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FINITE SUMS INVOLVING TRIGONOMETRIC FUNCTIONS AND SPECIAL POLYNOMIALS: ANALYSIS OF GENERATING FUNCTIONS AND p-ADIC INTEGRALS

Neslihan Kilar, Abdelmejid Bayad and Yilmaz Simsek*

By using trigonometric and generating functions, some formulas and relations involving sums of powers of consecutive positive integers and certain combinatorial sums are derived. By applying the derivative operator to some certain families of special functions and finite sums involving trigonometric functions, many novel relations related to the special numbers and polynomials are obtained. Moreover, by applying p-adic integrals to these finite sums, some p-adic integral representations of trigonometric functions are found.

1. INTRODUCTION

Special polynomials and special functions related to finite sums involving trigonometric functions have been used in many different areas such as analytic number theory, Fourier analysis, combinatorial analysis, engineering, mathematical physics, and many other sciences. Sums of powers of integers have also been used in many areas. Moreover, these types of sums have important applications and properties that make them useful in a variety of areas of mathematics. For example, sums of powers are related to the Diophantine equations, many special numbers and

*Corresponding author. Neslihan Kilar
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polynomials. One of them is the well-known Faulhaber's formula, which expresses power sums as polynomials whose coefficients include the Bernoulli numbers. Moreover, these types of sums can be used in the theory of the Dedekind sums, the Hardy sums, the character sums, the Kloosterman sums, and other special sums.

Our motivation of this paper is to give not only formulas and relations for finite sums involving some of trigonometric functions, consecutive positive integers, and combinatorial sums, but also p-adic integral formulas for some of trigonometric functions. The other motivation is to give identities and formulas involving both sums of powers of positive integers and finite sums with the trigonometric functions. These results are related to the Bernoulli numbers and polynomials, the Euler numbers and polynomials, the Stirling numbers, the Catalan numbers, the array polynomials, and combinatorial numbers.

Throughout this paper, we use the following notations and definitions:

Let $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$ and $\mathbb{C}$ denote the set of natural numbers, the set of integers, the set of real numbers and the set of complex numbers, respectively, and also $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $i^2 = -1$. We also use the following notations:

$$0^n = \begin{cases} 1, & (n = 0) \\ 0, & (n \in \mathbb{N}) \end{cases}$$

and

$$\binom{\lambda}{n} = \frac{\lambda (\lambda - 1) (\lambda - 2) \dots (\lambda - n + 1)}{n!},$$

where $\binom{\lambda}{0} = 0$ and $n \in \mathbb{N}$, $\lambda \in \mathbb{C}$ (*cf.* [**43**]).

The Bernoulli polynomials $B_n^{(r)}(y)$ of order r are defined by

$$\left(\frac{\omega}{e^{\omega} - 1}\right)^r e^{y\omega} = \sum_{n=0}^{\infty} B_n^{(r)}(y) \frac{\omega^n}{n!}, \tag{1}$$

where $|\omega| < 2\pi$ and $r \in \mathbb{Z}$ $(r \geq 0, r < 0)$ (*cf.* [**2**], [**3**], [**7**], [**16**], [**17**], [**37**], [**40**], [**42**], [**43**]; and references therein).

Substituting $y = 0$ into (1), we get

$$B_n^{(r)}(0) = B_n^{(r)},$$

which denotes the Bernoulli numbers of order r.

Moreover, the Bernoulli polynomials $B_n(y)$ and the Bernoulli numbers B_n are given by

$$B_n^{(1)}(y) = B_n(y) \qquad \text{and} \qquad B_n^{(1)} = B_n.$$

The Euler polynomials $E_n^{(r)}(y)$ of order r are defined by

$$\left(\frac{2}{e^{\omega} + 1}\right)^r e^{y\omega} = \sum_{n=0}^{\infty} E_n^{(r)}(y) \frac{\omega^n}{n!}, \tag{2}$$

where $|\omega| < \pi$ and $r \in \mathbb{Z}$ $(r \geq 0, r < 0)$ (*cf.* [**3**], [**7**], [**16**], [**17**], [**37**], [**40**], [**42**], [**43**]; and references therein).

Substituting $y = 0$ into (2), we have

$$E_n^{(r)}(0) = E_n^{(r)},$$

which denotes the Euler numbers of order r.

When $r = 1$ in (2) and the above equation, the Euler polynomials $E_n(y)$ and the Euler numbers E_n are given by

$$E_n^{(1)}(y) = E_n(y) \qquad \text{and} \qquad E_n^{(1)} = E_n.$$

The Catalan numbers C_n are defined by

$$\frac{1-\sqrt{1-4\omega}}{2\omega} = \sum_{n=0}^{\infty} C_n \omega^n,$$

where $0 < |\omega| \leq \frac{1}{4}$ and $C_0 = 1$. From the above generating function, we have

$$C_n = \frac{1}{n+1}\binom{2n}{n} \tag{3}$$

(*cf.* [**29**], [**30**], [**35**], [**41**], [**43**]).

The Stirling numbers of the second kind are defined by

$$\frac{(e^{\omega}-1)^k}{k!} = \sum_{n=0}^{\infty} S_2(n,k)\frac{\omega^n}{n!} \tag{4}$$

(*cf.* [**2**], [**7**], [**43**]).

By using (2) and (4), we have

$$E_n^{(r)}(y) = \sum_{j=0}^{n}\binom{n}{j} y^{n-j} \sum_{m=0}^{j} (-1)^m \binom{r+m-1}{m} 2^{-m} m! S_2(j,m)$$

(*cf.* [**43**, Eq. (52)]).

The array polynomials are defined by

$$\frac{(e^{\omega}-1)^k}{k!} e^{y\omega} = \sum_{n=0}^{\infty} S_k^n(y)\frac{\omega^n}{n!} \tag{5}$$

(*cf.* [**2**], [**4**], [**6**], [**33**], [**34**], [**38**]).

From (5), we get

$$S_k^n(y) = \frac{1}{k!}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}(y+j)^n,$$

and for $k > n$,

$$S_k^n(y) = 0$$

and also $S_0^0(y) = S_n^n(y) = 1$, $S_0^n(y) = y^n$ (*cf.* [**2**], [**4**], [**6**], [**33**], [**34**]).

One can easily see from (5) that

$$S_k^n(0) = S_2(n,k).$$

The combinatorial numbers $y_1(n,k;\lambda)$ are defined by

$$\frac{(\lambda e^{\omega}+1)^k}{k!} = \sum_{n=0}^{\infty} y_1(n,k;\lambda)\frac{\omega^n}{n!}, \tag{6}$$

where $k \in \mathbb{N}_0$ and $\lambda \in \mathbb{C}$ (*cf.* [**37**, Eq. (8)]). The numbers $y_1(n,k;\lambda)$, which have many applications, are also called Simsek numbers (see [**9**]; also [**24**], [**38**]).

By using (6), we get

$$y_1(n,k;\lambda) = \frac{1}{k!}\sum_{j=0}^{k}\binom{k}{j} j^n \lambda^j \tag{7}$$

(*cf.* [**37**, Eq. (9)]).

The numbers $B(n;k)$ are defined by

$$B(n;k) = \sum_{j=0}^{k}\binom{k}{j} j^n \tag{8}$$

with

$$B(n;k) = \frac{d^n}{d\omega^n}(e^{\omega}+1)^k \,|_{\omega=0}$$

(*cf.* [**13**]).

By using (7) and (8), we have

$$B(n;k) = k!\,y_1(n,k;1) \tag{9}$$

(*cf.* [**37**, Eq. (10)]).

The results of this paper are briefly summarized as follows:

In Section 2, by applying derivative operator to the some trigonometric identities, we give many new and interesting formulas, including the Bernoulli numbers and polynomials, the Euler numbers and polynomials, the Stirling numbers, the Catalan numbers, the array polynomials, and combinatorial numbers. Moreover, by using trigonometric relations, we give some finite sums of trigonometric functions.

In Section 3, by applying the p-adic integrals, which are the Volkenborn integral and the fermionic integral, to some trigonometric relations, we give some finite sums and p-adic integral formulas of the trigonometric functions.

In Section 4, we give some explicit formulas and relations in terms of the Bernoulli polynomials and numbers for the sums of powers of consecutive positive integers.

In Section 5, we give some observations and comments on our results.

2. APPLYING DERIVATIVE OPERATOR TO THE TRIGONOMETRIC FUNCTIONS AND SUMS

In this section, by applying the derivative operator to some trigonometric functions and combinatorial sums, we obtain many identities and relations for some special numbers and polynomials including the numbers $y_1(n,k;\lambda)$, the Catalan numbers, the Stirling numbers, the Bernoulli numbers and polynomials, the Euler polynomials, and the array polynomials.

Theorem 1. *Let $m, r \in \mathbb{N}_0$. Then we have*

$$S_{2r}^{m}\left(\frac{1-2r}{2}\right)+S_{2r}^{m}\left(-\frac{2r+1}{2}\right)$$
$$=\frac{(1+(-1)^m)}{(2r+1)!2^m}\sum_{j=0}^{r}(-1)^{r+j}\binom{2r+1}{r-j}(2j+1)^{m+1}. \tag{10}$$

Proof. Differentiating both sides of the well-known identity with respect to ω,

$$e^{-i\omega\left(\frac{2r+1}{2}\right)}\left(e^{i\omega}-1\right)^{2r+1}=\sum_{j=0}^{r}(-1)^{r+j}\binom{2r+1}{r-j}\left(e^{i\omega\left(\frac{2j+1}{2}\right)}-e^{-i\omega\left(\frac{2j+1}{2}\right)}\right)$$

(*cf.* [**3**], [**16**]), we have

$$\frac{d}{d\omega}\left\{e^{-i\omega\left(\frac{2r+1}{2}\right)}\left(e^{i\omega}-1\right)^{2r+1}\right\}=\sum_{j=0}^{r}(-1)^{r+j}\binom{2r+1}{r-j}\frac{d}{d\omega}\left\{e^{i\omega\left(\frac{2j+1}{2}\right)}-e^{-i\omega\left(\frac{2j+1}{2}\right)}\right\}.$$

Thus

$$(2r+1)\left(e^{i\omega}-1\right)^{2r}e^{-i\omega\left(\frac{2r+1}{2}\right)}\left(e^{i\omega}+1\right)$$
$$=\sum_{j=0}^{r}(-1)^{r+j}\binom{2r+1}{r-j}(2j+1)\left(e^{i\omega\left(\frac{2j+1}{2}\right)}+e^{-i\omega\left(\frac{2j+1}{2}\right)}\right). \tag{11}$$

Combining the above equation with (5), we get

$$\sum_{m=0}^{\infty}S_{2r}^{m}\left(\frac{-2r+1}{2}\right)\frac{(i\omega)^m}{m!}+\sum_{m=0}^{\infty}S_{2r}^{m}\left(-\frac{2r+1}{2}\right)\frac{(i\omega)^m}{m!}$$
$$=\frac{1}{(2r+1)!}\sum_{m=0}^{\infty}\sum_{j=0}^{r}(-1)^{r+j}\binom{2r+1}{r-j}\frac{(2j+1)^{m+1}(1+(-1)^m)}{2^m}\frac{(i\omega)^m}{m!}.$$

Comparing the coefficients of $\frac{(i\omega)^m}{m!}$ on both sides of the above equation, we arrive at the desired result. □

If m is replaced by $2m+1$, then Theorem 1 reduces to the following result:

Corollary 2. *Let $r, m \in \mathbb{N}_0$. Then we have*

$$S_{2r}^{2m+1}\left(\frac{1-2r}{2}\right) = -S_{2r}^{2m+1}\left(-\frac{2r+1}{2}\right).$$

Replacing m by $2m$ in Theorem 1, we have the following result:

Corollary 3. *Let $m, r \in \mathbb{N}_0$. Then we have*

$$\begin{aligned} & S_{2r}^{2m}\left(\frac{1-2r}{2}\right) + S_{2r}^{2m}\left(-\frac{2r+1}{2}\right) \\ = & \frac{2^{1-2m}}{(2r+1)!}\sum_{j=0}^{r}(-1)^{r+j}\binom{2r+1}{r-j}(2j+1)^{2m+1}. \end{aligned} \tag{12}$$

Combining (11) with the following identity

$$\cos(\omega) = \frac{e^{i\omega}+e^{-i\omega}}{2}, \tag{13}$$

we obtain

$$\begin{aligned} & (2r+1)\left(e^{i\omega}-1\right)^{2r} e^{-i\omega\left(\frac{2r+1}{2}\right)}\left(e^{i\omega}+1\right) \\ = & 2\sum_{j=0}^{r}(-1)^{r+j}\binom{2r+1}{r-j}(2j+1)\cos\left(\omega\left(\frac{2j+1}{2}\right)\right). \end{aligned} \tag{14}$$

Combining the above equation with (5), we get

$$\begin{aligned} & \sum_{m=0}^{\infty} S_{2r}^{m}\left(\frac{1-2r}{2}\right)\frac{(i\omega)^m}{m!} + \sum_{m=0}^{\infty} S_{2r}^{m}\left(-\frac{2r+1}{2}\right)\frac{(i\omega)^m}{m!} \\ = & \frac{2}{(2r+1)!}\sum_{j=0}^{r}(-1)^{r+j}\binom{2r+1}{r-j}(2j+1)\sum_{m=0}^{\infty}\frac{(-1)^m(2j+1)^{2m}}{2^{2m}}\frac{\omega^{2m}}{(2m)!}. \end{aligned}$$

Comparing the coefficients of $\frac{\omega^{2m}}{(2m)!}$ on both sides of the above equation, after some elementary calculations, we also arrive at the Corollary 3.

Theorem 4. *Let $m, r \in \mathbb{N}_0$. Then we have*

$$\begin{aligned} & \sum_{j=0}^{m}\binom{m}{j} S_2(j,2r)\, E_{m-j}^{(-1)}\left(-\frac{2r+1}{2}\right) \\ = & \frac{\left(1+(-1)^m\right)}{2^{m+1}(2r+1)!}\sum_{j=0}^{r}(-1)^{r+j}\binom{2r+1}{r-j}(2j+1)^{m+1}. \end{aligned} \tag{15}$$

Proof. Combining (11) with (2) and (4), we have

$$\begin{aligned}
&(2r+1)!\sum_{m=0}^{\infty} S_2(m,2r)\frac{(i\omega)^m}{m!}\sum_{m=0}^{\infty} E_m^{(-1)}\left(-\frac{2r+1}{2}\right)\frac{(i\omega)^m}{m!} \\
= &\sum_{m=0}^{\infty}\sum_{j=0}^{r}(-1)^{r+j}\binom{2r+1}{r-j}\frac{(2j+1)^{m+1}(1+(-1)^m)}{2^{m+1}}\frac{(i\omega)^m}{m!}.
\end{aligned}$$

Therefore

$$\begin{aligned}
&(2r+1)!\sum_{m=0}^{\infty}\sum_{j=0}^{m}\binom{m}{j}S_2(j,2r)E_{m-j}^{(-1)}\left(-\frac{2r+1}{2}\right)\frac{(i\omega)^m}{m!} \\
= &\sum_{m=0}^{\infty}\sum_{j=0}^{r}(-1)^{r+j}\binom{2r+1}{r-j}\frac{(2j+1)^{m+1}(1+(-1)^m)}{2^{m+1}}\frac{(i\omega)^m}{m!}.
\end{aligned}$$

Comparing the coefficients of $\frac{(i\omega)^m}{m!}$ on both sides of the above equation, we arrive at the desired result. □

If m is replaced by $2m+1$, then Eq. (15) reduces to the following corollary:

Corollary 5. *Let $m, r \in \mathbb{N}_0$. Then we have*

$$\sum_{j=0}^{2m+1}\binom{2m+1}{j}S_2(j,2r)E_{2m+1-j}^{(-1)}\left(-\frac{2r+1}{2}\right)=0.$$

If m is replaced by $2m$, then (15) reduces to the following corollary:

Corollary 6. *Let $m, r \in \mathbb{N}_0$. Then we have*

$$\begin{aligned}
&\sum_{j=0}^{2m}\binom{2m}{j}S_2(j,2r)E_{2m-j}^{(-1)}\left(-\frac{2r+1}{2}\right) \\
= &\frac{1}{2^{2m}(2r+1)!}\sum_{j=0}^{r}(-1)^{r+j}\binom{2r+1}{r-j}(2j+1)^{2m+1}. \quad (16)
\end{aligned}$$

Combining (16) with (12) yields the following result:

Corollary 7. *Let $m, r \in \mathbb{N}_0$. Then we have*

$$S_{2r}^{2m}\left(\frac{1-2r}{2}\right)+S_{2r}^{2m}\left(-\frac{2r+1}{2}\right)=2\sum_{j=0}^{2m}\binom{2m}{j}S_2(j,2r)E_{2m-j}^{(-1)}\left(-\frac{2r+1}{2}\right).$$

Theorem 8. *Let $m, r \in \mathbb{N}$. Then we have*

$$\begin{aligned}
&S_{2r-1}^{m}(1)-\frac{(-1)^r r^{m+1}}{(r!)^2} \\
= &\sum_{j=1}^{r}(-1)^{j+r}\binom{2r}{r-j}\frac{(2j)^m}{(2r)!}\left(2rE_m^{(-1)}\left(\frac{r-j}{2j}\right)+jmB_{m-1}^{(-1)}\left(\frac{r-j}{2j}\right)\right).
\end{aligned}$$

Proof. Using (13) and the following trigonometric identities

$$\left(2\sin\left(\frac{\omega}{2}\right)\right)^{2r}=\binom{2r}{r}+2\sum_{j=1}^{r}(-1)^{j}\binom{2r}{r-j}\cos\left(j\omega\right),$$

(*cf.* [**3**, Eq. (3.1.10)]), and

$$\sin\left(\omega\right)=\frac{e^{i\omega}-e^{-i\omega}}{2i}, \tag{17}$$

we have

$$\left(e^{i\omega}-1\right)^{2r}=(-1)^{r}\binom{2r}{r}e^{i\omega r}+\sum_{j=1}^{r}(-1)^{j+r}\binom{2r}{r-j}e^{i\omega r}\left(e^{i\omega j}+e^{-i\omega j}\right).$$

Differentiating both sides of the above equation with respect to ω, after some calculations, we get

$$\begin{aligned}&2re^{i\omega}\left(e^{i\omega}-1\right)^{2r-1}-(-1)^{r}\binom{2r}{r}re^{i\omega r}\\=&\sum_{j=1}^{r}(-1)^{j+r}\binom{2r}{r-j}\left(re^{i\omega r}\left(e^{i\omega j}+e^{-i\omega j}\right)+je^{i\omega r}\left(e^{i\omega j}-e^{-i\omega j}\right)\right).\end{aligned}$$

Combining the above equation with (1), (2) and (5), we get

$$\begin{aligned}&(2r)!\sum_{m=0}^{\infty}S_{2r-1}^{m}\left(1\right)\frac{(i\omega)^{m}}{m!}-\binom{2r}{r}(-1)^{r}\sum_{m=0}^{\infty}r^{m+1}\frac{(i\omega)^{m}}{m!}\\=&\sum_{m=0}^{\infty}\sum_{j=1}^{r}(-1)^{j+r}\binom{2r}{r-j}(2j)^{m}\\\times&\left(2rE_{m}^{(-1)}\left(\frac{r-j}{2j}\right)+jmB_{m-1}^{(-1)}\left(\frac{r-j}{2j}\right)\right)\frac{(i\omega)^{m}}{m!}.\end{aligned}$$

Comparing the coefficients of $\frac{(i\omega)^{m}}{m!}$ on both sides of the above equation, we arrive at the desired result. □

Theorem 9. *Let $r\in\mathbb{N}$ and $m\in\mathbb{N}_0$. Then we have*

$$y_1\left(m,2r;1\right)=\frac{(r+1)r^{m}C_r}{(2r)!}+\frac{2}{(2r)!}\sum_{j=1}^{r}\binom{2r}{r-j}(2j)^{m}E_{m}^{(-1)}\left(\frac{r-j}{2j}\right). \tag{18}$$

Proof. Combining the following identity

$$\left(e^{i\omega}+1\right)^{2r}=\binom{2r}{r}e^{i\omega r}+\sum_{j=1}^{r}\binom{2r}{r-j}e^{i\omega r}\left(e^{i\omega j}+e^{-i\omega j}\right) \tag{19}$$

(*cf.* [**16**]), with (2) and (6), we get

$$(2r)!\sum_{m=0}^{\infty} y_1(m,2r;1)\frac{(i\omega)^m}{m!} = (r+1)C_r\sum_{m=0}^{\infty} r^m\frac{(i\omega)^m}{m!} + 2\sum_{j=1}^{r}\binom{2r}{r-j}\sum_{m=0}^{\infty}(2j)^m E_m^{(-1)}\left(\frac{r-j}{2j}\right)\frac{(i\omega)^m}{m!}.$$

Comparing the coefficients of $\frac{(i\omega)^m}{m!}$ on both sides of the above equation, we arrive at the desired result. □

Theorem 10. *Let $m,r\in\mathbb{N}$. Then we have*

$$E_m^{(2r)}(r)+\sum_{j=1}^{r}\frac{\binom{2r}{r-j}}{\binom{2r}{r}}\left(E_m^{(2r)}(r+j)+E_m^{(2r)}(r-j)\right)=0.$$

Proof. By using (19), we get

$$1=\binom{2r}{r}\frac{e^{i\omega r}}{(e^{i\omega}+1)^{2r}}+\sum_{j=1}^{r}\binom{2r}{r-j}e^{i\omega r}\left(\frac{e^{i\omega j}+e^{-i\omega j}}{(e^{i\omega}+1)^{2r}}\right).$$

Combining the above equation with (2), we obtain

$$1=\frac{\binom{2r}{r}}{4^r}\sum_{m=0}^{\infty}E_m^{(2r)}(r)\frac{(i\omega)^m}{m!}+\sum_{j=1}^{r}\binom{2r}{r-j}\sum_{m=0}^{\infty}\left(\frac{E_m^{(2r)}(r+j)+E_m^{(2r)}(r-j)}{4^r}\right)\frac{(i\omega)^m}{m!}.$$

Comparing the coefficients of $\frac{(i\omega)^m}{m!}$ on both sides of the above equation, we arrive the desired result. □

Theorem 11. *Let $m\in\mathbb{N}_0$ and $r\in\mathbb{N}$. Then we have*

$$\sum_{j=1}^{r}\binom{2r}{r-j}\sum_{k=0}^{m}\binom{m}{k}r^{m-k}j^k\left(1+(-1)^k\right)=(2r)!y_1(m,2r;1)-\frac{r^m(2r)!}{(r!)^2}. \tag{20}$$

Proof. By using (6) and (19), we get

$$(2r)!\sum_{m=0}^{\infty} y_1(m,2r;1)\frac{(i\omega)^m}{m!} = \binom{2r}{r}\sum_{m=0}^{\infty} r^m\frac{(i\omega)^m}{m!}+\sum_{j=1}^{r}\binom{2r}{r-j}\sum_{m=0}^{\infty} r^m\frac{(i\omega)^m}{m!} \times\left(\sum_{m=0}^{\infty} j^m\frac{(i\omega)^m}{m!}+\sum_{m=0}^{\infty}(-j)^m\frac{(i\omega)^m}{m!}\right).$$

Thus

$$\begin{aligned}(2r)!\sum_{m=0}^{\infty} y_1\left(m,2r;1\right)\frac{(i\omega)^m}{m!} &= \binom{2r}{r}\sum_{m=0}^{\infty} r^m \frac{(i\omega)^m}{m!}\\ &+\sum_{j=1}^{r}\binom{2r}{r-j}\sum_{m=0}^{\infty}\sum_{k=0}^{m}\binom{m}{k} r^{m-k} j^k \frac{(i\omega)^m}{m!}\\ &+\sum_{j=1}^{r}\binom{2r}{r-j}\sum_{m=0}^{\infty}\sum_{k=0}^{m}(-1)^k\binom{m}{k} r^{m-k} j^k \frac{(i\omega)^m}{m!}.\end{aligned}$$

Comparing the coefficients of $\frac{(i\omega)^m}{m!}$ on both sides of the above equation, we arrive at the desired result. □

Combining (20) with (3) and (9), we have the following corollary:

Corollary 12. *Let $m \in \mathbb{N}_0$ and $r \in \mathbb{N}$. Then we have*

$$B\left(m;2r\right) - \left(r+1\right) r^m C_n = \sum_{j=1}^{r}\binom{2r}{r-j}\sum_{k=0}^{m}\binom{m}{k} r^{m-k} j^k \left(1+(-1)^k\right).$$

Theorem 13. *Let $m, r \in \mathbb{N}_0$. Then we have*

$$y_1\left(m,2r+1;1\right) = \frac{2}{(2r+1)!}\sum_{j=0}^{r}\binom{2r+1}{r-j}\left(2j+1\right)^m E_m^{(-1)}\left(\frac{r-j}{2j+1}\right).$$

Proof. By using (13) and the following trigonometric identity

$$(2\cos(\omega))^{2r+1} = 2\sum_{j=0}^{r}\binom{2r+1}{r-j}\cos\left(\left(2j+1\right)\omega\right) \tag{21}$$

(*cf.* [**3**, Eq. (3.1.13)]), we have

$$\left(e^{2i\omega}+1\right)^{2r+1} = \sum_{j=0}^{r}\binom{2r+1}{r-j} e^{2i\omega(r-j)}\left(e^{2i\omega(2j+1)}+1\right).$$

Combining the above equation with (2) and (6), we get

$$\begin{aligned}&(2r+1)!\sum_{m=0}^{\infty} y_1\left(m,2r+1;1\right)\frac{(2i\omega)^m}{m!}\\ = \;& 2\sum_{j=0}^{r}\binom{2r+1}{r-j}\sum_{m=0}^{\infty}\left(2j+1\right)^m E_m^{(-1)}\left(\frac{r-j}{2j+1}\right)\frac{(2i\omega)^m}{m!}.\end{aligned}$$

Comparing the coefficients of $\frac{(2i\omega)^m}{m!}$ on both sides of the above equation, we arrive at the desired result. □

2.1 Some finite sums including trigonometric functions

Here, by using some trigonometric identities, we obtain some finite sums and formulas, including trigonometric functions.

By using Theorem 8.30 in [**1**, Eqs. (10) and (14)], for every $x \neq k\pi$ with $k \in \mathbb{N}$, we have

$$\sum_{j=1}^{n} e^{i(2j-1)x} = e^{-ix} \sum_{j=1}^{n} e^{2ijx} = \frac{\sin(nx)}{\sin(x)} e^{inx}.$$

By taking real and imaginary parts of the above equation, we get the following results:

$$\sum_{j=1}^{n} \sin((2j-1)\omega) = \frac{\sin^2(n\omega)}{\sin(\omega)} \tag{22}$$

and

$$\sum_{j=1}^{n} \cos((2j-1)\omega) = \frac{\sin(2n\omega)}{2\sin(\omega)}, \tag{23}$$

where $n \in \mathbb{N}$ (*cf.* [**1**, Eqs. (15) and (16)]; see also [**10**, Eqs. (1.11) and (1.15)]).

Differentiating both sides of Eq. (22) with respect to ω, we have

$$\sum_{j=1}^{n} \frac{d}{d\omega} \left\{ \sin((2j-1)\omega) \right\} = \frac{d}{d\omega} \left\{ \frac{\sin^2(n\omega)}{\sin(\omega)} \right\}.$$

After some elementary calculations, we get

$$\sum_{j=1}^{n} (2j-1)\cos((2j-1)\omega) = \frac{2n\sin(n\omega)\cos(n\omega)}{\sin(\omega)} - \frac{\sin^2(n\omega)\cot(\omega)}{\sin(\omega)}.$$

Combining the above equation with (22) and (23), we obtain

$$\sum_{j=1}^{n} (2j-1)\cos((2j-1)\omega) = 2n \sum_{j=1}^{n} \cos((2j-1)\omega) - \cot(\omega) \sum_{j=1}^{n} \sin((2j-1)\omega).$$

After some elementary calculations in the above equation, for $n \in \mathbb{N}$, we have

$$\sum_{j=1}^{n} (2n-2j+1)\cos((2j-1)\omega) = \cot(\omega) \sum_{j=1}^{n} \sin((2j-1)\omega). \tag{24}$$

By combining (22) with (23), we also get

$$\sum_{j=1}^{n} \left(\cos((2j-1)\omega)\sin(n\omega) - \cos(n\omega)\sin((2j-1)\omega) \right) = 0.$$

Using the above equation, we have the following well-known result:

$$\sum_{j=1}^{n} \sin\left((n-2j+1)\,\omega\right) = 0.$$

3. p-ADIC INTEGRALS FORMULAS INCLUDING TRIGONOMETRIC FUNCTIONS

In this section, by using p-adic integrals and their integral equations, we give some formulas and finite sums including the trigonometric functions.

In order to give these results, we need the following notations, definitions, and properties of p-adic integrals, which have many applications in mathematics and mathematical physics.

Let p be a prime integer and $\mathbb{Z}_p$ denote the set of p-adic integers. Let $\mathbb{K}$ be a field with a complete valuation and $C^1\left(\mathbb{Z}_p \to \mathbb{K}\right)$ be a set of continuously differentiable functions. That is,

$$\left\{ g : \mathbb{Z}_p \to \mathbb{K} : \; g(u) \text{ is differentiable and } \frac{d}{du}\left\{g\left(u\right)\right\} \text{ is continuous} \right\}.$$

Further, let k be residue class field of $\mathbb{K}$. If $char\,(k) = p$ then

$$\mathbb{E} = \left\{ x \in \mathbb{K} : \; |x| < p^{\frac{1}{1-p}} \right\}$$

and if $char\,(k) = 0$ then

$$\mathbb{E} = \left\{ x \in \mathbb{K} : \; x < 1 \right\}$$

(*cf.* [**31**]).

The integral equation for the Volkenborn integral (or the bosonic p-adic integral) on $\mathbb{Z}_p$ is given by

$$\int_{\mathbb{Z}_p} g\left(u+n\right) d\mu_1\left(u\right) = \int_{\mathbb{Z}_p} g\left(u\right) d\mu_1\left(u\right) + \sum_{k=0}^{n-1} g'\left(k\right),$$

where

$$g'\left(k\right) = \frac{d}{du}\left\{g\left(u\right)\right\}|_{u=k}$$

(*cf.* [**31**]; see also [**19**], [**40**]).

The Volkenborn integral is related to the trigonometric functions. The Volkenborn integral for the cosine function is given as follows:

$$\begin{aligned} \int_{\mathbb{Z}_p} \cos\left(\beta u\right) d\mu_1\left(u\right) &= \frac{\beta \sin\left(\beta\right)}{2\left(1-\cos\left(\beta\right)\right)} \\ &= \frac{\beta}{2}\cot\left(\frac{\beta}{2}\right), \end{aligned} \tag{25}$$

where $\beta \in \mathbb{E}$ with $\beta \neq 0$ and $p \neq 2$ (*cf.* [**31**, p. 172]; see also [**21**], [**40**]).

The Volkenborn integral for the sine function is given as follows:

$$\int_{\mathbb{Z}_p} \sin(\beta u)\, d\mu_1(u) = -\frac{\beta}{2}, \tag{26}$$

where $\beta \in \mathbb{E}$ (*cf.* [**31**, p. 170]; see also [**21**], [**40**]).

The integral equation for the fermionic p-adic integral on $\mathbb{Z}_p$ is given by

$$\int_{\mathbb{Z}_p} g(u+n)\, d\mu_{-1}(u) + (-1)^{n+1} \int_{\mathbb{Z}_p} g(u)\, d\mu_{-1}(u) = 2\sum_{k=0}^{n-1} (-1)^{n-1-k} g(k),$$

where $n \in \mathbb{N}$ (*cf.* [**19**], [**20**], [**21**], [**40**]).

The fermionic p-adic integral is related to the trigonometric functions, which are given as follows:

$$\int_{\mathbb{Z}_p} \cos(\beta u)\, d\mu_{-1}(u) = 1, \tag{27}$$

where $\beta \in \mathbb{E}$ with $\beta \neq 0$ and $p \neq 2$, also

$$\int_{\mathbb{Z}_p} \sin(\beta u)\, d\mu_{-1}(u) = -\frac{\sin(\beta)}{\cos(\beta)+1} \tag{28}$$

(*cf.* [**19**], [**20**], [**21**], [**40**]).

Now, using the above properties and relations of the p-adic integrals, we give some formulas related to the trigonometric functions and finite sums.

Theorem 14. *Let $n \in \mathbb{N}$. Then we have*

$$\sum_{j=1}^{n} \int_{\mathbb{Z}_p} \sin((2j-1)\omega) \cot(\omega)\, d\mu_1(\omega) = \frac{1}{2} \sum_{j=1}^{n} (2n-2j+1)(2j-1) \cot\left(\frac{2j-1}{2}\right).$$

Proof. By applying the Volkenborn integral to Eq. (24), we get

$$\sum_{j=1}^{n} (2n-2j+1) \int_{\mathbb{Z}_p} \cos((2j-1)\omega)\, d\mu_1(\omega) = \sum_{j=1}^{n} \int_{\mathbb{Z}_p} \sin((2j-1)\omega) \cot(\omega)\, d\mu_1(\omega).$$

Combining the above equation with (25), we obtain

$$\frac{1}{2} \sum_{j=1}^{n} (2n-2j+1)(2j-1) \cot\left(\frac{2j-1}{2}\right) = \sum_{j=1}^{n} \int_{\mathbb{Z}_p} \sin((2j-1)\omega) \cot(\omega)\, d\mu_1(\omega).$$

Thus, the proof of the theorem is completed. □

Theorem 15. *Let $n \in \mathbb{N}$. Then we have*

$$\int_{\mathbb{Z}_p} \frac{\sin^2(n\omega)}{\sin(\omega)} d\mu_1(\omega) = -\frac{n^2}{2}.$$

Proof. By applying the Volkenborn integral to Eq. (22), we have

$$\sum_{j=1}^{n} \int_{\mathbb{Z}_p} \sin((2j-1)\omega)\, d\mu_1(\omega) = \int_{\mathbb{Z}_p} \frac{\sin^2(n\omega)}{\sin(\omega)} d\mu_1(\omega).$$

Combining the above equation with (26), we get

$$\sum_{j=1}^{n} \frac{(1-2j)}{2} = \int_{\mathbb{Z}_p} \frac{\sin^2(n\omega)}{\sin(\omega)} d\mu_1(\omega).$$

Thus,

$$\int_{\mathbb{Z}_p} \frac{\sin^2(n\omega)}{\sin(\omega)} d\mu_1(\omega) = -\frac{n^2}{2}.$$

Therefore, the proof of the theorem is completed. □

Theorem 16. *Let $n \in \mathbb{N}$. Then we have*

$$\int_{\mathbb{Z}_p} \frac{\sin(2n\omega)}{\sin(\omega)} d\mu_1(\omega) = \sum_{j=1}^{n} (2j-1) \cot\left(\frac{2j-1}{2}\right).$$

Proof. By applying the Volkenborn integral to Eq. (23), we have

$$\sum_{j=1}^{n} \int_{\mathbb{Z}_p} \cos((2j-1)\omega)\, d\mu_1(\omega) = \int_{\mathbb{Z}_p} \frac{\sin(2n\omega)}{2\sin(\omega)} d\mu_1(\omega).$$

Combining the above equation with (25), we get

$$\sum_{j=1}^{n} (2j-1) \cot\left(\frac{2j-1}{2}\right) = \int_{\mathbb{Z}_p} \frac{\sin(2n\omega)}{\sin(\omega)} d\mu_1(\omega).$$

Thus, we arrive at the desired result. □

Theorem 17. *Let $n \in \mathbb{N}$. Then we have*

$$\sum_{j=1}^{n} \int_{\mathbb{Z}_p} \sin((2j-1)\omega) \cot(\omega)\, d\mu_{-1}(\omega) = n^2.$$

Proof. By applying the fermionic p-adic integral to Eq. (24), we get

$$\sum_{j=1}^{n}(2n-2j+1)\int\limits_{\mathbb{Z}_p}\cos\left((2j-1)\,\omega\right)d\mu_{-1}\left(\omega\right)=\sum_{j=1}^{n}\int\limits_{\mathbb{Z}_p}\sin\left((2j-1)\,\omega\right)\cot\left(\omega\right)d\mu_{-1}\left(\omega\right).$$

Combining the above equation with (27), we obtain

$$\sum_{j=1}^{n}(2n-2j+1)=\sum_{j=1}^{n}\int\limits_{\mathbb{Z}_p}\sin\left((2j-1)\,\omega\right)\cot\left(\omega\right)d\mu_{-1}\left(\omega\right).$$

Hence

$$\sum_{j=1}^{n}\int\limits_{\mathbb{Z}_p}\sin\left((2j-1)\,\omega\right)\cot\left(\omega\right)d\mu_{-1}\left(\omega\right)=n^2.$$

Therefore, the proof of the theorem is completed. □

Theorem 18. *Let $n\in\mathbb{N}$. Then we have*

$$\int\limits_{\mathbb{Z}_p}\frac{\sin^2\left(n\omega\right)}{\sin\left(\omega\right)}d\mu_{-1}\left(\omega\right)=-\sum_{j=1}^{n}\tan\left(\frac{2j-1}{2}\right).$$

Proof. By applying the fermionic p-adic integral to Eq. (22), we have

$$\sum_{j=1}^{n}\int\limits_{\mathbb{Z}_p}\sin\left((2j-1)\,\omega\right)d\mu_{-1}\left(\omega\right)=\int\limits_{\mathbb{Z}_p}\frac{\sin^2\left(n\omega\right)}{\sin\left(\omega\right)}d\mu_{-1}\left(\omega\right).$$

Combining the above equation with (28), we get

$$-\sum_{j=1}^{n}\frac{\sin\left(2j-1\right)}{\cos\left(2j-1\right)+1}=\int\limits_{\mathbb{Z}_p}\frac{\sin^2\left(n\omega\right)}{\sin\left(\omega\right)}d\mu_{-1}\left(\omega\right).$$

After some elementary calculations, we arrive at the desired result. □

Theorem 19. *Let $n\in\mathbb{N}$. Then we have*

$$\int\limits_{\mathbb{Z}_p}\frac{\sin\left(2n\omega\right)}{\sin\left(\omega\right)}d\mu_{-1}\left(\omega\right)=2n.$$

Proof. By applying the fermionic p-adic integral to Eq. (23), we have

$$\sum_{j=1}^{n}\int\limits_{\mathbb{Z}_p}\cos\left((2j-1)\,\omega\right)d\mu_{-1}\left(\omega\right)=\int\limits_{\mathbb{Z}_p}\frac{\sin\left(2n\omega\right)}{2\sin\left(\omega\right)}d\mu_{-1}\left(\omega\right).$$

Combining the above equation with (27), after some calculations, we obtain

$$\frac{1}{2}\int_{\mathbb{Z}_p} \frac{\sin(2n\omega)}{\sin(\omega)} d\mu_{-1}(\omega) = n.$$

Thus, we arrive at the desired result. □

4. EXPLICIT FORMULAS FOR THE SUMS OF POWERS OF CONSECUTIVE POSITIVE INTEGERS DERIVED FROM FINITE SUMS INVOLVING TRIGONOMETRIC FUNCTIONS

In this section, using finite sums involving trigonometric functions, we give interesting new calculation formulas, especially on the sum of the powers of consecutive positive integers. The sum of the powers of the positive integers was started to be studied with the finding of the numbers. Especially, the most important calculation formulas were given by the famous German mathematician Johann Faulhaber at the beginning of the 17th century, and these are still called Faulhaber formulas today. The well-known Faulhaber's formula is given as follows:

$$\sum_{j=1}^{n-1} j^m = \frac{B_{m+1}(n) - B_{m+1}}{m+1}, \tag{29}$$

where $m \in \mathbb{N}$.

In [**12**], Gou and Shen studied on the following sums of powers of odd integers:

$$\sum_{j=1}^{n} (2j-1)^{2k-1} \qquad \text{and} \qquad \sum_{j=1}^{n} (2j-1)^{2k}.$$

They gave the following formulas for these sums:

$$\sum_{j=1}^{n} (2j-1)^{2k-1} = n^2 \sum_{j=1}^{k} c_j n^{2k-2j}$$

and

$$\sum_{j=1}^{n} (2j-1)^{2k} = n(2n-1)(2n+1) \sum_{j=1}^{k} d_j (2n-1)^{k-j} (2n+1)^{k-j},$$

where c_j and d_j $(j = 1, 2, ..., k)$ are undetermined constants (*cf.* [**12**, Eqs. (1.8) and (1.9)]). They also found some recurrence relations for c_j and d_j.

We give the following explicit and novel formulas in terms of the Bernoulli polynomials and numbers for the sums of powers of consecutive positive integers:

Theorem 20. *Let $m \in \mathbb{N}_0$ and $n \in \mathbb{N}$. Then we have*

$$\sum_{j=1}^{n} (2j-1)^{2m} = \frac{2^{2m} B_{2m+1}\left(\frac{2n+1}{2}\right)}{2m+1}. \tag{30}$$

Proof. By using (23), we have

$$2\sum_{j=1}^{n}\sum_{m=0}^{\infty} (-1)^m (2j-1)^{2m} \frac{\omega^{2m}}{(2m)!} = \frac{e^{(2n+1)i\omega} - e^{(1-2n)i\omega}}{e^{2i\omega}-1}. \tag{31}$$

Combining the above equation with (1), we get

$$\begin{aligned} &2\sum_{j=1}^{n}\sum_{m=0}^{\infty} (-1)^m (2j-1)^{2m} \frac{\omega^{2m}}{(2m)!} \\ =& \sum_{m=0}^{\infty} \frac{1}{m+1}\left(B_{m+1}\left(\frac{2n+1}{2}\right) - B_{m+1}\left(\frac{1-2n}{2}\right)\right)\frac{(2i\omega)^m}{m!}. \end{aligned}$$

Thus,

$$\begin{aligned} &2\sum_{j=1}^{n}\sum_{m=0}^{\infty} (-1)^m (2j-1)^{2m} \frac{\omega^{2m}}{(2m)!} \\ =& \sum_{m=0}^{\infty} \frac{1}{2m+1}\left(B_{2m+1}\left(\frac{2n+1}{2}\right) - B_{2m+1}\left(\frac{1-2n}{2}\right)\right)(2i)^{2m}\frac{\omega^{2m}}{(2m)!} \\ &+ \sum_{m=0}^{\infty} \frac{1}{2m+2}\left(B_{2m+2}\left(\frac{2n+1}{2}\right) - B_{2m+2}\left(\frac{1-2n}{2}\right)\right)(2i)^{2m+1}\frac{\omega^{2m+1}}{(2m+1)!}. \end{aligned}$$

By comparing the coefficients of the same powers of variable ω on both sides of the above equation, we have the following results:

$$\sum_{j=1}^{n} (2j-1)^{2m} = \frac{2^{2m-1}}{2m+1}\left(B_{2m+1}\left(\frac{2n+1}{2}\right) - B_{2m+1}\left(\frac{1-2n}{2}\right)\right) \tag{32}$$

and

$$B_{2m+2}\left(\frac{2n+1}{2}\right) = B_{2m+2}\left(\frac{1-2n}{2}\right).$$

Combining (32) with the following well-known identity,

$$B_k(1-x) = (-1)^k B_k(x),$$

we arrive at the desired result. □

Theorem 21. *Let $n \in \mathbb{N}$ and $m \in \mathbb{N}_0$. Then we have*

$$\begin{aligned}&\left(1+(-1)^m\right)\sum_{j=1}^{n}(2n-2j+1)(2j-1)^m\\ =&\sum_{j=1}^{n}\sum_{k=0}^{\left[\frac{m}{2}\right]}\binom{m}{2k}B_{2k}B_{m-2k}^{(-1)}\left(-\frac{1}{2}\right)2^{m+1}(2j-1)^{m-2k+1}.\end{aligned}$$

Proof. Combining (24) with (13), (17) and the following well-known identity

$$\omega\cot(\omega)=\sum_{m=0}^{\infty}(-1)^m 2^{2m}B_{2m}\frac{\omega^{2m}}{(2m)!}, \tag{33}$$

we have

$$\begin{aligned}&\frac{1}{2}\sum_{j=1}^{n}(2n-2j+1)\left(e^{i\omega(2j-1)}+e^{-i\omega(2j-1)}\right)\\ =&\frac{1}{2i\omega}\sum_{j=1}^{n}e^{-i\omega(2j-1)}\left(e^{2i\omega(2j-1)}-1\right)\sum_{m=0}^{\infty}(-1)^m 2^{2m}B_{2m}\frac{\omega^{2m}}{(2m)!}.\end{aligned}$$

Substituting (1) into the above equation, after some calculations, we get

$$\begin{aligned}&\sum_{m=0}^{\infty}\sum_{j=1}^{n}(2n-2j+1)\left(1+(-1)^m\right)(2j-1)^m i^m\frac{\omega^m}{m!}\\ =&\sum_{j=1}^{n}\sum_{m=0}^{\infty}\sum_{k=0}^{\left[\frac{m}{2}\right]}\binom{m}{2k}B_{m-2k}^{(-1)}\left(-\frac{1}{2}\right)(4j-2)^{m-2k+1}i^{m-2k}(-1)^k 2^{2k}B_{2k}\frac{\omega^m}{m!}.\end{aligned}$$

Comparing the coefficients of $\frac{\omega^m}{m!}$ on both sides of the above equation, after some elementary calculations, we arrive at the desired result. □

Lemma 22. *Let $m \in \mathbb{N}_0$ and $n \in \mathbb{N}$. Then we have*

$$\sum_{j=1}^{n}(2j-1)^{2m}=\sum_{k=0}^{m}\sum_{j=1}^{n}\binom{2m}{2k}\frac{2^{2k}n^{2k-1}(2j-1)^{2m-2k+1}B_{2k}}{(2m-2k+1)}.$$

Proof. By combining (22) with (23), we have

$$\sum_{j=1}^{n}\cos\left((2j-1)\omega\right)=\cot(n\omega)\sum_{j=1}^{n}\sin\left((2j-1)\omega\right).$$

From the above equation, we have

$$\sum_{m=0}^{\infty}\sum_{j=1}^{n}(-1)^m(2j-1)^{2m}\frac{\omega^{2m}}{(2m)!}=\cot(n\omega)\sum_{m=0}^{\infty}\sum_{j=1}^{n}(-1)^m(2j-1)^{2m+1}\frac{\omega^{2m+1}}{(2m+1)!}.$$

Substituting (33) into the above equation, after some calculations, we obtain

$$\sum_{m=0}^{\infty}\sum_{j=1}^{n}(-1)^{m}(2j-1)^{2m}\frac{\omega^{2m}}{(2m)!}$$
$$=\sum_{m=0}^{\infty}\sum_{k=0}^{m}\sum_{j=1}^{n}\frac{(-1)^{m}2^{2k}n^{2k-1}(2j-1)^{2m-2k+1}B_{2k}}{(2k)!\,(2m-2k+1)!}\omega^{2m}.$$

Comparing the coefficients of ω^{2m} on both sides of the above equation, after some calculations, we arrive at the desired result. □

Using (31), we get

$$2\sum_{j=1}^{n}\sum_{m=0}^{\infty}(-1)^{m}(2j-1)^{2m}\frac{\omega^{2m}}{(2m)!} = \frac{1}{2i\omega}\sum_{m=0}^{\infty}B_{m}\frac{(2i\omega)^{m}}{m!}\sum_{m=0}^{\infty}(2n+1)^{m}\frac{(i\omega)^{m}}{m!} -\frac{1}{2i\omega}\sum_{m=0}^{\infty}B_{m}\frac{(2i\omega)^{m}}{m!}\sum_{m=0}^{\infty}(1-2n)^{m}\frac{(i\omega)^{m}}{m!}.$$

Thus,

$$\sum_{j=1}^{n}\sum_{m=0}^{\infty}(-1)^{m}(2j-1)^{2m}\frac{\omega^{2m}}{(2m)!} = \sum_{m=0}^{\infty}\sum_{k=0}^{m+1}\binom{m+1}{k}\frac{2^{m-k-1}}{m+1} \times\left((2n+1)^{k}-(1-2n)^{k}\right)B_{m+1-k}\frac{(i\omega)^{m}}{m!}.$$

By comparing the coefficients of the same powers of variable ω on both sides of the above equation, after some elementary algebraic computations, we get the following lemmas:

Lemma 23. *Let $m\in\mathbb{N}_0$ and $n\in\mathbb{N}$. Then we have*

$$\sum_{j=1}^{n}(2j-1)^{2m}=\sum_{k=0}^{2m+1}\binom{2m+1}{k}\frac{2^{2m-k-1}}{2m+1}\left((2n+1)^{k}-(1-2n)^{k}\right)B_{2m+1-k}.$$

Lemma 24. *Let $m\in\mathbb{N}_0$ and $n\in\mathbb{N}$. Then we have*

$$\sum_{k=0}^{2m+2}\binom{2m+2}{k}2^{-k}\left((2n+1)^{k}-(1-2n)^{k}\right)B_{2m+2-k}=0.$$

Combining Theorem 20 with Lemma 22 and Lemma 23, we get the following results including the Bernoulli numbers and polynomials:

Corollary 25. *Let $m\in\mathbb{N}_0$ and $n\in\mathbb{N}$. Then we have*

$$\sum_{k=0}^{2m+1}\binom{2m+1}{k}\frac{2^{-k-1}}{2m+1}\left((2n+1)^{k}-(1-2n)^{k}\right)B_{2m+1-k}=\frac{B_{2m+1}\left(\frac{2n+1}{2}\right)}{2m+1},$$

$$\sum_{k=0}^{m}\sum_{j=1}^{n}\binom{2m}{2k}\frac{2^{2k-2m}n^{2k-1}\left(2j-1\right)^{2m-2k+1}B_{2k}}{(2m-2k+1)}=\frac{B_{2m+1}\left(\frac{2n+1}{2}\right)}{2m+1},$$

and

$$\begin{aligned}&\sum_{k=0}^{2m+1}\binom{2m+1}{k}\frac{2^{2m-k-1}}{2m+1}\left((2n+1)^{k}-(1-2n)^{k}\right)B_{2m+1-k}\\=\ &\sum_{k=0}^{m}\sum_{j=1}^{n}\binom{2m}{2k}\frac{2^{2k}n^{2k-1}\left(2j-1\right)^{2m-2k+1}B_{2k}}{(2m-2k+1)}.\end{aligned}$$

Theorem 26. *Let $m \in \mathbb{N}_0$ and $n \in \mathbb{N}$. Then we have*

$$\sum_{j=1}^{n}(2j-1)^{m}=2^{m}\sum_{j=0}^{m}\binom{m}{j}n^{j+1}B_{j}^{(-1)}B_{m-j}\left(\frac{1}{2}\right). \tag{34}$$

Proof. By combining the Euler's formula with (22) and (23), we get

$$\sum_{j=1}^{n}e^{(2j-1)i\omega}=\frac{\sin\left(n\omega\right)}{\sin\left(\omega\right)}e^{in\omega}.$$

Thus,

$$\sum_{j=1}^{n}\sum_{m=0}^{\infty}(2j-1)^{m}\,\frac{(i\omega)^{m}}{m!}=\frac{e^{i\omega}\left(e^{2in\omega}-1\right)}{e^{2i\omega}-1}.$$

Combining the above equation with (1), we have

$$\sum_{j=1}^{n}\sum_{m=0}^{\infty}(2j-1)^{m}\,\frac{(i\omega)^{m}}{m!}=n\sum_{m=0}^{\infty}B_{m}^{(-1)}\frac{(2in\omega)^{m}}{m!}\sum_{m=0}^{\infty}B_{m}\left(\frac{1}{2}\right)\frac{(2i\omega)^{m}}{m!}.$$

Hence,

$$\sum_{j=1}^{n}\sum_{m=0}^{\infty}(2j-1)^{m}\,\frac{(i\omega)^{m}}{m!}=\sum_{m=0}^{\infty}\sum_{j=0}^{m}\binom{m}{j}2^{m}n^{j+1}B_{j}^{(-1)}B_{m-j}\left(\frac{1}{2}\right)\frac{(i\omega)^{m}}{m!}.$$

Comparing the coefficients of $\frac{(i\omega)^m}{m!}$ on both sides of the above equation, we arrive at the desired result. □

Now we give some special cases of Eq. (34).

When $m = 1$ in (34), we have the following well-known sum:

$$\sum_{j=1}^{n}(2j-1)=n^{2},$$

for detail, see OEIS A000290 also [**11**].

When $m=2$ in (34), we have the following well-known sum:

$$\sum_{j=1}^{n}(2j-1)^2=\frac{n\left(4n^2-1\right)}{3},$$

for detail, see OEIS A000447 also [**11**].

Substituting $m=3$ into (34), we have the following well-known sum:

$$\sum_{j=1}^{n}(2j-1)^3=n^2\left(2n^2-1\right),$$

for detail, see OEIS A002593 also [**11**].

By using (34), we have

$$\sum_{j=1}^{n}\sum_{k=0}^{m}(-1)^{m-k}\binom{m}{k}2^k j^k=2^m\sum_{j=0}^{m}\binom{m}{j}n^{j+1}B_j^{(-1)}B_{m-j}\left(\frac{1}{2}\right).$$

Combining the above equation with (29), we get the following finite sum:

$$(-1)^m n+\sum_{k=1}^{m}(-1)^{m-k}\binom{m}{k}2^k\left(\frac{B_{k+1}(n+1)-B_{k+1}}{k+1}\right)$$
$$=2^m\sum_{j=0}^{m}\binom{m}{j}n^{j+1}B_j^{(-1)}B_{m-j}\left(\frac{1}{2}\right).$$

Hence, we arrive at the following corollary:

Corollary 27. *Let $m,n\in\mathbb{N}$. Then we have*

$$(-1)^m n+\sum_{k=1}^{m}(-1)^{m-k}\binom{m}{k}2^k\left(\frac{B_{k+1}(n+1)-B_{k+1}}{k+1}\right)$$
$$=2^m\sum_{j=0}^{m}\binom{m}{j}n^{j+1}B_j^{(-1)}B_{m-j}\left(\frac{1}{2}\right).$$

Replacing m by $2m$ in (34), and combining final equation with (30), we have the following corollary:

Corollary 28. *Let $m\in\mathbb{N}_0$ and $n\in\mathbb{N}$. Then we have*

$$\sum_{j=0}^{2m}\binom{2m}{j}n^{j+1}B_j^{(-1)}B_{2m-j}\left(\frac{1}{2}\right)=\frac{B_{2m+1}\left(\frac{2n+1}{2}\right)}{2m+1}.$$

Theorem 29. *Let $m\in\mathbb{N}_0$ and $n\in\mathbb{N}$. Then we have*

$$\sum_{j=1}^{n}(2j-1)^{2m+1}=(2m+1)\sum_{k=0}^{m}\sum_{j=0}^{k}\binom{2k}{2j}\binom{2m}{2k}\frac{n^{2k+2}4^{m-k}B_{2m-2k}\left(\frac{1}{2}\right)}{(2j+1)(2k-2j+1)}.$$

Proof. By using (22) and (1), we have

$$\begin{aligned}&\sum_{j=1}^{n}\sum_{m=0}^{\infty}(-1)^m(2j-1)^{2m+1}\frac{\omega^{2m+1}}{(2m+1)!}\\ =&\sum_{m=0}^{\infty}(-1)^m\frac{(n\omega)^{2m+1}}{(2m+1)!}\sum_{m=0}^{\infty}(-1)^m n^{2m+1}\frac{\omega^{2m}}{(2m+1)!}\sum_{m=0}^{\infty}B_m\left(\frac{1}{2}\right)\frac{(2i\omega)^m}{m!}.\end{aligned}$$

Therefore,

$$\begin{aligned}&\sum_{j=1}^{n}\sum_{m=0}^{\infty}(-1)^m(2j-1)^{2m+1}\frac{\omega^{2m+1}}{(2m+1)!}\\ =&\sum_{m=0}^{\infty}\sum_{k=0}^{\left[\frac{m}{2}\right]}\sum_{j=0}^{k}\frac{(-1)^k n^{2k+2}(2i)^{m-2k}B_{m-2k}\left(\frac{1}{2}\right)}{(2j+1)!\,(2k-2j+1)!\,(m-2k)!}\omega^{m+1}.\end{aligned}$$

By comparing the coefficients of the same powers of variable ω on both sides of the above equation, we get

$$\frac{(-1)^m}{(2m+1)!}\sum_{j=1}^{n}(2j-1)^{2m+1}=\sum_{k=0}^{m}\sum_{j=0}^{k}\frac{(-1)^m n^{2k+2}2^{2m-2k}B_{2m-2k}\left(\frac{1}{2}\right)}{(2j+1)!\,(2k-2j+1)!\,(2m-2k)!}.$$

Thus, the proof of the theorem is completed. □

5. CONCLUSIONS

In this paper, many new and interesting formulas and relations were given using special functions, trigonometric functions, and some finite sums. These formulas and relations also included well-known numbers and polynomials. Some of these were given as follows: the Bernoulli numbers and polynomials, the Euler numbers and polynomials, the Stirling numbers, the Catalan numbers, the array polynomials and combinatorial numbers, certain finite sums. In addition to these, new formulas on the sum of powers of positive integers, called Faulhaber formulas, were given by the famous German mathematician Johann Faulhaber in the early 17th century. A few new calculation formulas associated with the Faulhaber formulas were also given, especially for the sum of the powers of consecutive positive integers. Furthermore, integral representations, containing the Volkenborn integral and the fermionic p-adic integral, were given for the trigonometric functions and sums. Moreover, our future studies will be investigate relations between the sums of the powers of consecutive positive integers, certain family of finite sums related to trigonometric functions, the Dedekind sums, the Hardy sums, the character sums, and the Kloosterman sums.

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Neslihan Kilar
Department of Computer Technologies,
Bor Vocational School,
Niğde Ömer Halisdemir University,
TR-51700 Niğde, Turkey
E-mail: *neslihankilar@ohu.edu.tr; neslihankilar@gmail.com*

Abdelmejid Bayad
Laboratoire de Mathématiques et Modélisation d'Évry (LAMME),
Université Paris-Saclay,
CNRS (UMR 8071), Bâtiment I.B.G.B.I.,
23 Boulevard de France, CEDEX, 91037 Evry, France
E-mail: *abdelmejid.bayad@univ-evry.fr*

Yilmaz Simsek
Department of Mathematics,
Faculty of Science,
University of Akdeniz,
TR-07058 Antalya, Turkey
E-mail: *ysimsek@akdeniz.edu.tr; ysimsek63@gmail.com*