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APPLICATIONS OF THE GENERALIZED FUNCTION-TO-SEQUENCE TRANSFORM

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We deal with applications of the transform $\mathcal{T}_\alpha$ we introduced in our paper *On a generalized function-to-sequence transform*, Appl. Anal. Disc. Math. Vol. 14 No 2 (2020) 300-316. Taking different sequences $\{\alpha_n\}_{n\in\mathbb{N}_0}$ linked to a generalized linear difference operator $\mathcal{D}_\alpha$ gives rise to a family of transforms $\mathcal{T}_\alpha$ that enables the mapping of a differential equation and its solutions to a difference equation and its solutions. It can map a differential operator to a difference one as well.

1. INTRODUCTION AND PRELIMINARIES

For for an arbitrary sequence $\{t_n\}_{n\in\mathbb{N}_0}$, we refer to the Newton binomial formula

$$t_n = \sum_{k=0}^{n} \binom{n}{k} \Delta^k t_0 \quad (\Delta t_n = t_{n+1} - t_n), \tag{1}$$

whence, on account of the linearity of the difference operator Δ, there follows

$$\Delta t_n = \sum_{k=0}^{n} \binom{n}{k} \Delta^{k+1} t_0. \tag{2}$$

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By virtue of (1), for $p \in \mathbb{N}_0$, there holds as well

$$t_{n+p} = \sum_{k=0}^{p} \binom{p}{k} \Delta^k t_n = t_n + \binom{p}{1} \Delta t_n + \binom{p}{2} \Delta^2 t_n + \cdots + \Delta^p t_n. \tag{3}$$

Let $\{\alpha_n\}_{n\in\mathbb{N}_0}$ be a sequence of real numbers satisfying $\alpha_0 = 1$, $\alpha_n \neq 0$, used in the definition of a family of linear operators on linear spaces of sequences in [**1**], we multiply by c_k the terms $\binom{n}{k}\Delta^{k+1} t_0$ in the linear difference operator (2), where $c_k = -\frac{\alpha_{k+1}}{\alpha_k}$, to obtain a more general linear difference operator $\mathcal{D}_\alpha$, defined as follows

$$\mathcal{D}_\alpha t_n = \sum_{k=0}^{n} c_k \binom{n}{k} \Delta^{k+1} t_0. \tag{4}$$

Thus, for $c_k = 1$, i.e. $\alpha_k = (-1)^k$, (4) becomes the forward difference operator Δ.

In [**16**] we derived the inverse transform of $\mathcal{D}_\alpha t_n$, i.e.

$$\mathcal{D}_\alpha^{-1} t_n = \sum_{k=1}^{n} \frac{1}{c_{k-1}} \binom{n}{k} \Delta^{k-1} t_0 = \sum_{k=1}^{n} \frac{\alpha_{k-1}}{\alpha_k} \binom{n}{k} \Delta^{k-1} t_0, \tag{5}$$

and the formula

$$\mathcal{D}_\alpha^{m} t_n = (-1)^m \sum_{k=0}^{n} \frac{\alpha_{k+m}}{\alpha_k} \binom{n}{k} \Delta^{k+m} t_0, \quad m \in \mathbb{N}.$$

The binomial transform takes the sequence $\{s_n\}$ to the sequence $\{t_n\}$ via the transformation [**13**]

$$t_n = \sum_{k=0}^{n} \binom{n}{k} s_k,$$

and presents an infinite-dimensional linear operator.

Since $\mathcal{D}_\alpha$ reduces to Δ for $\alpha_k = (-1)^k$, motivated by that important case we introduced in [**16**] the **generalized binomial transform**

$$t_n = \sum_{k=0}^{n} \frac{(-1)^k}{\alpha_k} \binom{n}{k} s_k, \quad T = \begin{pmatrix} \frac{1}{\alpha_0} & 0 & 0 & \dots & 0 \\ \frac{1}{\alpha_0} & -\frac{1}{\alpha_1} & 0 & \dots & 0 \\ \frac{1}{\alpha_0} & -\frac{2}{\alpha_1} & \frac{1}{\alpha_2} & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ \frac{1}{\alpha_0} & -\frac{\binom{n}{1}}{\alpha_1} & \frac{\binom{n}{2}}{\alpha_2} & \dots & \frac{(-1)^n\binom{n}{n}}{\alpha_n} \end{pmatrix}. \tag{6}$$

The transform (6) comprises variations of the binomial transform. They are considered in [**13**] and called the k-binomial transform for $\alpha_j = \dfrac{(-1)^j}{k^n}$, the rising k-binomial transform for $\alpha_j = \dfrac{(-1)^j}{k^j}$, and the falling k-binomial transform for $\alpha_j = \dfrac{(-1)^j}{k^{n-j}}$.

The k-binomial transforms relate many sequences listed in the On-Line Encyclopedia of Integer Sequences [**12**]. Several of these relationships are given in Layman [**11**] and listed in numerous tables of integer sequences related by repeated applications of the binomial transform and thus (via [**13**, Theorem 3.2]) by the falling k-binomial transform.

Let S_c be a set of real functions f having continuous derivatives of all orders at $x = 0$ for which there exists a constant $M > 0$, such that $|f^{(k)}(0)| \leqslant M$ for every $k \in \mathbb{N}_0$, and S_q be a set of one-parametric sequences $\{t_n^{(m)}\}$, with $m \in \mathbb{N}_0$ as a parameter, so that there holds $|\mathcal{D}_\alpha^k t_0| \leqslant M$ for every $k \in \mathbb{N}_0$.

On the basis of the generalized binomial transform (6), in [**16**] the generalized function-to-sequence transform was introduced.

Definition 1. *The transform $\mathcal{T}_\alpha$ mapping a function $f \in S_c$ to a sequence $\{t_n^{(m)}\} \in S_q$, determined by the equalities*

$$\mathcal{T}_\alpha x^m f(x) = \{t_n^{(m)}\}, \quad t_n^{(m)} = \sum_{k=m}^{n} \frac{(-1)^{k-m}}{\alpha_{k-m}} \binom{n}{k} \frac{d^k}{dx^k} \big(x^m f(x)\big)_{x=0} \tag{7}$$

is called $\mathcal{T}_\alpha$-transform of the function f.

The binomial transform is a sequence transformation, however, after replacing the sequence $\{s_n\}$ with the sequence $\{f^{(n)}(0)\}$ of an infinitely differentiable function $f(x)$ in (6), we obtain a more general form of a sequence transformation, a specific linear transform mapping a set of functions into a set of sequences. We denote it by $\mathcal{T}_\alpha$.

In the case $m = 0$, the sequence $\{t_n^{(0)}\}$ is denoted by $\{t_n\}$, and (7) takes the form

$$\mathcal{T}_\alpha f(x) = \{t_n\}, \quad t_n = \sum_{k=0}^{n} \frac{(-1)^k}{\alpha_k} \binom{n}{k} f^{(k)}(0),$$

and its matrix is (6). So $\mathcal{T}_\alpha$ is regarded as a differential operator.

In order to study properties of $\mathcal{T}_\alpha$, we made use of $\mathcal{D}_\alpha$, and proved in [**16**] these basic properties of the differential operator $\mathcal{T}_\alpha$, so that for $p \in \mathbb{N}$, there holds

$$1^\circ\ \mathcal{T}_\alpha f^{(p)}(x) = \{\mathcal{D}_\alpha^p t_n\}, \qquad 2^\circ\ \mathcal{T}_\alpha \int_0^x f(t)dt = \{\mathcal{D}_\alpha^{-1} t_n\}, \tag{8}$$

where $\mathcal{D}_\alpha^{-1} t_n$ is given by (5) and for $m, p \in \mathbb{N}_0$, we have

$$\mathcal{T}_\alpha x^m f^{(p)}(x) = \{s_n^{(m)}\}, \quad s_n^{(m)} = n^{(m)} \mathcal{D}_\alpha^p t_{n-m}. \tag{9}$$

For $f(x) = C$, then $\mathcal{T}_\alpha C = \{t_n\}$, $t_n = C$, $n \in \mathbb{N}_0$. Also, if $\mathcal{T}_\alpha f(x) = \{t_n\}$, then $\mathcal{T}_\alpha C f(x) = \{C t_n\}$. Knowing that $n^{(m)} = n(n-1)\cdots(n-m+1)$, from (9) for $p = 0$, there holds

$$t_n^{(m)} = n^{(m)} \sum_{k=0}^{n-m} \frac{(-1)^k}{\alpha_k} \binom{n-m}{k} f^{(k)}(0) = n^{(m)} t_{n-m}.$$

We provide the $\mathcal{T}_\alpha$-transforms of some basic functions in Appendix.

Definition 2. *For any sequence $\{t_n\} \in S_q$ and the linear operator $\mathcal{D}_\alpha$ introduced by (4), the function $f(x)$ defined by*

$$\mathcal{B}_\alpha\{t_n\} = f(x), \quad f(x) = \sum_{k=0}^{\infty} \mathcal{D}_\alpha^k t_0 \frac{x^k}{k!} = \sum_{k=0}^{\infty} \alpha_k (-1)^k \Delta^k t_0 \frac{x^k}{k!}. \tag{10}$$

is called the $\mathcal{B}_\alpha$-transform.

In [**16**] we proved that $\mathcal{B}_\alpha$ is the inverse linear transform of $\mathcal{T}_\alpha$, having an infinite dimensional matrix (11) the inverse of T, which is (6). So expressing $\Delta^k t_0$, $k \in \mathbb{N}_0$, in terms of $t_0, t_1, t_2, \dots$, the equality (10) can be obtained in the following matrix form

(11)

$$\begin{pmatrix} 1 & \frac{x}{1!} & \frac{x^2}{2!} & \dots & \frac{x^n}{n!} & \dots \end{pmatrix} \begin{pmatrix} \alpha_0 & 0 & 0 & \dots & 0 \\ \alpha_1 & -\alpha_1 & 0 & \dots & 0 \\ \alpha_2 & -2\alpha_2 & \alpha_2 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ \alpha_n & -\binom{n}{1}\alpha_n & \binom{n}{2}\alpha_n & \dots & (-1)^n\binom{n}{n}\alpha_n \\ \dots & \dots & \dots & \dots & \dots \end{pmatrix} \begin{pmatrix} t_0 \\ t_1 \\ t_2 \\ \dots \\ t_n \\ \dots \end{pmatrix}.$$

Definition 3. *The convolution of the sequences $\{t_n\}, \{s_n\} \in S_q$, is defined by*

$$r_n = t_n * s_n = \sum_{k=0}^{n} \binom{n}{k} \sum_{j=0}^{k} \binom{k}{j} \frac{\alpha_j \alpha_{k-j}}{\alpha_k} \Delta^j t_0 \Delta^{k-j} s_0. \tag{12}$$

Here we give some properties of convolutions. For the sequences $\{r_n\}$, $\{s_n\}$, $\{t_n\} \in S_q$ and $c \in \mathbb{R}$, the following relations are valid

1° $c * t_n = c t_n$,

2° $t_n * s_n = s_n * t_n$,

3° $r_n * (s_n + t_n) = r_n * s_n + r_n * t_n$.

In [**16**] we proved for the sequences $\{t_n\}, \{s_n\} \in S_q$ and $\mathcal{B}_\alpha$-transform, that the equality

$$\mathcal{B}_\alpha\{t_n * s_n\} = \mathcal{B}_\alpha\{t_n\}\mathcal{B}_\alpha\{s_n\}$$

holds true if and only if the convolution $t_n * s_n$ is defined by (12). Making use of this result, for $\mathcal{T}_\alpha e^{ax} = \{s_n\}$, $\mathcal{T}_\alpha f(x) = \{t_n\}$ and $\alpha_k = (-a)^k$, we obtain

$$\begin{aligned} e^{ax} f(x) &= \mathcal{B}_\alpha\{s_n\}\mathcal{B}_\alpha\{t_n\} = \mathcal{B}_\alpha\{s_n * t_n\} \\ &= \sum_{k=0}^{\infty} \sum_{j=0}^{k} \frac{(e^{ax})^{(k-j)}\big|_{x=0} f^{(j)}(0)}{j!(k-j)!} x^k = \sum_{k=0}^{\infty} \frac{x^k}{k!} \sum_{j=0}^{k} \binom{k}{j} a^{k-j} f^{(j)}(0) \\ &= \sum_{k=0}^{\infty} \frac{x^k}{k!} a^k \sum_{j=0}^{k} \frac{1}{a^j} \binom{k}{j} f^{(j)}(0) = \sum_{k=0}^{\infty} t_k \frac{(ax)^k}{k!}, \end{aligned}$$

which is the Borel transform of $e^{ax}f(x)$ (see [**6**]), whence we get

$$f(x) = e^{-ax}\sum_{k=0}^{\infty} t_k \frac{(ax)^k}{k!}.$$

2. APPLICATIONS

The transform $\mathcal{T}_\alpha$ has numerous applications. In the next subsection we deal with Genocchi polynomials. Afterwards we consider applications of $\mathcal{T}_\alpha$ to differential equations for solving difference equations.

2.1 TWO-DIMENSIONAL GENOCCHI SEQUENCES

The Genocchi polynomials are defined through the generating function [**10**]

$$\frac{2te^{xt}}{e^t+1} = \sum_{m=0}^{\infty} G_m(x)\frac{t^m}{m!}. \tag{13}$$

By differentiating both sides of (13) with respect to x, we come to the following differential equation of the Genocchi polynomials [**4**]

$$\frac{d}{dx}G_m(x) = mG_{m-1}(x). \tag{14}$$

The Fourier series representations of the Genocchi polynomials [**8**] are

$$G_{2m-1}(x) = \frac{4(-1)^{m-1}(2m-1)!}{\pi^{2m-1}}\sum_{n=1}^{\infty}\frac{\sin(2n-1)\pi x}{(2n-1)^{2m-1}},$$

$$G_{2m}(x) = \frac{4(-1)^{m}(2m)!}{\pi^{2m}}\sum_{n=1}^{\infty}\frac{\cos(2n-1)\pi x}{(2n-1)^{2m}}, \quad m\in\mathbb{N}.$$

In the paper [**17**] we derived the closed form expression for trigonometric series in terms of the Dirichlet lambda function defined by

$$\lambda(s) = \sum_{n=1}^{\infty}\frac{1}{(2n-1)^s}.$$

By substituting πx for x in these series, one obtains

(15)

$$\sum_{n=1}^{\infty}\frac{\sin(2n-1)\pi x}{(2n-1)^{2m-1}} = \frac{(-1)^{m-1}\pi(\pi x)^{2m-2}}{4(2m-2)!} + \sum_{k=0}^{m-1}\frac{(-1)^k\lambda(2m-2k-2)}{(2k+1)!}(\pi x)^{2k+1},$$

$$\sum_{n=1}^{\infty}\frac{\cos(2n-1)\pi x}{(2n-1)^{2m}} = \frac{(-1)^{m}\pi(\pi x)^{2m-1}}{4(2m-1)!} + \sum_{k=0}^{m}\frac{(-1)^k\lambda(2m-2k)}{(2k)!}(\pi x)^{2k}.$$

Thus, taking account of (15) we can express the Genocchi polynomials in terms of the Dirichlet lambda function

$$G_{2m-1}(x) = (2m-1)x^{2m-2} - 4(2m-1)!\sum_{k=0}^{m-1} \frac{(-1)^{m+k}\lambda(2m-2k-2)}{(2k+1)!\pi^{2m-2k-2}}x^{2k+1},$$

(16)

$$G_{2m}(x) = 2mx^{2m-1} + 4(2m)!\sum_{k=0}^{m} \frac{(-1)^{m+k}\lambda(2m-2k)}{(2k)!\pi^{2m-2k}}\,x^{2k}.$$

Making use of the relation [**3**]

$$\lambda(2m) = \frac{(-1)^m \pi^{2m} G_{2m}}{4(2m)!}, \quad m \in \mathbb{N},$$

where G_{2m} denotes Genocchi numbers, then applying $\mathcal{T}_\alpha$ to (16), we come to **the generalized two-dimensional Genocchi sequences**

$$G^\alpha_{2m-1,n} = \frac{(2m-1)!}{\alpha_{2m-2}}\binom{n}{2m-2} - \sum_{k=0}^{m-1}\binom{2m-1}{2k+1}G_{2m-2k-2}\frac{(2k+1)!}{\alpha_{2k+1}}\binom{n}{2k+1},$$

$$G^\alpha_{2m,n} = -\frac{(2m)!}{\alpha_{2m-1}}\binom{n}{2m-1} + \sum_{k=0}^{m}\binom{2m}{2k}G_{2m-2k}\frac{(2k)!}{\alpha_{2k}}\binom{n}{2k}.$$

Let $\mathcal{T}_\alpha G_m(x) = G^\alpha_{m,n}$. Applying the $\mathcal{T}_\alpha$-transform to the equation (14), then referring to the property 1° in (8), we obtain the partial difference equation

$$\mathcal{D}_\alpha G^\alpha_{m,n} = mG^\alpha_{m-1,n}. \tag{17}$$

Knowing that for $\alpha_k = (-1)^k$, the operator $\mathcal{D}_\alpha$ reduces to Δ, the equation (17) becomes

$$G_{m,n+1} - G_{m,n} = mG_{m-1,n}, \tag{18}$$

and the generalized two-dimensional Genocchi sequences for $\alpha_k = (-1)^k$ are solutions of (18).

2.2 DIFFERENTIAL OPERATORS AND SOLVING LINEAR DIFFERENCE EQUATIONS

The transforms $\mathcal{T}_\alpha$ and $\mathcal{B}_\alpha$ provide a useful method for solving a difference equation by mapping it first by $\mathcal{B}_\alpha$ to the corresponding linear differential equation that is often easier to solve, then its solution is mapped by $\mathcal{T}_\alpha$ to a sequence, giving a solution of the difference equation.

Lemma 4. *In the special case of* $\alpha_k = (-1)^k$*, the operator* $\mathcal{D}_\alpha$ *becomes* Δ*, and applying* $\mathcal{T}_\alpha$ *to the Bessel operator* $\frac{d}{dx}x\frac{d}{dx}$ *yields*

$$\mathcal{T}_\alpha \frac{d}{dx}x\frac{d}{dx}f(x) = \{\Delta n \Delta t_{n-1}\} = \{\Delta n(t_n - t_{n-1})\} = \{\Delta n \nabla t_n\},$$

i.e. the $\mathcal{T}_\alpha$*-transform maps Bessel's operator* $\frac{d}{dx}x\frac{d}{dx}$ *to the operator* $\Delta n \nabla$*. Also, the eigenvector of the Bessel operator is mapped to the eigenvector of the operator* $\Delta n \nabla$*.*

Proof. Since (4) reduces to (2), in view of 1° in (8), there holds $\mathcal{T}_\alpha f'(x) = \frac{d}{dx}f(x) = \{\Delta t_n\}$ and relying on (9) we have $\mathcal{T}_\alpha x\frac{d}{dx}f(x) = \{n\Delta t_{n-1}\}$, implying

$$\mathcal{T}_\alpha \frac{d}{dx}x\frac{d}{dx}f(x) = \{\Delta n \Delta t_{n-1}\} = \{\Delta n \Delta(t_n - t_{n-1})\} = \{\Delta n \nabla t_n\}. \tag{19}$$

Applying the linear Bessel differential operator $\frac{d}{dx}x\frac{d}{dx}$ to the Laguerre-type exponential function [**16**]

$$e_1(x) = \sum_{k=0}^{\infty} \frac{x^k}{(k!)^2},$$

by the uniform convergence everywhere of the series on right-hand side, we are allowed to exchange summation and differentiation, and find

$$\frac{d}{dx}x\frac{d}{dx}e_1(x) = e_1(x), \tag{20}$$

which means that the L-exponential function is its eigenvector. Let $\mathcal{T}_\alpha e_1(x) = \{s_n\}$. Then, we have

$$s_n = \sum_{k=0}^{n} \binom{n}{k} e_1^{(k)}(0) = \sum_{k=0}^{n} \binom{n}{k} \frac{k^{(k)}}{(k!)^2} = \sum_{k=0}^{n} \frac{1}{k!}\binom{n}{k} = \sum_{k=0}^{n} \frac{n^{(k)}}{(k!)^2}. \tag{21}$$

Now applying the $\mathcal{T}_\alpha$-transform to (20), because of (19), we have for $\alpha_k = (-1)^k$

$$\mathcal{T}_\alpha \frac{d}{dx}x\frac{d}{dx}e_1(x) = \{\Delta n \Delta s_{n-1}\} = \{\Delta n \nabla s_n\} = \mathcal{T}e_1(x) = \{s_n\},$$

that is,

$$\Delta n \nabla s_n = s_n. \tag{22}$$

So, s_n is the eigenvector of the discrete Bessel operator $\Delta n \nabla$. □

Example 5. *We make use of Lemma 4 to give a discrete model of the Laguerre-Malthus population growth. The continuous model is defined in* [**2**] *by the equation*

$$t\frac{d^2N(t)}{dt^2} + \frac{dN(t)}{dt} = rN(t) \quad \Leftrightarrow \quad \frac{d}{dt}t\frac{d}{dt}N(t) = rN(t), \tag{23}$$

where a positive constant r presents the growth rate. Assuming the initial conditions

$$N(0) = N_0, \quad N'(0) = rN_0,$$

we find its solution

$$N(t) = N_0 e_1(rt) = N_0 \sum_{k=0}^{\infty} r^k \frac{t^k}{(k!)^2}.$$

Denote $\mathcal{T}_\alpha N(t) = \{t_n\}$. Applying the $\mathcal{T}_\alpha$-transform to the right-hand side of the equivalence (23)*, on account of* (22) *and* (21)*, we obtain a discrete Laguerre-Malthus model and its solution*

$$\Delta n \nabla t_n = rt_n, \quad t_n = N_0 \sum_{k=0}^{n} r^k \frac{n^{(k)}}{(k!)^2}.$$

Example 6. *The solution of Bessel's differential equation $x^2y'' + xy' + (x^2 - m^2)y = 0$ ($m \in \mathbb{N}$) is the well-known Bessel function* (see [**5**]) *of the first kind and order m*

$$J_m(x) = \sum_{k=0}^{\infty} \frac{(-1)^k (\frac{x}{2})^{m+2k}}{k!\Gamma(m+k+1)}. \tag{24}$$

Applying Table in Appendix, we map (24) *to the **Bessel-α sequence***

$$J_{m,n} = \frac{(-1)^m}{2^m} \sum_{k=0}^{[\frac{n-m}{2}]} \frac{(-1)^k n^{(m+2k)}}{2^{2k} k!(m+k)!\alpha_{m+2k}}, \quad n \geqslant m+2k,\ 2\Big[\frac{n-m}{2}\Big] \leqslant n-m,$$

*which is a solution of the **Bessel-α difference equation***

$$n^{(2)}\mathcal{D}_\alpha^2 t_{n-2} + n\mathcal{D}_\alpha t_{n-1} + n^{(2)} t_{n-2} - m^2 t_n = 0,$$

obtained by applying (9) *to the Bessel differential equation.*

Replacing x by px in (24)*, we have*

$$\mathcal{T}_\alpha J_m(px) = J_{m,n}(p) = \sum_{k=0}^{[\frac{n-m}{2}]} \frac{(-1)^{m+k}(\frac{p}{2})^{m+2k} n^{(m+2k)}}{k!(m+k)!\alpha_{m+2k}}, \quad n \geqslant m+2k,\ 2\Big[\frac{n-m}{2}\Big] \leqslant n-m.$$

In [**14**] *we derived a closed form formula for the series in terms of Bessel functions. Setting there $\nu = m$, that formula becomes*

$$\sum_{p=1}^{\infty} \frac{1}{p^{2n+m}} J_m(px) = \frac{(-1)^{n+1} x^m}{(2n)! 2^{m+1}\sqrt{\pi}} \sum_{k=0}^{2n} \frac{(2\pi)^k \Gamma(n - \frac{k-1}{2})}{\Gamma(n+m+1-\frac{k}{2})} \binom{2n}{k} x^{2n-k} B_k,$$

where B_k are the Bernoulli numbers. Because of the uniform convergence of the left-hand side series, we can take the term-by-term derivative. We apply $\mathcal{T}_\alpha$ looking up in the appendix, p. 429, and obtain the series in terms of Bessel sequences $J_{m,n}(p)$ in the closed form

$$\sum_{p=1}^{\infty} \frac{1}{p^{2n+m}} J_{m,n}(p) = \frac{(-1)^{n+m+1} n^{(m)}}{(2n)! 2^{m+1}\alpha_{m+2k}\sqrt{\pi}} \sum_{k=0}^{2n} \frac{(2\pi)^k \Gamma(n - \frac{k-1}{2})}{\Gamma(n+m+1-\frac{k}{2})} \binom{2n}{k} \frac{(-1)^{2n-k} n^{(2n-k)} B_k}{\alpha_{n-2k}}.$$

2.3 APPLICATION OF THE CAUCHY METHOD TO SOLVING LINEAR DIFFERENCE EQUATIONS

By using the theory of residues, in the paper [**7**] (see also [**9**]) Cauchy obtained a general solution of the linear differential equation that does not require to search for a particular solution. For instance, consider the equation

$$b_0 y^{(m)}(x) + b_1 y^{(m-1)}(x) + \cdots + b_{m-1} y'(x) + b_m y(x) = F(x). \tag{25}$$

By virtue of the Cauchy method, its solution is

$$y(x) = \sum_{p=1}^{s} \operatorname*{Res}_{z=z_p} \Big(\frac{f(z)}{g(z)} e^{zx} \Big) + \sum_{p=1}^{s} \operatorname*{Res}_{z=z_p} \Big(\frac{e^{zx}}{g(z)} \int_{x_0}^{x} e^{-zt} F(t) dt \Big), \tag{26}$$

where $f(z)$ is an arbitrary regular function, the zeros of which do not coincide with those of the polynomial

$$g(z) = b_0 z^n + b_1 z^{n-1}(x) + \cdots + b_n = b_0 (z - z_1)^{d_1} \cdots (z - z_s)^{d_s} \tag{27}$$

where $s < n$, and $d_1 + \cdots + d_s = n$.

Notice that B. Tortolini (see [**15**]) obtained this result but in another way. If we apply the $\mathcal{T}_\alpha$-transform to (26), and take account of linearity of $\mathcal{T}_\alpha$, setting $x_0 = 0$, from (26), there follows

$$\mathcal{T}_\alpha y(x) = \sum_{p=1}^{s} \mathcal{T}_\alpha \operatorname*{Res}_{z=z_p} \Big(\frac{f(z)}{g(z)} e^{zx} \Big) + \sum_{p=1}^{s} \mathcal{T}_\alpha \operatorname*{Res}_{z=z_p} \Big(\frac{e^{zx}}{g(z)} \int_{0}^{x} e^{-zt} F(t) dt \Big).$$

Lemma 7. *If $f(z)$ is an arbitrary regular function, the zeros of which do not coincide with the ones of the polynomial* (27)*, then for $\alpha_k = (-1)^k$, there holds*

$$\sum_{p=1}^{s} \mathcal{T}_\alpha \operatorname*{Res}_{z=z_p} \Big(\frac{f(z)}{g(z)} e^{zx} \Big) = \sum_{p=1}^{s} \operatorname*{Res}_{z=z_p} \Big(\frac{f(z)}{g(z)} (1+z)^n \Big). \tag{28}$$

Proof. According to the definition of the $\mathcal{T}_a$-transform, we have

$$\begin{aligned}
\sum_{p=1}^{s} \mathcal{T}_\alpha \operatorname*{Res}_{z=z_p} \Big(\frac{f(z)}{g(z)} e^{zx} \Big) &= \sum_{p=1}^{s} \mathcal{T}_\alpha \frac{1}{(d_p - 1)!} \frac{\partial^{d_p - 1}}{\partial z^{d_p - 1}} \Big(\frac{f(z)}{g_p(z)} e^{zx} \Big)_{z=z_p} \\
&= \sum_{p=1}^{s} \frac{1}{(d_p - 1)!} \sum_{k=0}^{n} \binom{n}{k} \frac{\partial^{d_p - 1}}{\partial z^{d_p - 1}} \Big(\frac{\partial^k}{\partial x^k} \Big(\frac{f(z)}{g_p(z)} e^{zx} \Big)_{x=0} \Big)_{z=z_p} \\
&= \sum_{p=1}^{s} \frac{1}{(d_p - 1)!} \frac{\partial^{d_p - 1}}{\partial z^{d_p - 1}} \Big(\mathcal{T}_\alpha \frac{f(z)}{g_p(z)} e^{zx} \Big)_{z=z_p} \\
&= \sum_{p=1}^{s} \operatorname*{Res}_{z=z_p} \Big(\mathcal{T}_\alpha \frac{f(z)}{g(z)} e^{zx} \Big).
\end{aligned}$$

Further, it follows

$$\sum_{p=1}^{s} \operatorname*{Res}_{z=z_p} \Big(\mathcal{T}_\alpha \frac{f(z)}{g(z)} e^{zx}\Big) = \sum_{p=1}^{s} \operatorname*{Res}_{z=z_p} \Big(\sum_{k=0}^{n} \binom{n}{k} \frac{\partial^k}{\partial x^k} \Big(\frac{f(z)}{g(z)} e^{zx}\Big)_{x=0}\Big)$$
$$= \sum_{p=1}^{s} \operatorname*{Res}_{z=z_p} \Big(\frac{f(z)}{g(z)} \sum_{k=0}^{n} \binom{n}{k} z^k\Big) = \sum_{p=1}^{s} \operatorname*{Res}_{z=z_p} \Big(\frac{f(z)}{g(z)} (1+z)^n\Big).$$

Thereby we have proved (28). □

Lemma 8. *There holds*

$$\sum_{p=1}^{s} \mathcal{T}_\alpha \operatorname*{Res}_{z=z_p} \Big(\frac{e^{zx}}{g(z)} h(x)\Big) = \sum_{p=1}^{s} \operatorname*{Res}_{z=z_p} \left(\frac{(1+z)^n}{g(z)} \sum_{k=0}^{n} \frac{h^{(k)}(0)}{(1+z)^k}\right), \tag{29}$$

where $h(x) = \int_0^x e^{-zt} F(t)dt$ *with* $h^{(k)}(0) = \sum_{j=0}^{k-1} \binom{k}{j} (-z)^{k-j-1} F^{(j)}(0)$.

Proof. First, let $g_p(z)$ mean that we omit the factor $(z-z_s)^{d_p}$ in the polynomial (27). Then, we have

$$\sum_{p=1}^{s} \mathcal{T}_\alpha \operatorname*{Res}_{z=z_p} \Big(\frac{e^{zx}}{g(z)} \int_0^x e^{-zt} F(t)dt\Big) = \sum_{p=1}^{s} \mathcal{T}_\alpha \frac{1}{(d_p-1)!} \frac{\partial^{d_p-1}}{\partial z^{d_p-1}} \Big(\frac{e^{zx}}{g_p(z)} \int_0^x e^{-zt} F(t)dt\Big)_{z=z_p}$$
$$= \sum_{p=1}^{s} \frac{1}{(d_p-1)!} \sum_{k=0}^{n} \binom{n}{k} \frac{\partial^k}{\partial x^k} \Big(\frac{\partial^{d_p-1}}{\partial z^{d_p-1}} \Big(\frac{e^{zx}}{g_p(z)} \int_0^x e^{-zt} F(t)dt\Big)_{z=z_p}\Big)_{x=0}$$
$$= \sum_{p=1}^{s} \frac{1}{(d_p-1)!} \frac{\partial^{d_p-1}}{\partial z^{d_p-1}} \Big(\sum_{k=0}^{n} \binom{n}{k} \frac{\partial^k}{\partial x^k} \Big(\frac{e^{zx}}{g_p(z)} \int_0^x e^{-zt} F(t)dt\Big)_{x=0}\Big)_{z=z_p}$$
$$= \sum_{p=1}^{s} \operatorname*{Res}_{z=z_p} \Big(\mathcal{T}_\alpha \frac{e^{zx}}{g(z)} \int_0^x e^{-zt} F(t)dt\Big) = \sum_{p=1}^{s} \operatorname*{Res}_{z=z_p} \Big(\frac{1}{g(z)} \mathcal{T}_\alpha e^{zx} \int_0^x e^{-zt} F(t)dt\Big).$$

Here we regard $\int_0^x e^{-zt} F(t)dt$ as a function $h(x)$, and treat z as a parameter. By differentiating the function $e^{zx}h(x)$ k times at $x=0$, one gets

$$\frac{d^k}{dx^k} e^{zx} h(x)\Big|_{x=0} = \sum_{j=0}^{k} \binom{k}{j} (e^{zx})^{(k-j)} h^{(j)}(x)\Big|_{x=0} = \sum_{j=0}^{k} \binom{k}{j} z^{k-j} h^{(j)}(0),$$

so that we find

$$\mathcal{T}_\alpha e^{zx} h(x) = \sum_{k=0}^{n} \binom{n}{k} \sum_{j=0}^{k} \binom{k}{j} z^{k-j} h^{(j)}(0) = \sum_{k=0}^{n} z^k \binom{n}{k} \sum_{j=0}^{k} \binom{k}{j} \frac{h^{(j)}(0)}{z^j}.$$

Changing the order of summation yields

$$\begin{aligned}\mathcal{T}_\alpha e^{zx}h(x) &= \sum_{j=0}^{n}\binom{n}{j}\frac{h^{(j)}(0)}{z^j}\sum_{k=0}^{n-j}\binom{n-j}{k}z^{n-k}\\ &= \sum_{j=0}^{n}\binom{n}{j}z^{n-j}h^{(j)}(0)\sum_{k=0}^{n-j}\binom{n-j}{k}\frac{1}{z^k}\\ &= \sum_{j=0}^{n}\binom{n}{j}z^{n-j}h^{(j)}(0)\Big(1+\frac{1}{z}\Big)^{n-j} = (1+z)^n\sum_{j=0}^{n}\binom{n}{j}\frac{h^{(j)}(0)}{(1+z)^j}.\end{aligned}$$

Finally, we obtain

$$\sum_{p=1}^{s}\operatorname*{Res}_{z=z_p}\Big(\frac{1}{g(z)}\mathcal{T}_\alpha e^{zx}h(x)\Big) = \sum_{p=1}^{s}\operatorname*{Res}_{z=z_p}\Big(\frac{(1+z)^n}{g(z)}\sum_{k=0}^{n}\binom{n}{k}\frac{h^{(k)}(0)}{(1+z)^k},$$

where $h^{(k)}(0) = \sum_{j=0}^{k-1}\binom{k-1}{j}(-z)^{k-j-1}F^{(j)}(0)$, so we arrive at (29). □

We apply the Cauchy method (26), Lemma 7 and Lemma 8 to solve linear difference equations.

Theorem 9. *Using $g(z)$ defined by* (27) *and $F(x)$ on the right-hand side of* (25)*, the solution of the linear difference equation*

$$t_{n+m} + a_1 t_{n+m-1} + \cdots + a_m t_n = e_n \quad a_i \in \mathbb{R},\ i = 1,\ldots,m, \tag{30}$$

where $e_n = \mathcal{T}_\alpha F(x)$, is given by

$$\begin{aligned}t_n = &\sum_{p=1}^{s}\operatorname*{Res}_{z=z_p}\Big(\frac{f(z)}{g(z)}(1+z)^n\Big)\\ &+\sum_{p=1}^{s}\operatorname*{Res}_{z=z_p}\Big(\sum_{k=0}^{n}\frac{(1+z)^{n-k}}{g(z)}\binom{n}{k}\sum_{j=0}^{k-1}\binom{k-1}{j}(-z)^{k-j-1}F^{(j)}(0)\Big).\end{aligned}$$

Proof. We deal first with $\mathcal{T}_a$ for $\alpha_k = (-1)^k$ and make use of (3), so the left-hand side of (30) becomes

$$\sum_{k=0}^{m}\binom{m}{k}\Delta^k t_n + a_1\sum_{k=0}^{m-1}\binom{m-1}{k}\Delta^k t_n + \cdots + a_m t_n = \sum_{p=0}^{m}a_p\sum_{k=0}^{m-p}\binom{m-p}{k}\Delta^k t_n.$$

We recall that the operator $\mathcal{D}_\alpha$ reduces to the difference operator Δ for $\alpha_k = (-1)^k$, and referring to the statement 1° of (8), for a function $y(x)$ we find its kth derivative $\mathcal{T}_\alpha y^{(k)}(x) = \{\Delta^k t_n\}$, but by applying the inverse transform $\mathcal{B}_\alpha$, we have

$$\mathcal{B}_\alpha\mathcal{T}_\alpha y^{(k)}(x) = y^{(k)}(x) = \mathcal{B}_\alpha\{\Delta^k t_n\}, \qquad k = 0,1,\ldots,m.$$

In view of that, by applying the $\mathcal{B}_\alpha$-transform (30), the difference equation is mapped to the linear differential equation

$$\sum_{p=0}^{m} a_p \sum_{k=0}^{m-p} \binom{m-p}{k} y^{(k)}(x) = \sum_{k=0}^{m} y^{(k)}(x) \sum_{p=0}^{m-k} a_p \binom{m-p}{k} = F(x),$$

which is a differential equation of the form (25), where

$$F(x) = \mathcal{B}_\alpha\{e_n\}, \quad b_{m-k} = \sum_{p=0}^{m-k} a_p \binom{m-p}{k}.$$

So, in order to find its general solution, we apply the Cauchy method yielding as a solution (26), and relying on the results of Lemma 7 and Lemma 8 the solution of the difference equation (30) is obtained by summing (28) and (29). □

Appendix - Table of $\mathcal{T}_a$-Transform pairs

$f(x)$	$\mathcal{T}_\alpha f(x)$
x^r	$\frac{(-1)^r}{\alpha_r} n^{(r)}, \quad n^{(r)} = n(n-1)\cdots(n-r+1),\ n, r \in \mathbb{N},\ r \leqslant n$
$(1+x)^a$	$\sum_{k=0}^{n} \frac{(-1)^k}{\alpha_k} \binom{n}{k} a^{(k)}, \quad a^{(k)} = a(a-1)\cdots(a-k+1),\ a \in \mathbb{R}$
e^{ax}	$\sum_{k=0}^{n} \frac{(-1)^k}{\alpha_k} \binom{n}{k} a^k$
$\ln(1+x)$	$\sum_{k=1}^{n} \frac{(-1)^{k-1} n^{(k)}}{k\, \alpha_k}$
$\sin ax$	$\sum_{k=0}^{[\frac{n-1}{2}]} \frac{(-1)^{k+1}}{\alpha_{2k+1}} \binom{n}{2k+1} a^{2k+1}$
$\cos ax$	$\sum_{k=0}^{[\frac{n}{2}]+1} \frac{(-1)^k}{\alpha_{2k}} \binom{n}{2k} a^{2k}$

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