(\prod_{l=0}^{j-1} \frac{k-l}{n+cl} \right)^{1/j}.$$

There are several facts motivating the introduction of the new operators.

a) They are constructed using the classical basis functions $p_{n,k}^{[c]}(x)$. These functions were deeply investigated and are well known, which facilitates the study of the new operators. See, e.g., the proof of Lemma 1. The more complicated nodes $t_{n,k,j}^{[c]}$ are responsible for the essential new property, the preservation of e_0 and e_j for an arbitrary $j \in \mathbb{N}$.

b) $B_{n,1}^{[c]} = B_n^{[c]}$, hence the family $(B_{n,j}^{[c]})$ generalizes the family $(B_n^{[c]})$.

c) While $B_n^{[c]}$ preserves the functions e_0 and e_1, $B_{n,j}^{[c]}$ preserves e_0 and e_j. This gives a hint about how to choose the family $(B_{n,j}^{[c]})$ which is suitable to approximate a given function f. For example, suppose that after an initial rough examination we find that $ae_0 + be_l \leq f \leq pe_0 + qe_l$ for some real numbers a, b, p, q, then we will choose the sequence $(B_{n,l}^{[c]})$ for approximating f.

d) According to Remark 4 (iii), the space E_j on which Voronovskaja formula works for $B_{n,j}^{[c]}$ increases with j.

An essential property, the preservation of e_0 and e_j, is presented in the following lemma.

Lemma 1. *Let $j \in \mathbb{N}$. Then*

$$B_{n,j}^{[c]} e_0 = e_0 \text{ and } B_{n,j}^{[c]} e_j = e_j.$$

Proof. The proof follows easily by using $B_n^{[c]} e_0 = e_0$ and

$$\prod_{l=0}^{j-1} \frac{k-l}{n+cl} p_{n,k}^{[c]}(x) = x^j p_{n+cj,k-j}^{[c]}(x). \tag{4}$$

□

The main result of the paper is Theorem 2. It is proved in Section 2. In Section 3 we apply it in order to prove the Voronovskaja formula for the operators $B_{n,j}^{[c]}$.

2. A GENERAL VORONOVSKAJA TYPE RESULT

In this section we prove a general Voronovskaja type result related to [**9**, Theorem 2.1], but here we consider positive linear operators acting on the infinite interval $[0, \infty)$. Our theorem below should be also considered in the context of [**4**, Proposition 2.1] and [**3**, Theorem 1.5.2]

Let $\varphi \in C^2[0,\infty)$, $\varphi(0)=0$, $\varphi'(t)>0$, $t\in(0,\infty)$, $\lim\limits_{t\to\infty}\varphi(t)=\infty$. Denote

$$E_\varphi := \left\{ f \in C[0,\infty) \;\middle|\; \sup_{t\geq 0} \frac{|f(t)|}{1+\varphi^2(t)} < \infty \right\}$$

and $\|f\|_\varphi := \sup\limits_{t\geq 0} \dfrac{|f(t)|}{1+\varphi^2(t)}$, $f \in E_\varphi$.

Theorem 2. *Let $x>0$ be given and let $\Psi_x(t):=\varphi(t)-\varphi(x)$, $t\geq 0$. Denote by E_φ^x a linear subspace of $C[0,\infty)$ such that $E_\varphi \subset E_\varphi^x$ and $\Psi_x^4 \in E_\varphi^x$. Let $L_n : E_\varphi^x \to C[0,\infty)$ be a sequence of positive linear operators such that*

(i) $\lim\limits_{n\to\infty} n\left(L_n(e_0;x)-1\right)=0$,

(ii) $\lim\limits_{n\to\infty} nL_n(\Psi_x;x)=b(x)$,

(iii) $\lim\limits_{n\to\infty} nL_n(\Psi_x^2;x)=2a(x)$,

(iv) $\lim\limits_{n\to\infty} nL_n(\Psi_x^4;x)=0$.

If $f\in E_\varphi$ and there exists $f''(x)\in\mathbb{R}$, then

$$\lim_{n\to\infty} n\left(L_n(f;x)-f(x)\right) = \frac{a(x)}{\varphi'(x)^2} f''(x) + \frac{b(x)\varphi'(x)^2 - a(x)\varphi''(x)}{\varphi'(x)^3} f'(x). \tag{5}$$

Proof. With Taylor's formula we can write for $t\geq 0$

$$\begin{aligned} f(t) &= \left(f\circ\varphi^{-1}\right)(\varphi(t)) = \left(f\circ\varphi^{-1}\right)(\varphi(x)) + D^1\left(f\circ\varphi^{-1}\right)(\varphi(x))\left(\varphi(t)-\varphi(x)\right) \\ &\quad + \frac{1}{2}D^2\left(f\circ\varphi^{-1}\right)(\varphi(x))\left(\varphi(t)-\varphi(x)\right)^2 + h_x(t)\left(\varphi(t)-\varphi(x)\right)^2, \end{aligned}$$

where $h_x\in C[0,+\infty)$, $h_x(x)=0$. This leads to

$$f(t) = f(x) + D^1\left(f\circ\varphi^{-1}\right)(\varphi(x))\Psi_x(t) + \frac{1}{2}D^2\left(f\circ\varphi^{-1}\right)(\varphi(x))\Psi_x^2(t) + h_x(t)\Psi_x^2(t),$$

and so

$$f(t) = f(x) + \frac{f'(x)}{\varphi'(x)}\Psi_x(t) + \frac{f''(x)\varphi'(x)-f'(x)\varphi''(x)}{2\varphi'(x)^3}\Psi_x^2(t) + h_x(t)\Psi_x^2(t). \tag{6}$$

For $t\geq 0$, $t\neq x$ we have

$$h_x(t) = \frac{1}{\Psi_x^2(t)}\left[f(t)-f(x)-\frac{f'(x)}{\varphi'(x)}\Psi_x(t) - \frac{f''(x)\varphi'(x)-f'(x)\varphi''(x)}{2\varphi'(x)^3}\Psi_x^2(t)\right].$$

Consequently,

(7)
$$h_x(t) = \frac{f(t)}{1+\varphi^2(t)}\frac{1+\varphi^2(t)}{(\varphi(t)-\varphi(x))^2} - \frac{f(x)}{\Psi_x^2(t)} - \frac{f'(x)}{\varphi'(x)}\frac{1}{\Psi_x(t)} - \frac{f''(x)\varphi'(x)-f'(x)\varphi''(x)}{2\varphi'(x)^3}.$$

Let us prove that

(8)
$$h_x\Psi_x^2 \in E_\varphi.$$

From (iii) it follows that $\exists A > 0$ such that $nL_n(\Psi_x^2;x) \leq A$, $n \geq 1$.

Let $\varepsilon > 0$. Since $h_x(x) = 0$, there exists $\delta > 0$ such that

(9)
$$|h_x(t)| \leq \frac{\varepsilon}{A+1}, \text{ for all } t \geq 0 \text{ with } |\varphi(t)-\varphi(x)| = |\Psi_x(t)| \leq \delta.$$

Set $s := \varphi(t) \in [0,\infty)$, $y := \varphi(x) \in (0,\infty)$, $q(s) := \dfrac{1+s^2}{(s-y)^2}$. Then $q : [0, y-\delta] \cup [y+\delta,\infty) \to (0,\infty)$ is a continuous function, with $\lim\limits_{s\to\infty} q(s) = 1$. It follows that q is bounded, and consequently

$$M_\delta := \sup_{|\varphi(t)-\varphi(x)|\geq\delta} \frac{1+\varphi^2(t)}{(\varphi(t)-\varphi(x))^2} < \infty.$$

Now suppose that $|\Psi_x(t)| \geq \delta$. Then, by (7),

$$\begin{aligned}|h_x(t)| &\leq \|f\|_\varphi M_\delta + \frac{|f(x)|}{\delta^2} + \frac{|f'(x)|}{\delta\varphi'(x)} + \frac{|f''(x)\varphi'(x)-f'(x)\varphi''(x)|}{2\varphi'(x)^3} =: K_\delta \\ &\leq \frac{K_\delta}{\delta^2}\Psi_x^2(t).\end{aligned}$$

To resume, we have

(10)
$$|h_x(t)| \leq \frac{\varepsilon}{A+1} + K_\delta,\ t \geq 0,$$

(11)
$$|h_x(t)| \leq \frac{\varepsilon}{A+1} + \frac{K_\delta}{\delta^2}\Psi_x^2(t),\ t \geq 0.$$

From (10) we see that

$$\frac{|h_x(t)\Psi_x^2(t)|}{1+\varphi^2(t)} \leq \left(\frac{\varepsilon}{A+1} + K_\delta\right)\frac{(\varphi(t)-\varphi(x))^2}{1+\varphi^2(t)},\ t \geq 0.$$

Since $\lim\limits_{t\to\infty}\dfrac{(\varphi(t)-\varphi(x))^2}{1+\varphi^2(t)} = 1$, we get $\sup\limits_{t\geq 0}\dfrac{|h_x(t)\Psi_x^2(t)|}{1+\varphi^2(t)} < \infty$. This proves (8)

Now using (6)we get

$$\begin{aligned}n\left(L_n(f;x) - f(x)\right) &= n\left(L_n(e_0;x) - 1\right)f(x) + \frac{f'(x)}{\varphi'(x)}nL_n(\Psi_x;x) \\ &+ \frac{f''(x)\varphi'(x)-f'(x)\varphi''(x)}{2\varphi'(x)^3}nL_n(\Psi_x^2;x) + nL_n\left(h_x\Psi_x^2;x\right).\end{aligned}$$

Combined with (i), (ii), (iii), this leads to

$$\lim_{n\to\infty} n\left(L_n(f;x)-f(x)\right)$$

$$=\frac{f'(x)}{\varphi'(x)}b(x)+\frac{f''(x)\varphi'(x)-f'(x)\varphi''(x)}{\varphi'(x)^3}a(x)+\lim_{n\to\infty} nL_n(h_x\Psi_x^2;x).$$

It remains to prove that

$$\lim_{n\to\infty} nL_n\left(h_x\Psi_x^2;x\right)=0. \tag{12}$$

By virtue of (11) we have

$$|h_x\Psi_x^2|\le\frac{\varepsilon}{A+1}\Psi_x^2+\frac{K_\delta}{\delta^2}\Psi_x^4,$$

so that

$$n\left|L_n(h_x\Psi_x^2;x)\right|\le\frac{\varepsilon}{A+1}nL_n\left(\Psi_x^2;x\right)+\frac{K_\delta}{\delta^2}nL_n(\Psi_x^4;x).$$

The relation (iv) shows that $\exists n_\varepsilon$ such that

$$\frac{K_\delta}{\delta^2}nL_n(\Psi_x^4;x)\le\frac{\varepsilon}{A+1},\ n\ge n_\varepsilon.$$

Since $nL_n(\Psi_x^2;x)\le A$, we get

$$n\left|L_n(h_x\Psi_x^2;x)\right|\le\varepsilon,\ n\ge n_\varepsilon,$$

and this proves (12).

□

3. VORONOVSKAJA TYPE RESULTS FOR $B_{n,j}^{[c]}$ OPERATORS

In this section we apply the general Theorem 2 to the operators $B_{n,j}^{[c]}$, $c\ge 0$.

Theorem 3. *Let $c\ge 0$ and $f\in C[0,\infty)$ with $\sup_{t\ge 0}\frac{|f(t)|}{1+t^{2j}}<\infty$. Suppose that $x\in(0,\infty)$ and there exists $f''(x)\in\mathbb{R}$. Then,*

$$\lim_{n\to\infty} n\left(B_{n,j}^{[c]}(f;x)-f(x)\right)=\frac{x(1+cx)}{2}f''(x)-\frac{(j-1)(1+cx)}{2}f'(x).$$

Proof. First we observe that for $i\in\mathbb{N}$

$$\prod_{l=0}^{j-1}(k-l)^i=\sum_{\mu=0}^{ij}C_\mu\prod_{l=0}^{\mu-1}(k-l). \tag{13}$$

The coefficients of k^{ij} in the two sides of (13) are 1, respectively, C_{ij}. The coefficients of k^{ij-1} are $-\dfrac{ij(j-1)}{2}$, respectively, $C_{ij-1}-\dfrac{ij(ij-1)}{2}$. Therefore,

$$C_{ij}=1,\ \ C_{ij-1}=\frac{i(i-1)j^2}{2}.$$

In order to apply Theorem 2 we use $\varphi(t)=t^j$ and $\Psi_x(t)=t^j-x^j$. For $m\in\mathbb{N}$ we have

$$\begin{aligned}
&B_{n,j}^{[c]}\left(\Psi_x^m;x\right)=\sum_{i=0}^{m}\binom{m}{i}(-1)^{m-i}x^{j(m-i)}B_{n,j}^{[c]}(t^{ij};x)\\
&=\sum_{i=0}^{m}\binom{m}{i}(-1)^{m-i}x^{j(m-i)}\sum_{k=0}^{\infty}p_{n,k}^{[c]}(x)\left(\prod_{l=0}^{j-1}\frac{k-l}{n+cl}\right)^i\\
&=\sum_{i=0}^{m}\binom{m}{i}(-1)^{m-i}x^{j(m-i)}\prod_{l=0}^{j-1}(n+cl)^{-i}\sum_{\mu=0}^{ij}C_\mu\prod_{l=0}^{\mu-1}(n+cl)\sum_{k=\mu}^{\infty}p_{n,k}(x)\prod_{l=0}^{\mu-1}\frac{k-l}{n+cl}\\
&=\sum_{i=0}^{m}\binom{m}{i}(-1)^{m-i}x^{j(m-i)}\prod_{l=0}^{j-1}(n+cl)^{-i}\sum_{\mu=0}^{ij}C_\mu\prod_{l=0}^{\mu-1}(n+cl)x^\mu,
\end{aligned}$$

where we used (4).

In order to apply Theorem 2 we only need the leading terms in n, i.e.,

$$B_{n,j}^{[c]}(\Psi_x^m;x)=S_1+S_2+\mathcal{O}\left(n^{-2}\right),$$

where

$$\begin{aligned}
S_1&:=\sum_{i=0}^{m}\binom{m}{i}(-1)^{m-i}x^{jm}\frac{\prod_{l=0}^{ij-1}(n+cl)}{\prod_{l=0}^{j-1}(n+cl)^i},\\
S_2&:=\sum_{i=0}^{m}\binom{m}{i}(-1)^{m-i}x^{jm-1}\frac{j^2i(i-1)}{2}\frac{\prod_{l=0}^{ij-2}(n+cl)}{\prod_{l=0}^{j-1}(n+cl)^i}.
\end{aligned}$$

By elementary calculations we obtain

$$\lim_{n\to\infty}nS_1=cj^2x^{2j},\ \text{ for } m=2, \tag{14}$$

$$\lim_{n\to\infty}nS_1=0,\ \text{ for } m=4, \tag{15}$$

$$\begin{aligned}
\lim_{n\to\infty}nS_2&=\frac{j^2x^{jm-1}}{2}\sum_{i=2}^{m}\binom{m}{i}(-1)^{m-i}i(i-1)\\
&=\frac{j^2x^{jm-1}}{2}m(m-1)\sum_{i=0}^{m-2}\binom{m-2}{i}(-1)^{m-2-i}=0,\ \text{ for } m\neq 2,
\end{aligned} \tag{16}$$

$$\lim_{n\to\infty}nS_2=x^{2j-1}j^2,\ \text{ for } m=2. \tag{17}$$

From Lemma 1 we get

$$\lim_{n\to\infty} n\left(B_{n,j}^{[c]}(e_0;x) - e_0(x)\right) = 0,$$
$$\lim_{n\to\infty} nB_{n,j}^{[c]}(\Psi_x;x) = 0 = b(x),$$

and from (14)-(17) it follows

$$\lim_{n\to\infty} nB_{n,j}^{[c]}(\Psi_x^2;x) = x^{2j-1}j^2(1+cx) = 2a(x),$$
$$\lim_{n\to\infty} nB_{n,j}^{[c]}(\Psi_x^4;x) = 0.$$

Substituting $a(x)$ and $b(x)$ in the Voronovskaja type result of Theorem 2 completes the proof. □

Remark 4. *(i) For $c = 0$ it is also possible to prove a Voronovskaja type result using the technique given by Birou [**6**]. An inequality similar to [**6**, Lemma 1] in this case is given by*

$$\sqrt{k(k-j+1)} \le \left\{\prod_{l=0}^{j-1}(k-l)\right\}^{1/j} \le \frac{2k-j+1}{2},$$

which is valid for all k with $1 \le j \le k$. The corresponding auxiliary operators are given by

$$P_{n,1}(f;x) = \sum_{k=j}^{\infty} p_{n,k}^{[0]} f\left(\sqrt{\frac{k(k-j+1)}{n^2}}\right),$$
$$P_{n,2}(f;x) = \sum_{k=j}^{\infty} p_{n,k}^{[0]} f\left(\frac{2k-j+1}{2n}\right).$$

*(ii) For $c > 0$ it is not possible to use Birou's technique because an analogous inequality to [**6**, Lemma 1] does not hold.*

(iii) Let $E_j = \left\{ f \in C[0,\infty) \,\middle|\, \sup_{t\ge 0} \frac{f(t)|}{1+t^{2j}} < \infty \right\}$. This is the set for which Theorem 3 can be applied. Obviously, for $j_1 < j_2$ we have $E_{j_1} \subset E_{j_2}$. So, the space E_j increases with increasing j.

*(iv) For $c < 0$ one can apply our technique in combination with [**9**, Theorem 2.1] with a slight modification to cover the interval $I_c = [0, -1/c]$.*

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Ana-Maria Acu
Department of Mathematics and Informatics,
Lucian Blaga University of Sibiu,
Sibiu, Romania
E-mail: *anamaria.acu@ulbsibiu.ro*

Margareta Heilmann
School of Mathematics and Natural Sciences,
University of Wuppertal,
Wuppertal, Germany
E-mail: *heilmann@math.uni-wuppertal.de*

Ioan Rasa
Department of Mathematics,
Technical University of Cluj-Napoca,
Cluj-Napoca, Romania
E-mail: *ioan.rasa@math.utcluj.ro*

Ancuţa Emilia Şteopoaie
Department of Mathematics,
Technical University of Cluj-Napoca,
Cluj-Napoca, Romania
E-mail: *ancasteopoaie@yahoo.ro*