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OSCILLATORY BEHAVIOR OF SEMI-CANONICAL THIRD-ORDER DELAY DIFFERENTAL EQUATIONS WITH A SUPERLINEAR NEUTRAL TERM

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A class of third-order semi-canonical differential equations with a superlinear neutral term of the type

$$\Big(a(t)[(b(t)(x(t)+p(t)x^{\alpha}(\tau(t))')']^{\beta})' + q(t)x^{\lambda}(\sigma(t)) = 0$$

is considered. Some oscillation conditions are presented which are new in form and complement those already reported in the literature. Some examples illustrating the main results are provided.

1. INTRODUCTION

The study of neutral type differential equations has received a great deal of attention in recent years due in part to their many applications in the natural sciences and technology. For example, they occur in the study of distributed networks containing lossless transmission lines and as the Euler equations for variational problems involving a delay (see [**9**]). In this paper, we investigate the oscillatory behavior of the third-order neutral differential equation

$$(E_1) \qquad (a(t)[(b(t)z'(t))']^{\beta})' + q(t)x^{\lambda}(\sigma(t)) = 0, \quad t \geq t_0 > 0,$$

where $z(t) = x(t) + p(t)x^{\alpha}(\tau(t))$. Throughout this paper, we assume that:

(A_1) α, β, and λ are ratios of odd positive integers with $\alpha \geq 1$;

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(A_2) a, b, p, q, $\tau \in C([t_0,\infty),\mathbb{R})$, $a(t) > 0$, $b(t) > 0$, $q(t) > 0$, and $p(t) \geq 0$ with $\lim_{t\to\infty} p(t) = 0$;

(A_3) $\sigma \in C'([t_0,\infty),\mathbb{R})$, $\tau(t) \leq t$, $\sigma(t) \leq t$, and

$$\lim_{t\to\infty} \tau(t) = \lim_{t\to\infty} \sigma(t) = \infty;$$

(A_4) equation (E_1) is of semi-canonical type in the sense that

$$\int_{t_0}^{\infty} \frac{1}{a^{1/\beta}(t)} dt = \infty \quad \text{and} \quad \int_{t_0}^{\infty} \frac{1}{b(t)} dt < \infty.$$

By a *solution* of (E_1), we mean a nontrivial function $x \in C([t_x,\infty),\mathbb{R})$, $t_x \geq t_0$, with $z \in C'([t_x,\infty),\mathbb{R})$, $b(t)z'(t) \in C'([t_0,\infty),\mathbb{R})$, $a(t)[(b(t)z'(t))']^\beta \in C'([t_x,\infty),\mathbb{R})$ and which satisfies (E_1) on $[t_x,\infty)$. We consider only those solutions $x(t)$ of (E_1) that are nontrivial in the sense that $sup\{|x(t)| : t \geq T\} > 0$ for all $T \geq t_x$, and we tacitly assume that (E_1) possesses such solutions. A solution of (E_1) is called *oscillatory* if it is neither eventually positive nor eventually negative on $[t_x,\infty)$; otherwise, it is termed *nonoscillatory*.

The study of qualitative behavior of solutions of third order differential equations is very helpful in predicting the dynamic behavior of solutions of third order partial differential equations (see [**12, 16**]). In recent years, many researchers have been interested in the oscillatory behavior of third order differential equations; see, for example, [**1, 4, 5, 7, 8, 17, 22, 25**] and the references therein.

In [**11**], the authors studied the oscillatory behavior of solutions of (E_1) in the special case $b(t) \equiv 1$ and $\alpha = \beta = \lambda = 1$. In [**13, 14**], the authors investigated equation (E_1) in the case $\alpha = \beta = \lambda = 1$ under the three assumptions

$$(1) \qquad \int_{t_0}^{\infty} \frac{1}{a(t)} dt = \infty, \quad \int_{t_0}^{\infty} \frac{1}{b(t)} dt = \infty,$$

$$(2) \qquad \int_{t_0}^{\infty} \frac{1}{a(t)} dt < \infty, \quad \int_{t_0}^{\infty} \frac{1}{b(t)} dt = \infty,$$

$$(3) \qquad \int_{t_0}^{\infty} \frac{1}{a(t)} dt < \infty, \quad \int_{t_0}^{\infty} \frac{1}{b(t)} dt < \infty,$$

and established conditions guaranteeing that a solution either oscillates or tends to zero.

In [**21**], the authors studied equation (E_1) in the case where $\beta = 1$ and $0 < \alpha \leq 1$ under each of the conditions (1)–(3) and

$$\int_{t_0}^{\infty} \frac{1}{a(t)} dt = \infty, \quad \int_{t_0}^{\infty} \frac{1}{b(t)} dt < \infty. \tag{4}$$

In [**3**] and [**24**], the authors considered the equation (E_1) with $a(t) \equiv 1 \equiv b(t)$ and $\alpha = \beta = 1$ under the assumption that

$$p(t) \geq 1 \quad \text{and} \quad p(t) \not\equiv 1 \quad \text{for large } t, \tag{5}$$

and obtained conditions for the oscillation of all solutions. In [**2, 20**], the authors extended the results obtained in [**3**] for equation (E_1) in the case $\alpha = 1$ and $b(t) \equiv 1$. In [**23**], the authors studied equation (E_1) for the case $a(t) \equiv b(t) \equiv 1$ under condition (5) and obtained conditions which ensure that a solution either oscillates or tends to zero. In [**6**] and [**15**], the authors obtained similar results for equation (E_1) with $b(t) \equiv 1$ under the conditions (5) and either

$$\int_{t_0}^{\infty} a^{-1/\beta}(t) dt = \infty,$$

or

$$\int_{t_0}^{\infty} a^{-1/\beta}(t) dt < \infty.$$

It appears that there are no known results regarding the oscillatory behavior of solutions of third order differential equations with a super-linear neutral term and $p(t) \to 0$ as $t \to \infty$. Motivated by the above observations, in this paper we investigate the oscillatory and asymptotic behavior of solutions of equation (E_1) by first transforming the equation into a canonical type equation and then applying a Ricatti type transformation. In view of this, our results obtained here are new and complement existing results reported in the literature.

2. MAIN RESULTS

To present our results in a compact form, we adopt the following notations:

$$B(t) = \int_{t}^{\infty} \frac{1}{b(s)} ds, \quad d(t) = b(t) B^2(t), \quad c(t) = \frac{a(t)}{B^{\beta}(t)},$$

$$\mu(t, t_*) = \int_{t_*}^{t} c^{-1/\beta}(s) ds, \quad \text{and} \quad \eta(t, t_*) = \int_{t_*}^{t} \frac{\mu(s, t_*)}{d(s)} ds$$

for $t \geq t_*$ for any $t_* \geq t_0$.

Theorem 1. *Assume that*

$$\int_{t_0}^{\infty} c^{-1/\beta}(t)dt = \infty. \tag{6}$$

Then the semi-canonical operator $\mathcal{L}$ defined by

$$\mathcal{L}z(t) = (a(t)[(b(t)z'(t))']^{\beta})'$$

can be written as the canonical operator

$$\mathcal{L}z(t) = \left(\frac{a(t)}{B^{\beta}(t)}\left[\left(b(t)B^2(t)\left(\frac{z(t)}{B(t)}\right)'\right)'\right]^{\beta}\right)'.$$

Proof. The proof is similar to the proof of Theorem 1 in [**18**] and hence the details are omitted. □

The following result should be clear.

Theorem 2. *Let* (6) *hold. Then the semi-canonical differential equation* (E_1) *has the unique (up to positive multiplicative constants with product 1) canonical representation*

$$(c(t)((d(t)y'(t))')^{\beta})' + q(t)x^{\lambda}(\sigma(t)) = 0, \tag{E_2}$$

where $y(t) = \dfrac{z(t)}{B(t)}$.

Corollary 3. *If* (6) *holds, then the semi-canonical differential equation* (E_1) *has an eventually positive solution if and only if the canonical equation* (E_2) *has an eventually positive solution.*

Corollary 3 clearly simplifies the examination of (E_2) since for (E_2) we are concerned with only two classes of positive solutions, that is, either

Case I: $y(t) > 0$, $y'(t) < 0$, $(d(t)y'(t))' > 0$, $(c(t)[(d(t)y'(t))']^{\beta})' < 0$, and we say $y \in \mathcal{N}_0$,

Case II: $y(t) > 0$, $y'(t) > 0$, $(d(t)y'(t))' > 0$, $(c(t)[(d(t)y'(t))']^{\beta})' < 0$, and we say $y \in \mathcal{N}_2$.

The following lemma will prove to be useful in some of the calculations involved in our proofs of the main results in this paper.

Lemma 4. *Let $x(t)$ be a positive solution of* (E_1) *and the corresponding function* $y(t) \in \mathcal{N}_2$. *Then for any* $t_* \geq t_0$, *(i)* $\dfrac{d(t)y'(t)}{\mu(t,t_*)}$ *is decreasing,*

(ii) $\dfrac{y(t)}{\eta(t,t_*)}$ *is decreasing,*

(iii) $y'(t) \geq \dfrac{\mu(t,t_*)}{d(t)} c^{1/\beta}(t)(d(t)y'(t))'$,

(iv) $y(t) \geq d(t)y'(t)\dfrac{\eta(t,t_*)}{\mu(t,t_*)}$.

Proof. Let $y(t) \in \mathcal{N}_2$ for $t \geq t_* \geq t_0$. Since $c(t)[(d(t)y'(t))']^\beta$ is decreasing,

$$d(t)y'(t) \geq \int_{t_*}^{t} c^{1/\beta}(s)\frac{(d(s)y'(s))'}{c^{1/\beta}(s)}ds \geq c^{1/\beta}(t)(d(t)y'(t))'\mu(t,t_*)$$

so

$$y'(t) \geq c^{1/\beta}(t)\frac{(d(t)y'(t))'}{d(t)}\mu(t,t_*),$$

which proves (iii). This implies

$$\left(\frac{d(t)y'(t)}{\mu(t,t_*)}\right)' = \frac{(d(t)y'(t))'\mu(t,t_*) - d(t)y'(t)c^{-\beta}(t)}{\mu^2(t,t_*)} \leq 0.$$

Thus, $\dfrac{d(t)y'(t)}{\mu(t,t_*)}$ is decreasing, proving (i). Moreover, this yields

$$y(t) \geq \int_{t_*}^{t} \frac{d(s)y'(s)}{d(s)\mu(s,t_*)}\mu(s,t_*)ds \geq \frac{d(t)y'(t)}{\mu(t,t_*)}\eta(t,t_*),$$

which proves (iv). Finally,

$$\left(\frac{y(t)}{\eta(t,t_*)}\right)' = \frac{y'(t)\eta(t,t_*) - y(t)\mu(t,t_*)d^{-1}(t)}{\eta^2(t,t_*)} \leq 0$$

which shows that $y(t)/\eta(t,t_*)$ is decreasing and proves (ii). □

To prove our main results, we make the additional assumption that

(A_5) $\quad \lim_{t\to\infty} p(t)\dfrac{B^\alpha(\tau(t))}{B(t)}\eta^{\alpha-1}(t,t_*) = 0.$

Notice that (A_5) implies $\lim_{t\to\infty} p(t)\dfrac{B^\alpha(\tau(t))}{B(t)} = 0$.
For constants $d_1 > 0$ and $d_2 > 0$, define

$$Q(t) = \begin{cases} 1, & \text{if } \lambda = \beta, \\ d_1, & \text{if } \lambda > \beta, \\ d_2\eta^{\lambda-\beta}(t,t_*), & \text{if } \lambda < \beta, \end{cases}$$

and set $\Omega(t) = q(t)B^\lambda(\sigma(t))Q(\sigma(t))$.

Lemma 5. *Let condition* (6) *hold and assume that x is an eventually positive solution of* (E_1) *with $y \in \mathcal{N}_0$. If either*

$$\int_{t_0}^{\infty} q(t)B^{\lambda}(\sigma(t))dt = \infty, \tag{7}$$

or

$$\int_{t_0}^{\infty} \frac{1}{d(v)} \int_{v}^{\infty} \left(\frac{1}{c(u)} \int_{u}^{\infty} q(s)B^{\lambda}(\sigma(s))ds \right)^{1/\beta} dudv = \infty, \tag{8}$$

then

$$\lim_{t\to\infty} \frac{x(t)}{B(t)} = 0.$$

Proof. Let $x(t)$ be an eventually positive solution of (E_1), say $x(t) > 0, x(\tau(t)) > 0$, and $x(\sigma(t)) > 0$ for $t \geq t_1$ for some $t_1 \geq t_0$. By Corollary 3, the corresponding function $y(t) = \dfrac{1}{B(t)}(x(t)+p(t)x^{\alpha}(\tau(t)))$ is a positive solution of (E_2). Then, since $z(t) \geq x(t)$ and $y(t) = z(t)/B(t)$, we obtain

$$\begin{aligned} x(t) &\geq B(t)y(t) - p(t)x^{\alpha}(\tau(t)) \geq B(t)y(t) - p(t)z^{\alpha}(\tau(t)) \\ &\geq B(t)y(t) - p(t)B^{\alpha}(\tau(t))y^{\alpha}(\tau(t)). \end{aligned} \tag{9}$$

Since $y \in \mathcal{N}_0$, we have $y > 0$ and $y' < 0$, so there exists a constant $l \geq 0$ such that

$$\lim_{t\to\infty} y(t) = l < \infty.$$

Suppose $l > 0$. Then there exists $t_2 \geq t_1$ such that, for $t \geq t_2$,

$$l + \epsilon > y(t) > l.$$

Using this in (9), we obtain

$$x(t) \geq lB(t)(1 - p(t)\frac{B^{\alpha}(\tau(t))}{lB(t)}(l+\epsilon)^{\alpha}),\ t \geq t_2. \tag{10}$$

From (A_5), there exist $\epsilon_1 \in (0,1)$ and $t_3 \geq t_2$ such that

$$p(t)\frac{B^{\alpha}(\tau(t))}{lB(t)}(l+\epsilon)^{\alpha} \leq (1-\epsilon_1),\ t \geq t_3.$$

Using this in (10) gives

$$x(t) \geq l\epsilon_1 B(t),\ t \geq t_3. \tag{11}$$

From (E_2) and (11), we have

$$(c(t)((d(t)y'(t))')^{\beta})' + (l\epsilon_1)^{\lambda}q(t)B^{\lambda}(\sigma(t)) \le 0,\ t \ge t_3. \tag{12}$$

Integrating (12) from t_3 to ∞ yields

$$\int_{t_3}^{\infty} q(t)B^{\lambda}(\sigma(t))dt \le \frac{(c(t_3)((d(t_3)y'(t_3))')^{\beta})}{(l\epsilon_1)^{\lambda}} < \infty,$$

which contradicts (7). Therefore, $l = 0$, and so $\lim_{t\to\infty} y(t) = 0$. Since $0 < \dfrac{x(t)}{B(t)} \le y(t)$, we see that $\lim_{t\to\infty} \dfrac{x(t)}{B(t)} = 0$.

Now consider the case where condition (7) is not satisfied. Integrating (12) from t to ∞ twice gives

$$\frac{1}{d(t)}\int_{t}^{\infty}\left(\frac{1}{c(u)}\int_{u}^{\infty} q(s)B^{\lambda}(\sigma(s))ds\right)^{1/\beta} du \le \frac{-y'(t)}{(l\epsilon_1)^{\lambda}}.$$

Integrating again from t_3 to ∞ gives

$$\int_{t_3}^{\infty}\frac{1}{d(v)}\int_{v}^{\infty}\left(\frac{1}{c(u)}\int_{u}^{\infty} q(s)B^{\lambda}(\sigma(s))ds\right)^{1/\beta} dudv \le \frac{y(t_3)}{(l\epsilon_1)^{\lambda}},$$

which contradicts (8). This shows that $l = 0$ and completes the proof of the lemma. □

Lemma 6. *Let* (6) *hold and let x be an eventually positive solution of* (E_1) *with $y \in \mathcal{N}_2$. Then there exists $t_* \ge t_0$ and three constants $M_1 > 0$, $d_1 > 0$, and $d_2 > 0$ such that*

$$(c(t)((d(t)y'(t))')^{\beta})' + M_1\Omega(t)y^{\beta}(\sigma(t)) \le 0 \tag{13}$$

for large t.

Proof. Assume that $x(t)$ is an eventually positive solution of (E_1) such that $x(t) > 0$, $x(\tau(t)) > 0$, and $x(\sigma(t)) > 0$ for $t \ge t_1$ for some $t_1 \ge t_0$. Proceeding as in the proof of Lemma 5, we see that (9) holds. Since $y' > 0$, (9) implies

$$x(t) \ge B(t)\left[1 - p(t)\frac{B^{\alpha}(\tau(t))}{B(t)}y^{\alpha-1}(t)\right]y(t). \tag{14}$$

From Lemma 4 (ii), we see that $\dfrac{y(t)}{\eta(t,t_1)}$ is decreasing and therefore there exist $M > 0$ and $t_2 \ge t_1$,

$$y(t) \le M\eta(t,t_1), \quad t \ge t_2. \tag{15}$$

Using (15) in (14), we obtain

$$x(t) \geq B(t)\left[1 - M^{\alpha-1}p(t)\frac{B^{\alpha}(\tau(t))}{B(t)}\eta^{\alpha-1}(t,t_1)\right]y(t). \tag{16}$$

From (A_5), there exists $\epsilon_2 \in (0,1)$ and $t_3 \geq t_2$ such that

$$M^{\alpha-1}p(t)\frac{B^{\alpha}(\tau(t))}{B(t)}\eta^{\alpha-1}(t,t_1) \leq (1-\epsilon_2),\ t \geq t_3.$$

Using this in (16) gives

$$x(t) \geq \epsilon_2 B(t)y(t). \tag{17}$$

From (E_2) and (17),

$$(c(t)((d(t)y'(t))')^{\beta})' + (\epsilon_2)^{\lambda}q(t)B^{\lambda}(\sigma(t))y^{\lambda}(\sigma(t)) \leq 0.$$

or

$$(c(t)((d(t)y'(t))')^{\beta})' + (\epsilon_2)^{\lambda}q(t)B^{\lambda}(\sigma(t))y^{\lambda-\beta}(\sigma(t))y^{\beta}(\sigma(t)) \leq 0. \tag{18}$$

Since $y \in \mathcal{N}_2$, y is increasing and $y(t)/\eta(t,t_*)$ is decreasing by Lemma 4(ii). Therefore there are constants $c_1 > 0$ and $c_2 > 0$ such that $y(t) \geq c_1$ and $y(t) \leq c_2\eta(t,t_*)$. Now for $\lambda = \beta$, $y^{\lambda-\beta}(t) = 1$, for $\lambda > \beta$, $y^{\lambda-\beta}(t) \geq c_1^{\lambda-\beta} = d_1$, and for $\lambda < \beta$, $y^{\lambda-\beta}(t) \geq c_2^{\lambda-\beta}\eta^{\lambda-\beta}(t,t_*) = d_2\eta^{\lambda-\beta}(t,t_*)$. Hence, $y^{\lambda-\beta}(t) \geq Q(t)$. Using this in (18), we obtain

$$(c(t)((d(t)y'(t))')^{\beta})' + M_1\Omega(t)y^{\beta}(\sigma(t)) \leq 0.$$

That is, inequality (13) holds, and this completes the proof of the lemma. □

We are now ready to prove our first oscillation type result.

Theorem 7. *Let condition* (6) *hold and assume that either* (7) *or* (8) *holds. If there exists a non-decreasing function* $g \in C'([t_0,\infty),(0,\infty))$ *such that for all constants* $M_1 > 0$, $d_1 > 0$, *and* $d_2 > 0$ *and all sufficiently large* $t_* \geq t_0$ *and for any* $T \geq t_*$

$$\lim_{t\to\infty}\int_T^t \left[M_1 g(s)\Omega(s)\frac{\eta^{\beta}(\sigma(s),t_*)}{\mu^{\beta}(s,t_*)} - \frac{g'(s)}{\mu^{\beta}(s,t_*)}\right]ds = \infty, \tag{19}$$

then a solution x *of* (E_1) *either oscillates or satisfies*

$$\lim_{t\to\infty}\frac{x(t)}{B(t)} = 0.$$

Proof. Assume that (E_1) has an eventually positive nonoscillatory solution x. Then, by Corollary 3, the function

$$y(t) = \frac{1}{B(t)}(x(t) + p(t)x^{\alpha}(\tau(t)))$$

is an eventually positive solution of (E_2), and either $y(t) \in \mathcal{N}_0$ or $y(t) \in \mathcal{N}_2$ for $t \geq t_1 \geq t_0$. If $y \in \mathcal{N}_0$, then as in Lemma 5,

$$\frac{x(t)}{B(t)} \to 0 \text{ as } t \to \infty.$$

Next, let $y(t) \in \mathcal{N}_2$. For $t \geq T \geq t_1$, define

$$w(t) = g(t)c(t)\frac{((d(t)y'(t))')^{\beta}}{(d(t)y'(t))^{\beta}}; \tag{20}$$

then $w(t) > 0$ and from (12) and (20), we have

$$w'(t) \leq g'(t)c(t)\frac{((d(t)y'(t))')^{\beta}}{(d(t)y'(t))^{\beta}} - \frac{M_1 g(t)\Omega(t)y^{\beta}(\sigma(t))}{(d(t)y'(t))^{\beta}} - \beta g(t)c(t)\frac{((d(t)y'(t))')^{\beta+1}}{(d(t)y'(t))^{\beta+1}}. \tag{21}$$

It follows from Lemma 4 (iii) and (21) that

$$w'(t) \leq \frac{g'(t)}{\mu^{\beta}(t,t_*)} - \frac{M_1 g(t)\Omega(t)y^{\beta}(\sigma(t))}{y^{\beta}(t)}\frac{y^{\beta}(t)}{(d(t)y'(t))^{\beta}}. \tag{22}$$

Now $\sigma(t) \leq t$ and in view of the fact that $\dfrac{y(t)}{\eta(t,t_*)}$ is nonincreasing for large t, we see that

$$\frac{y(\sigma(t))}{y(t)} \geq \frac{\eta(\sigma(t),t_*)}{\eta(t,t_*)}. \tag{23}$$

Using Lemma 4 (iv) and (23) in (22) yields

$$w'(t) \leq \frac{g'(t)}{\mu^{\beta}(t,t_*)} - \frac{M_1 g(t)\Omega(t)\eta^{\beta}(\sigma(t),t_*)}{\mu^{\beta}(t,t_*)}.$$

Integrating from T to t gives

$$\int_T^t \left[\frac{M_1 g(s)\Omega(s)\eta^{\beta}(\sigma(s),t_*)}{\mu^{\beta}(s,t_*)} - \frac{g'(s)}{\mu^{\beta}(s,t_*)}\right] ds \leq w(T),$$

which contradicts (19). This completes the proof of the theorem. □

In a similar vein we have the following theorem; first, we have a lemma to be used in its proof.

Lemma 8. ([**10**], [**19**]) *For* $\gamma > 1$, $B > 0$, $C > 0$, *and* $u > 0$,

$$Bu - Cu^{\frac{\gamma+1}{\gamma}} \leq \frac{\gamma^\gamma}{(\gamma+1)^{\gamma+1}} \frac{B^{\gamma+1}}{C^\gamma}.$$

Theorem 9. *Let conditions* (6) *and either* (7) *or* (8) *hold. If there exists a non-decreasing function* $g \in C'([t_0, \infty), (0, \infty))$ *such that for all constants* $M_1 > 0$, $d_1 > 0$, *and* $d_2 > 0$, *for all sufficiently large* $t_* \geq t_0$, *and for any* $T \geq t_*$,

$$\limsup_{t\to\infty} \int_T^t \left[M_1 g(s)\Omega(s) \frac{\eta^\beta(\sigma(s), t_*)}{\mu^\beta(s, t_*)} - \frac{1}{(\beta+1)^{\beta+1}} \frac{c(s)(g'(s))^{\beta+1}}{g^\beta(s)} \right] ds = \infty, \tag{24}$$

then a solution x *of* (E_1) *either oscillates or satisfies*

$$\lim_{t\to\infty} \frac{x(t)}{B(t)} = 0.$$

Proof. Assume that x is an eventually positive solution of (E_1). Then by Corollary 3, the function

$$y(t) = \frac{1}{B(t)}(x(t) + p(t)x^\alpha(\tau(t)))$$

is an eventually positive solution of (E_2), and either $y(t) \in \mathcal{N}_0$ or $y(t) \in \mathcal{N}_2$ for $t \geq t_1 \geq t_0$. The proof if $y \in \mathcal{N}_0$ is the same as that in the proof of Theorem 7. So we consider the case $y \in \mathcal{N}_2$. Defining w as in (20) and arguing as in the proof of Theorem 7, we again arrive at (21). Inequality (21) can be written as

$$w'(t) \leq \frac{g'(t)}{g(t)} w(t) - \frac{M_1 g(t)\Omega(t) y^\beta(\sigma(t)) y^\beta(t)}{y^\beta(t)(d(t)y'(t))^\beta} - \beta \frac{w^{(\beta+1)/\beta}(t)}{(g(t)c(t))^{1/\beta}} \tag{25}$$

and using Lemma 4 (ii) and (23) in (25), we obtain

$$w'(t) \leq \frac{g'(t)}{g(t)} w(t) - \frac{M_1 g(t)\Omega(t)\eta^\beta(\sigma(t), t_*)}{\mu^\beta(t, t_*)} - \beta \frac{w^{(\beta+1)/\beta}(t)}{(g(t)c(t))^{1/\beta}}. \tag{26}$$

for $t \geq T$. By applying Lemma 8 with $\gamma = \beta$,

$$B = \frac{g'(t)}{g(t)}, \quad C = \frac{\beta}{(c(t)g(t))^{1/\beta}}, \quad \text{and} \quad u = w,$$

(26) becomes

$$w'(t) \leq -M_1 g(t)\Omega(t) \frac{\eta^\beta(\sigma(t), t_*)}{\mu^\beta(t, t_*)} + \frac{1}{(\beta+1)^{\beta+1}} \frac{c(t)(g'(t))^{\beta+1}}{g^\beta(t)}. \tag{27}$$

Integrating from T to t yields

$$\int_T^t \left[\frac{M_1 g(s)\Omega(s)\eta^\beta(\sigma(s), t_*)}{\mu^\beta(s, t_*)} - \frac{1}{(\beta+1)^{\beta+1}} \frac{c(s)(g'(s))^{\beta+1}}{g^\beta(s)} \right] ds \leq w(T),$$

which contradicts (24) and completes the proof of the theorem. □

3. EXAMPLES

In this section, we present two examples to illustrate our main results.

Example 10. Consider the third-order differential equation with a super-linear neutral term

$$\left(\frac{1}{t^{1/3}}\left(\left(t^2 z'(t)\right)'\right)^{1/3}\right)' + 2t^2 x^3(t/3) = 0,\ t \geq 1, \tag{28}$$

where $z(t) = x(t) + \frac{1}{t^3} x^3(t/2)$. Here $a(t) = \frac{1}{t^{1/3}}$, $b(t) = t^2$, $p(t) = \frac{1}{t^3}$, $q(t) = 2t^2$, $\tau(t) = \frac{t}{2}$, $\sigma(t) = t/3$, $\alpha = 3$, $\beta = 1/3$, and $\lambda = 3$. Simple calculations show that

$$B(t) = \frac{1}{t},\ d(t) = 1,\ c(t) = 1,\ \mu(t,1) \approx t,\ \eta(t,1) \approx t^2/2.$$

Condition (6) is clearly satisfied, and the transformed equation in canonical form becomes

$$((y''(t))^{1/3})' + 2t^2 x^3(t/3),\ t \geq 1, \tag{29}$$

where $y(t) = tz(t)$. It is easy to see that conditions (A_1)–(A_4) hold. Condition (A_5) becomes

$$\lim_{t\to\infty} \frac{2}{t} = 0,$$

so it holds as well. Now

$$Q(t) = d_1 \quad \text{and} \quad \Omega(t) = \frac{54 d_1}{t}.$$

Condition (7) becomes

$$\int_1^\infty \frac{54}{t} dt = \infty,$$

and by taking $g(t) = 1$, condition (19) becomes

$$\int_1^\infty \frac{54 M_1 d_1}{2^{1/3} 3^{2/3}} \frac{1}{t^{2/3}} dt = \infty.$$

That is, conditions (7) and (19) are satisfied. Hence, by Theorem 7, a solution x of (28) is either oscillatory or satisfies $\lim_{t\to\infty} tx(t) = 0$.

Example 11. Consider the equation

$$\left(\frac{1}{t^{1/5}}\left(\left(t^2 z'(t)\right)'\right)^{1/5}\right)' + 2t x^{1/5}(t/3) = 0,\ t \geq 1, \tag{30}$$

with $z(t) = x(t) + \frac{1}{t}x^{5/3}(t/2)$. Here $a(t) = \frac{1}{t^{1/5}}$, $b(t) = t^2$, $p(t) = \frac{1}{t}$, $q(t) = 2t$, $\tau(t) = \frac{t}{2}$, $\sigma(t) = t/3$, $\alpha = 5/3$, $\beta = 1/5$, and $\lambda = 1/5$. It is easy to see that

$$B(t) = \frac{1}{t},\ d(t) = 1,\ c(t) = 1,\ \mu(t,1) \approx t,\ \eta(t,1) \approx t^2/2.$$

Conditions (A_1)–(A_4) and (6) are clearly satisfied. The transformed equation is

$$((y''(t))^{1/5})' + 2tx^{1/5}(t/3),\ t \geq 1, \tag{31}$$

where $y(t) = tz(t)$. Condition (A_5) becomes

$$\lim_{t\to\infty} \frac{2}{t^{1/3}} = 0,$$

meaning that it holds. Now

$$Q(t) = 1 \quad \text{and} \quad \Omega(t) = 2(3)^{1/5}t^{4/5}.$$

Condition (7) becomes

$$\int_1^\infty 2(3)^{1/5}t^{4/5}dt = \infty,$$

and by taking $g(t) = 1$, condition (24) becomes

$$\int_1^\infty \frac{M_1 2^{4/5}t}{3^{1/5}}dt = \infty.$$

That is, conditions (7) and (24) are satisfied. Hence, by Theorem 9, a solution x of (30) will either oscillate or satisfy $\lim_{t\to\infty} tx(t) = 0$.

4. CONCLUSION

The results presented in this paper provided a new technique for studying the oscillatory and asymptotic behavior of solutions of third-order semi-canonical delay differential equations with a superlinear neutral term. The results already known for this type of equation [**6, 15, 23**] are applicable only if the neutral coefficient p satisfies $p(t) \to \infty$ as $t \to \infty$; our results apply to the case $p(t) \to 0$ as $t \to \infty$. Thus, the results obtained here are new and further contribute to the oscillation theory of third-order delay differential equations with super-linear neutral terms.

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