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SHARP WILKER, HUYGENS, ADAMOVIĆ-MITRINOVIĆ, CUSA AND SHAFER-FINK TYPE INEQUALITIES INVOLVING THE SINE INTEGRAL

Jun-Ling Sun and Chao-Ping Chen[*]

The aim of the present paper is to present sharp Wilker, Huygens, Adamović-Mitrinović, Cusa and Shafer-Fink type inequalities involving the sine integral.

1. INTRODUCTION

Throughout this paper, $\mathbb{N}$ represents the set of positive integers and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. Wilker [40] proposed, then Sumner et al. [39] proved the following inequalities:

$$\left(\frac{\sin x}{x}\right)^2 + \frac{\tan x}{x} > 2 \quad \text{for} \quad 0 < x < \frac{\pi}{2} \tag{1}$$

and

$$2 + \left(\frac{2}{\pi}\right)^4 x^3 \tan x < \left(\frac{\sin x}{x}\right)^2 + \frac{\tan x}{x} < 2 + \frac{8}{45} x^3 \tan x \quad \text{for} \quad 0 < x < \frac{\pi}{2}, \tag{2}$$

where the constants $\left(\frac{2}{\pi}\right)^4$ and $\frac{8}{45}$ are best possible. Wilker type inequalities (1) and (2) have attracted much interest of many mathematicians and have motivated

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a large number of papers involving different proofs and various generalizations and improvements (cf. [17, 28–30, 33, 36, 41, 42, 44, 45, 47, 54–57] and the references cited therein).

Another inequality which is of interest to us is Huygens inequality [18], which asserts that

$$(3) \qquad 2\left(\frac{\sin x}{x}\right)+\frac{\tan x}{x}>3 \qquad \text{for} \quad 0<|x|<\frac{\pi}{2}.$$

In analogy with (2), Chen and Cheung [13] developed (3) to produce a double inequality

$$(4) \qquad 3+\frac{3}{20}x^3\tan x<2\left(\frac{\sin x}{x}\right)+\frac{\tan x}{x}<3+\left(\frac{2}{\pi}\right)^4 x^3\tan x$$

for $0<|x|<\frac{\pi}{2}$, where the constants $\frac{3}{20}$ and $\left(\frac{2}{\pi}\right)^4$ are the best possible.

In view of (2) and (4), we obtain the following inequality chain:

$$\frac{3}{20}<\frac{2\left(\frac{\sin x}{x}\right)+\frac{\tan x}{x}-3}{x^3\tan x}<\left(\frac{2}{\pi}\right)^4<\frac{\left(\frac{\sin x}{x}\right)^2+\frac{\tan x}{x}-2}{x^3\tan x}<\frac{8}{45}$$

for $0<x<\pi/2$. The hyperbolic versions of the Wilker and Huygens type inequalities were established in [33, 43, 51, 58]. The Wilker and Huygens type inequalities for inverse trigonometric and inverse hyperbolic functions were presented in [8, 11, 24, 46]. The Wilker and Huygens type inequalities have also been established for the lemniscate functions and Jacobian elliptic and theta functions. For more details see [9, 10, 32] and [31], respectively.

The sine integral $\mathrm{Si}(x)$ is defined by

$$\mathrm{Si}(x)=\int_0^x \frac{\sin t}{t}\mathrm{d}t.$$

The sine integral $\mathrm{Si}(x)$ has the series expansion

$$\mathrm{Si}(x)=\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n+1)!(2n+1)}.$$

The series converge for all finite values of x.

The following double inequality was presented in [20]:

$$\arctan x<\mathrm{Si}(x)<\pi-\arctan x \quad \text{for} \quad x>0,$$

which implies the well-known integral

$$\int_0^\infty \frac{\sin t}{t}\mathrm{d}t=\frac{\pi}{2}.$$

Fejér conjectured in 1910 that the inequality

$$0 \leq S_n(x) = \sum_{k=1}^{n} \frac{\sin(kx)}{x} < \int_0^{\pi} \frac{\sin(kx)}{x} \mathrm{d}x = 1.85193\ldots$$

holds for all $n \in \mathbb{N}$ and $x \in [0, \pi]$. In 1911, D. Jackson [19] published the first proof of the above inequality. The left side of the above inequality is known as the Fejér-Jackson inequality. Alzer and Kwong [1] presented a variant of the Fej'er-Jackson inequality. Alzer and Fuglede [2] proved that the following inequality:

$$\sum_{k=1}^{n} \binom{m+n-k}{m} \sin(kx) > ax^2 + bx + c > 0$$

hold for all integers $n \geq 2$ and real numbers $x \in (0, \pi)$ if and only if

$$-\frac{m-1}{\pi} \leq a < 0, \quad b = -a\pi, \quad c = 0.$$

This refines a result due to Turán.

In this paper, we establish sharp Wilker and Huygens type inequalities involving the sine integral (Theorems 1 and 2). Also, we establish sharp Adamović-Mitrinović and Cusa type inequalities involving the sine integral (Theorem 3). We establish sharp Shafer-Fink type inequality involving the sine integral (Theorem 4).

The numerical values, which we have given in this article, were computed by using the computer program *MAPLE* 11.

2. LEMMAS

It is well known that

$$\sum_{n=0}^{2N+1} \frac{(-1)^n t^{2n}}{(2n+1)!} < \frac{\sin t}{t} < \sum_{n=0}^{2N} \frac{(-1)^n t^{2n}}{(2n+1)!} \tag{5}$$

for $t \neq 0$ and $N \in \mathbb{N}_0$, which yields

$$\sum_{n=0}^{2N+1} \frac{(-1)^n x^{2n}}{(2n+1)!(2n+1)} < \frac{\mathrm{Si}(x)}{x} < \sum_{n=0}^{2N} \frac{(-1)^n x^{2n}}{(2n+1)!(2n+1)} \tag{6}$$

for $x \neq 0$ and $N \in \mathbb{N}_0$. From (6) we obtain that for $0 < x < \pi/2$,

$$0 < 1 - \frac{1}{18}x^2 < \frac{\mathrm{Si}(x)}{x} < 1 - \frac{1}{18}x^2 + \frac{1}{600}x^4, \tag{7}$$

$$(8)\quad 0 < x - \frac{1}{18}x^3 + \frac{1}{600}x^5 - \frac{1}{35280}x^7 + \frac{1}{3265920}x^9 - \frac{1}{439084800}x^{11} + \frac{1}{80951270400}x^{13} - \frac{1}{19615115520000}x^{15} < \mathrm{Si}(x) < x - \frac{1}{18}x^3 + \frac{1}{600}x^5 - \frac{1}{35280}x^7 + \frac{1}{3265920}x^9 - \frac{1}{439084800}x^{11} + \frac{1}{80951270400}x^{13}$$

and

$$\begin{aligned}(9)\quad \mathrm{Si}^2(x) &< \left(x - \frac{1}{18}x^3 + \frac{1}{600}x^5 - \frac{1}{35280}x^7 + \frac{1}{3265920}x^9 - \frac{1}{439084800}x^{11} + \frac{1}{80951270400}x^{13}\right)^2 \\ &= x^2 - \frac{1}{9}x^4 + \frac{13}{2025}x^6 - \frac{8}{33075}x^8 + \frac{146}{22325625}x^{10} - \frac{647}{4862521125}x^{12} + \frac{28211}{13422179145375}x^{14} \\ &\quad - x^{16}\left(\frac{775237}{29452095953280000} - \frac{10078339}{38169916355450880000}x^2\right) \\ &\quad - x^{20}\left(\frac{1777}{848220363454464000} - \frac{5609}{43986284561995776000}x^2\right) \\ &\quad - x^{24}\left(\frac{1}{17772236186664960000} - \frac{1}{6553108179373916160000}x^2\right) \\ &< x^2 - \frac{1}{9}x^4 + \frac{13}{2025}x^6 - \frac{8}{33075}x^8 + \frac{146}{22325625}x^{10} - \frac{647}{4862521125}x^{12} + \frac{28211}{13422179145375}x^{14}.\end{aligned}$$

Lemma 1. *For $0 < x < \pi/2$, we have*

$$(10)\quad \sin^2 x \cos x > x^2 - \frac{5}{6}x^4 + \frac{91}{360}x^6 - \frac{41}{1008}x^8 + \frac{7381}{1814400}x^{10} - \frac{949}{3421440}x^{12} + \frac{597871}{43589145600}x^{14} - \frac{134521}{261534873600}x^{16},$$

$$(11)\quad 5\sin x \cos x < 5x - \frac{10}{3}x^3 + \frac{2}{3}x^5 - \frac{4}{63}x^7 + \frac{2}{567}x^9 - \frac{4}{31185}x^{11} + \frac{4}{1216215}x^{13}$$

and

$$(12)\quad 3x\sin x\cos x + \cos^2 x > 1 + 2x^2 - \frac{5}{3}x^4 + \frac{16}{45}x^6 - \frac{11}{315}x^8 + \frac{4}{2025}x^{10} - \frac{34}{467775}x^{12} + \frac{16}{8513505}x^{14} - \frac{23}{638512875}x^{16}.$$

Proof. We only prove inequality (10). The proofs of (11) and (12) are analogous. By using the power series expansion of $\cos x$, we obtain

$$\begin{aligned}(13)\quad \sin^2 x\cos x &= \frac{1}{2}\sin x\sin(2x) = \frac{1}{4}\cos(x) - \frac{1}{4}\cos(3x)\\ &= \sum_{n=1}^{\infty}(-1)^{n-1}\frac{3^{2n}-1}{4\cdot(2n)!}x^{2n}\\ &= x^2 - \frac{5}{6}x^4 + \frac{91}{360}x^6 - \frac{41}{1008}x^8 + \frac{7381}{1814400}x^{10} - \frac{949}{3421440}x^{12}\\ &\quad + \frac{597871}{43589145600}x^{14} - \frac{134521}{261534873600}x^{16} + \sum_{n=9}^{\infty}(-1)^{n-1}u_n(x),\end{aligned}$$

where

$$u_n(x) = \frac{3^{2n}-1}{4\cdot(2n)!}x^{2n}.$$

Noting that $x^2 < (\pi/2)^2 < 3$ holds for $0 < x < \pi/2$, we find that for $0 < x < \pi/2$ and $n \geq 9$,

$$\frac{u_{n+1}(x)}{u_n(x)} = \frac{x^2(9^{n+1}-1)}{2(2n+1)(n+1)(9^n-1)} < \frac{3\cdot 9^{n+1}}{2(2n+1)(n+1)(9^n-1)} < 1.$$

Therefore, for every $x \in (0, \pi/2)$, the sequence $n \longmapsto u_n(x)$ is strictly decreasing for $n \geq 9$. From (13) we obtain the desired result (10). The proof of Lemma 1 is complete. □

Lemma 2. *For $0 < x < \pi/2$, we have*

$$\begin{aligned}(14)\quad \sin^2 x\cos x\,\mathrm{Si}(x) > x^3 &- \frac{8}{9}x^5 + \frac{203}{675}x^7 - \frac{1114}{19845}x^9 + \frac{30242}{4465125}x^{11}\\ &- \frac{26794}{46309725}x^{13} + \frac{1068146263}{28761812454375}x^{15}\\ &- \frac{53697806}{28761812454375}x^{17}\end{aligned}$$

and

$$\begin{aligned}(15)\quad (5\sin x\cos x + x)\mathrm{Si}(x)^2 < 6x^3 - 4x^5 + \frac{242}{225}x^7 - \frac{47752}{297675}x^9 + \frac{12988}{826875}x^{11}\\ - \frac{1801796}{1620840375}x^{13} + \frac{3314211818}{54908914685625}x^{15}.\end{aligned}$$

Proof. Using (8) and (10), we obtain that for $0 < x < \pi/2$,

$$
\begin{aligned}
&\sin^2 x\cos x\,\mathrm{Si}(x) > \sin^2 x\cos x\left(x-\frac{1}{18}x^3+\frac{1}{600}x^5-\frac{1}{35280}x^7+\frac{1}{3265920}x^9\right.\\
&\qquad\qquad\left.-\frac{1}{439084800}x^{11}+\frac{1}{80951270400}x^{13}-\frac{1}{19615115520000}x^{15}\right)\\
&>\left(x^2-\frac{5}{6}x^4+\frac{91}{360}x^6-\frac{41}{1008}x^8+\frac{7381}{1814400}x^{10}-\frac{949}{3421440}x^{12}\right.\\
&\quad\left.+\frac{597871}{43589145600}x^{14}-\frac{134521}{261534873600}x^{16}\right)\left(x-\frac{1}{18}x^3+\frac{1}{600}x^5-\frac{1}{35280}x^7\right.\\
&\quad\left.+\frac{1}{3265920}x^9-\frac{1}{439084800}x^{11}+\frac{1}{80951270400}x^{13}-\frac{1}{19615115520000}x^{15}\right)\\
&=x^3-\frac{8}{9}x^5+\frac{203}{675}x^7-\frac{1114}{19845}x^9+\frac{30242}{4465125}x^{11}-\frac{26794}{46309725}x^{13}\\
&\quad+\frac{1068146263}{28761812454375}x^{15}-\frac{53697806}{28761812454375}x^{17}\\
&\quad+x^{19}\left(\frac{2826089297}{46605514475520000}-\frac{848085187}{632550697178400000}x^2\right)\\
&\quad+x^{23}\left(\frac{38135421743}{1959389039579811840000}-\frac{38830277}{2018601345720960000000}x^2\right)\\
&\quad+x^{27}\left(\frac{2366709229}{17466553724254322688000000}-\frac{9799481}{1389384955338412032000000}x^2\right)\\
&\quad+\frac{134521}{5130036758172598272000000}x^{31}\\
&>x^3-\frac{8}{9}x^5+\frac{203}{675}x^7-\frac{1114}{19845}x^9+\frac{30242}{4465125}x^{11}-\frac{26794}{46309725}x^{13}\\
&\quad+\frac{1068146263}{28761812454375}x^{15}-\frac{53697806}{28761812454375}x^{17}.
\end{aligned}
$$

Using (9) and (11), we obtain that for $0 < x < \pi/2$,

$$
\begin{aligned}
&(5\sin x\cos x + x)\mathrm{Si}(x)^2\\
&< \left(6x - \frac{10}{3}x^3 + \frac{2}{3}x^5 - \frac{4}{63}x^7 + \frac{2}{567}x^9 - \frac{4}{31185}x^{11} + \frac{4}{1216215}x^{13}\right)\\
&\quad \cdot \left(x^2 - \frac{1}{9}x^4 + \frac{13}{2025}x^6 - \frac{8}{33075}x^8 + \frac{146}{22325625}x^{10}\right.\\
&\qquad \left. - \frac{647}{4862521125}x^{12} + \frac{28211}{13422179145375}x^{14}\right)\\
&= 6x^3 - 4x^5 + \frac{242}{225}x^7 - \frac{47752}{297675}x^9 + \frac{12988}{826875}x^{11}\\
&\quad - \frac{1801796}{1620840375}x^{13} + \frac{3314211818}{54908914685625}x^{15}\\
&\quad - x^{17}\left(\frac{1541988416}{603998061541875} - \frac{154119146}{1811994184625625}x^2\right)\\
&\quad - x^{21}\left(\frac{1276887946}{570778168157071875} - \frac{288743194}{6278559849727790625}x^2\right)\\
&\quad - x^{25}\left(\frac{2664152}{3767135909836674375} - \frac{112844}{16324255609292255625}x^2\right)\\
&< 6x^3 - 4x^5 + \frac{242}{225}x^7 - \frac{47752}{297675}x^9 + \frac{12988}{826875}x^{11}\\
&\quad - \frac{1801796}{1620840375}x^{13} + \frac{3314211818}{54908914685625}x^{15}.
\end{aligned}
$$

The proof of Lemma 2 is complete. □

Lemma 3 ([3–5]). *Let $-\infty < a < b < \infty$, and $f, g : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable in (a, b). Suppose $g' \neq 0$ on (a, b). If $f'(x)/g'(x)$ is increasing (decreasing) on (a, b), then so are*

$$[f(x) - f(a)]/[g(x) - g(a)] \quad \textit{and} \quad [f(x) - f(b)]/[g(x) - g(b)].$$

If $f'(x)/g'(x)$ is strictly monotone, then the monotonicity in the conclusion is also strict.

3. SHARP WILKER AND HUYGENS TYPE INEQUALITIES

Theorem 1 presents a sharp Wilker type inequality involving the sine integral.

Theorem 1. *For $0 < x < \pi/2$, we have*

$$4 + \frac{103}{675}x^3\tan x < 3\left(\frac{\mathrm{Si}(x)}{x}\right)^2 + \frac{\tan x}{x} < 4 + \left(\frac{2}{\pi}\right)^4 x^3 \tan x, \tag{16}$$

where the constants $\frac{103}{675}$ and $\left(\frac{2}{\pi}\right)^4$ are the best possible.

Proof. The double inequality (16) is obtained by considering the function $f(x)$ which is defined by

$$f(x) = \frac{3\left(\frac{\mathrm{Si}(x)}{x}\right)^2 + \frac{\tan x}{x} - 4}{x^3 \tan x}, \qquad 0 < x < \frac{\pi}{2}.$$

Direct computation yields

$$\lim_{x\to 0^+} f(x) = \frac{103}{675} = 0.15259259\ldots \quad \text{and} \quad \lim_{x\to(\pi/2)^-} f(x) = \left(\frac{2}{\pi}\right)^4 = 0.1642557\ldots.$$

In order to prove Theorem 1, it suffices to show that $f(x)$ is strictly increasing for $0 < x < \pi/2$. Differentiating $f(x)$ and applying (14), (15) and (12), we obtain for $0 < x < \pi/2$,

$$\begin{aligned}
x^6 \sin^2 x f'(x) &= 6\sin^2 x \cos x \mathrm{Si}(x) - 3(5\sin x\cos x + x)\mathrm{Si}(x)^2 \\
&\quad + 4x(3x\sin x\cos x + \cos^2 x) + 4x^3 - 4x \\
&> 6\bigg(x^3 - \frac{8}{9}x^5 + \frac{203}{675}x^7 - \frac{1114}{19845}x^9 + \frac{30242}{4465125}x^{11} - \frac{26794}{46309725}x^{13} \\
&\qquad + \frac{1068146263}{28761812454375}x^{15} - \frac{53697806}{28761812454375}x^{17}\bigg) \\
&\quad - 3\bigg(6x^3 - 4x^5 + \frac{242}{225}x^7 - \frac{47752}{297675}x^9 + \frac{12988}{826875}x^{11} \\
&\qquad - \frac{1801796}{1620840375}x^{13} + \frac{3314211818}{54908914685625}x^{15}\bigg) \\
&\quad + 4x\bigg(1 + 2x^2 - \frac{5}{3}x^4 + \frac{16}{45}x^6 - \frac{11}{315}x^8 + \frac{4}{2025}x^{10} - \frac{34}{467775}x^{12} \\
&\qquad + \frac{16}{8513505}x^{14} - \frac{23}{638512875}x^{16}\bigg) + 4x^3 - 4x \\
&= x^9\left(\frac{472}{99225} + \frac{10544}{7441875}x^2 - \frac{230864}{540280125}x^4\right) \\
&\quad + x^{15}\left(\frac{9919325048}{201332687180625} - \frac{108776992}{9587270818125}x^2\right) > 0.
\end{aligned}$$

Hence, $f(x)$ is strictly increasing for $0 < x < \pi/2$. The proof of Theorem 1 is complete. □

Theorem 2 presents a sharp Huygens type inequality involving the sine integral.

Theorem 2. *For* $0 < x < \pi/2$, *we have*

$$7 + \frac{43}{300}x^3\tan x < 6\left(\frac{\mathrm{Si}(x)}{x}\right) + \left(\frac{\tan x}{x}\right) < 7 + \left(\frac{2}{\pi}\right)^4 x^3 \tan x, \tag{17}$$

where the constants $\frac{43}{300}$ *and* $\left(\frac{2}{\pi}\right)^4$ *are the best possible.*

Proof. The double inequality (17) is obtained by considering the function $H(x)$ which is defined by

$$H(x) = \frac{6\left(\frac{\mathrm{Si}(x)}{x}\right) + \left(\frac{\tan x}{x}\right) - 7}{x^3 \tan x} = \frac{6\mathrm{Si}(x) + \tan x - 7x}{x^4 \tan x} = \frac{H_1(x)}{H_2(x)}, \qquad 0 < x < \frac{\pi}{2},$$

where

$$H_1(x) = 6\mathrm{Si}(x) + \tan x - 7x \quad \text{and} \quad H_2(x) = x^4 \tan x.$$

Direct computation yields

$$\lim_{x \to 0^+} H(x) = \frac{43}{300} = 0.1433333\ldots \quad \text{and} \quad \lim_{x \to (\pi/2)^-} H(x) = \left(\frac{2}{\pi}\right)^4 = 0.1642557\ldots.$$

In order to prove Theorem 2, it suffices to show that $H(x)$ is strictly increasing for $0 < x < \pi/2$. Elementary calculations show that

$$\frac{H_1'(x)}{H_2'(x)} = \frac{6 \sin x \cos^2 x - 7x \cos^2 x + x}{x^4(4 \sin x \cos x + x)} =: H_3(x)$$

and

$$\begin{aligned}
\frac{x^8(4\sin x \cos x + x)^2}{2x^3 \cos x} H_3'(x) &= 12x \sin x \cos^3 x + 42x \sin x \cos^2 x \\
&\quad - 27x \sin x \cos x + 48 \cos^4 x \\
&\quad + (9x^2 - 48)\cos^2 x + (7x^3 - 6x)\sin x \\
&\quad + 24x^2 \cos x - 6x^2 \\
&= \frac{3}{2}x \sin(4x) + \frac{21}{2}x \sin(3x) - \frac{21}{2}x \sin(2x) \\
&\quad + \frac{1}{2}x(9 + 14x^2)\sin(x) + 6\cos(4x) \\
&\quad + \frac{9}{2}x^2 \cos(2x) + 24x^2 \cos x - \frac{3}{2}x^2 - 6 \\
&= \frac{59}{280}x^8 - \frac{10229}{151200}x^{10} + \frac{104213}{19958400}x^{12} \\
&\quad + \frac{183221}{454053600}x^{14} - \frac{25237379}{217945728000}x^{16} \\
&\quad + \sum_{n=9}^{\infty}(-1)^{n-1}v_n(x),
\end{aligned} \tag{18}$$

where

$$v_n(x) = \frac{(3n-24)16^n + 28n \cdot 9^n + 3n(6n-17)\cdot 4^n - 4n(56n^2 - 180n + 67)}{4 \cdot (2n)!}x^{2n}.$$

Noting that $2x^2 < 2(\pi/2)^2 < 5$, we find that for $0 < x < \pi/2$ and $n \geq 9$,

$$\begin{aligned}\frac{v_{n+1}(x)}{v_n(x)} &= \frac{2x^2[(12n-84)16^n+63(n+1)9^n+3(n+1)(6n-11)4^n-(n+1)(56n^2-68n-57)]}{(2n+1)(n+1)[(3n-24)16^n+28n\cdot 9^n+3n(6n-17)4^n-4n(56n^2-180n+67)]}\\ &< \frac{5\big[(12n-84)16^n+63(n+1)9^n+3(n+1)(6n-11)4^n\big]}{(2n+1)(n+1)[(3n-24)16^n-4n(56n^2-180n+67)]}\\ &= \frac{5\big(12n-84+\frac{63(n+1)9^n}{16^n}+\frac{3(n+1)(6n-11)4^n}{16^n}\big)}{(2n+1)(n+1)\big(3n-24-\frac{4n(56n^2-180n+67)}{16^n}\big)}.\end{aligned}$$

It is easy to see that the sequences

$$x_n = \frac{63(n+1)9^n}{16^n} + \frac{3(n+1)(6n-11)4^n}{16^n} \quad \text{and} \quad y_n = \frac{4n(56n^2-180n+67)}{16^n}$$

are both strictly decreasing for $n \geq 9$, and we have

$$0 < x_n \leq x_9 = \frac{122206536915}{34359738368} \quad \text{and} \quad 0 < y_n \leq y_9 = \frac{26847}{17179869184} \quad \text{for} \quad n \geq 9.$$

Hence, for $0 < x < \pi/2$ and $n \geq 9$, we have

$$\begin{aligned}\frac{v_{n+1}(x)}{v_n(x)} &< \frac{5\big(12n-84+\frac{122206536915}{34359738368}\big)}{(2n+1)(n+1)\big(3n-24-\frac{26847}{17179869184}\big)}\\ &= \frac{5(412316860416n-2864011485997)}{6(2n+1)(1+n)(17179869184n-137438962421)} < 1.\end{aligned}$$

Therefore, for every $x \in (0, \pi/2)$, the sequence $n \longmapsto v_n(x)$ is strictly decreasing for $n \geq 9$. We then obtain from (18) that

$$\begin{aligned}\frac{x^8(4\sin x\cos x + x)^2}{2x^3\cos x}H_3'(x) &> x^8\left(\frac{59}{280} - \frac{10229}{151200}x^2\right) + \frac{104213}{19958400}x^{12}\\ &\quad + x^{14}\left(\frac{183221}{454053600} - \frac{25237379}{217945728000}x^2\right) > 0\end{aligned}$$

for $0 < x < \pi/2$. Hence, $H_3(x) = \frac{H_1'(x)}{H_2'(x)}$ is strictly increasing on $(0, \pi/2)$. By Lemma 3, the function

$$\frac{H_1(x) - H_1(0)}{H_2(x) - H_2(0)} = \frac{H_1(x)}{H_2(x)} = H(x)$$

is strictly increasing on $(0, \pi/2)$. The proof of Theorem 2 is complete. □

From (16) we obtain that for $0 < x < \pi/2$,

$$3\left(\frac{\mathrm{Si}(x)}{x}\right)^2 + \frac{\tan x}{x} > 4, \tag{19}$$

which is an analogue of inequality (1). From (17) we obtain that for $0 < x < \pi/2$,

$$6\left(\frac{\mathrm{Si}(x)}{x}\right) + \left(\frac{\tan x}{x}\right) > 7, \tag{20}$$

which is an analogue of inequality (3).

From the left-hand side of (6), we obtain that for $0 < x < \pi/2$,

$$\frac{\mathrm{Si}(x)}{x} > 1 - \frac{1}{18}x^2 + \frac{1}{600}x^4 - \frac{1}{35280}x^6 > 0,$$

which yields, for $0 < x < \pi/2$,

$$\begin{aligned} \left(\frac{\mathrm{Si}(x)}{x}\right)^2 &> \left(1 - \frac{1}{18}x^2 + \frac{1}{600}x^4 - \frac{1}{35280}x^6\right)^2 \\ &= 1 - \frac{1}{9}x^2 + \frac{13}{2025}x^4 - \frac{8}{33075}x^6 \\ &\quad + x^8\left(\frac{941}{158760000} - \frac{1}{10584000}x^2\right) + \frac{1}{1244678400}x^{12} \\ &> 1 - \frac{1}{9}x^2 + \frac{13}{2025}x^4 - \frac{8}{33075}x^6. \end{aligned} \tag{21}$$

From the power series expansion of $\tan x$:

$$\tan x = \sum_{n=1}^{\infty} \frac{2^{2n}(2^{2n}-1)|B_{2n}|}{(2n)!}x^{2n-1}, \qquad |x| < \frac{\pi}{2},$$

where B_n $(n = 0, 1, 2, \ldots)$ are Bernoulli numbers, defined by

$$\frac{t}{e^t - 1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!},$$

we obtain

$$\frac{\tan x}{x} > 1 + \frac{1}{3}x^2 + \frac{2}{15}x^4 + \frac{17}{315}x^6, \qquad 0 < x < \frac{\pi}{2}. \tag{22}$$

By using (21), (22) and (7), we obtain that for $0 < x < \pi/2$,

$$\begin{aligned} &\frac{3\left(\frac{\mathrm{Si}(x)}{x}\right)^2 + \frac{\tan x}{x}}{4} - \frac{6\left(\frac{\mathrm{Si}(x)}{x}\right) + \left(\frac{\tan x}{x}\right)}{7} \\ &= \frac{3}{4}\left(\frac{\mathrm{Si}(x)}{x}\right)^2 + \frac{3}{28}\left(\frac{\tan x}{x}\right) - \frac{6}{7}\left(\frac{\mathrm{Si}(x)}{x}\right) \\ &> \frac{3}{4}\left(1 - \frac{1}{9}x^2 + \frac{13}{2025}x^4 - \frac{8}{33075}x^6\right) + \frac{3}{28}\left(1 + \frac{1}{3}x^2 + \frac{2}{15}x^4 + \frac{17}{315}x^6\right) \\ &\quad - \frac{6}{7}\left(1 - \frac{1}{18}x^2 + \frac{1}{600}x^4\right) = \frac{167}{9450}x^4 + \frac{247}{44100}x^6 > 0. \end{aligned}$$

Hence, we have, for $0 < x < \pi/2$,

$$\frac{3\left(\frac{\mathrm{Si}(x)}{x}\right)^2 + \frac{\tan x}{x}}{4} > \frac{6\left(\frac{\mathrm{Si}(x)}{x}\right) + \left(\frac{\tan x}{x}\right)}{7} > 1,$$

which shows that the inequality (20) is sharper than the inequality (19).

4. SHARP ADAMOVIĆ-MITRINOVIĆ AND CUSA TYPE INEQUALITIES

It is known in the literature that

$$(\cos x)^{1/3} < \frac{\sin x}{x} < \frac{2 + \cos x}{3} \tag{23}$$

for $0 < |x| < \pi/2$. The left-hand side inequality (23) was obtained by Adamović and Mitrinović (see [26, p. 238]), while the right-hand side inequality (23) was first mentioned by the German philosopher and theologian Nicolaus de Cusa (1401-1464), by a geometrical method. A rigorous proof of the right-hand side inequality (23) was given by Huygens [18], who used the right-hand side of (23) to estimate the number π. The left-hand side inequality (23) is now known as Adamović-Mitrinović inequality, while the right-hand side inequality (23) is now known as Cusa's inequality [7, 12, 28, 33, 38, 52, 53, 56]. Further interesting historical facts about Cusa's inequality can be found in [38].

The inequality (23) can be written in the following form:

$$\big(f(x)\big)^{1/3} < \frac{F(x)}{x} < \frac{2 + f(x)}{3}, \tag{24}$$

where $F'(x) = f(x)$. In analogy with (23) and (24), we here establish sharp Adamović-Mitrinović and Cusa type inequalities involving the sine integral given by Theorem 3.

Theorem 3. *For $0 < x < \pi/2$, we have*

$$\left(\frac{\sin x}{x}\right)^{\xi} < \frac{\mathrm{Si}(x)}{x} < (1 - \eta) + \eta\left(\frac{\sin x}{x}\right), \tag{25}$$

where the constants $\xi = \frac{1}{3}$ and $\eta = \frac{1}{3}$ are the best possible, in the sense that $\xi = \frac{1}{3}$ can not be replaced by a smaller number, and $\eta = \frac{1}{3}$ can not be replaced by a greater number.

Proof. First of all, we show that the left-hand side of (25) with $\xi = 1/3$ is valid for $0 < x < \pi/2$. It suffices to show that

$$\left(\frac{\sin x}{x}\right)^{1/3} < 1 - \frac{1}{18}x^2 < \frac{\mathrm{Si}(x)}{x}, \qquad 0 < x < \frac{\pi}{2}. \tag{26}$$

By using the right-hand side of (5), we obtain that for $0 < x < \pi/2$,

$$\left(1 - \frac{1}{18}x^2\right)^3 - \frac{\sin x}{x} > 1 - \frac{1}{6}x^2 + \frac{1}{108}x^4 - \frac{1}{5832}x^6 - \left(1 - \frac{1}{6}x^2 + \frac{1}{120}x^4\right)$$
$$= x^4\left(\frac{1}{1080} - \frac{1}{5832}x^2\right) > 0 \Longrightarrow \left(\frac{\sin x}{x}\right)^{1/3} < 1 - \frac{1}{18}x^2.$$

The right-hand side of (26) has been shown, see (7). Hence, the left-hand side of (25) with $\xi = 1/3$ is valid for $0 < x < \pi/2$.

Next, we assume that the left-hand side of (25) holds for $0 < x < \pi/2$. Then we have

$$I(x) = \frac{\mathrm{Si}(x)}{x} - \left(\frac{\sin x}{x}\right)^{\xi} > 0, \qquad 0 < x < \frac{\pi}{2}.$$

We find that

$$I(x) = \frac{3\xi - 1}{18}x^2 + O(x^4) \quad \text{as} \quad x \to 0.$$

It then follows that it is necessary to have $\xi \geq \frac{1}{3}$ for $I(x)$ to be positive on $(0, \pi/2)$. Hence, the left-hand side of (25) holds for $0 < x < \pi/2$, and the constant $\xi = \frac{1}{3}$ is the best possible.

Now, we show that the right-hand side of (25) with $\eta = 1/3$ is valid for $0 < x < \pi/2$. For $0 < x < \pi/2$, let

$$J(x) = \frac{2}{3} + \frac{1}{3}\left(\frac{\sin x}{x}\right) - \frac{\mathrm{Si}(x)}{x}.$$

Using (5) and (7), we find that for $0 < x < \pi/2$,

$$J(x) > \frac{2}{3} + \frac{1}{3}\left(1 - \frac{1}{6}x^2 + \frac{1}{120}x^4 - \frac{1}{5040}x^6\right) - \left(1 - \frac{1}{18}x^2 + \frac{1}{600}x^4\right)$$
$$= x^4\left(\frac{1}{900} - \frac{1}{15120}x^2\right) > 0.$$

Hence, the right-hand side of (25) with $\eta = 1/3$ is valid for $0 < x < \pi/2$.

If we write the right-hand side of (25) as

$$\frac{x - \mathrm{Si}(x)}{x - \sin x} > \eta,$$

we find that

$$\lim_{x \to 0} \frac{x - \mathrm{Si}(x)}{x - \sin x} = \frac{1}{3}.$$

Hence, the right-hand side of (25) holds for $0 < x < \pi/2$, and the constant $\eta = \frac{1}{3}$ is the best possible. The proof of Theorem 3 is complete. □

In particular, the choice $(\xi, \eta) = (\frac{1}{3}, \frac{1}{3})$ in (25) yields

$$\left(\frac{\sin x}{x}\right)^{1/3} < \frac{\mathrm{Si}(x)}{x} < \frac{2 + \frac{\sin x}{x}}{3}$$

for $0 < |x| < \pi/2$. In fact, we have the following inequality chain:

$$(\cos x)^{1/9} < \left(\frac{\sin x}{x}\right)^{1/3} < \frac{\mathrm{Si}(x)}{x} < \frac{2 + \frac{\sin x}{x}}{3} < \frac{8 + \cos x}{9} \tag{27}$$

for $0 < |x| < \pi/2$. Clearly, the first inequality in (27) holds for $0 < |x| < \pi/2$. The last inequality in (27) is equivalent to the second inequality in (23).

5. SHARP SHAFER-FINK TYPE INEQUALITY

For $0 \le x \le 1$, the following double inequality holds:

$$\frac{3x}{2 + \sqrt{1 - x^2}} \le \arcsin x \le \frac{\pi x}{2 + \sqrt{1 - x^2}}. \tag{28}$$

The left-hand side inequality was presented by Shafer (see, *e.g.*, [27, p. 247]), while the right-hand side inequality was established by Fink [14]. Shafer-Fink's inequalities have attracted much interest of many mathematicians and have motivated a large number of papers involving various generalizations and improvements [16, 21–23, 25, 35, 37, 48, 49]. Zhu [50] provided a solution to an open problem posed by Oppenheim in [34], and deduced some Shafer-Fink inequalities from the solution of Oppenheim's problem. Chen and Cheung [11] gave a concise proof to Oppenheim's problem.

We here establish a sharp Shafer-Fink type inequality involving the sine integral given by Theorem 4.

Theorem 4. *For* $0 \le x \le 1$*, we have*

$$\frac{9x}{10 - \sqrt{1 - x^2}} \le \mathrm{Si}(x) \le \frac{10\mathrm{Si}(1)x}{10 - \sqrt{1 - x^2}}, \tag{29}$$

where the constants 9 *and* $10\mathrm{Si}(1) = 9.4608307\ldots$ *are the best possible.*

Proof. Noting that $\mathrm{Si}(1) = 0.94608307\ldots$, we see that the double inequality (29) is valid for $x = 0$ and $x = 1$. We now prove that the double inequality (29) holds for $0 < x < 1$. To this end, we consider the function $U(x)$ defined by

$$U(x) = \frac{\mathrm{Si}(x)}{x}(10 - \sqrt{1 - x^2}), \qquad 0 < x < 1.$$

Differentiating $U(x)$ and applying (5) and (7), we obtain for $0 < x < 1$,

$$\begin{aligned} x\sqrt{1-x^2}U'(x) &= (10\sqrt{1-x^2}-1+x^2)\frac{\sin x}{x} - (10\sqrt{1-x^2}-1)\frac{\mathrm{Si}(x)}{x} \\ &> (10\sqrt{1-x^2}-1+x^2)\left(1-\frac{1}{6}x^2\right) \\ &\quad - (10\sqrt{1-x^2}-1)\left(1-\frac{1}{18}x^2+\frac{1}{600}x^4\right) \\ &= x^2\left(\frac{10}{9}-\frac{33}{200}x^2\right) - \left(\frac{1}{60}x^4+\frac{10}{9}x^2\right)\sqrt{1-x^2}. \end{aligned}$$

Elementary calculations show that for $0 < x < 1$,

$$\left(x^2\left(\frac{10}{9}-\frac{33}{200}x^2\right)\right)^2 - \left(\left(\frac{1}{60}x^4+\frac{10}{9}x^2\right)\sqrt{1-x^2}\right)^2 = \frac{673}{810}x^6 + \frac{69103}{1080000}x^8 + \frac{1}{3600}x^{10} > 0,$$

which implies

$$x^2\left(\frac{10}{9}-\frac{33}{200}x^2\right) > \left(\frac{1}{60}x^4+\frac{10}{9}x^2\right)\sqrt{1-x^2}$$

for $0 < x < 1$. We then obtain $U'(x) > 0$ for $0 < x < 1$. Hence, $U(x)$ is strictly increasing on $(0, 1)$, and we have

$$9 = \lim_{t\to 0^+} U(t) < U(x) = \frac{\mathrm{Si}(x)}{x}(10-\sqrt{1-x^2}) < \lim_{t\to 1^-} U(t) = 10\mathrm{Si}(1), \qquad 0 < x < 1.$$

The proof of Theorem 4 is complete. □

The sine integral is in fact a special case of an integral involving Bessel function of the first kind which has been investigated for example by Askey and Steinig [6], Gasper [15]. Here we conclude this paper by using this observation to pose an open problem as follows:

Open problem 1. *Whether the inequalities of the present paper can be extended to integrals of Bessel functions of the first kind.*

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REFERENCES

1. H. Alzer, M. K. Kwong: *A variant of the Fejér-Jackson inequality.* Studia Sci. Math. Hungar. **56** (4) (2019), 500–509.

2. H. Alzer, B. Fuglede: *On a trigonometric inequality of Turán*. J. Approx. Theory **164** (11) (2012), 1496–1500.

3. G. D. Anderson, S. L. Qiu, M. K. Vamanamurthy, M. Vuorinen: *Generalized elliptic integral and modular equations*. Pacific J. Math. **192** (2000), 1–37.

4. G. D. Anderson, M. K. Vamanamurthy, M. Vuorinen: *Conformal Invariants, Inequalities, and Quasiconformal Maps*. New York, 1997.

5. G. D. Anderson, M. K. Vamanamurthy, M. Vuorinen: *Monotonicity rules in calculus*. Amer. Math. Monthly **113** (2006), 805–816.

6. R. Askey, J. Steinig: *Some positive trigonometric sums*. Trans. Amer. Math. Soc. **187** (1974), 295–307.

7. Y. J. Bagul, C. Chesneau, M. Kostić: *On the Cusa-Huygens inequality*. RACSAM Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. **115** (2021), Article 29 `https://doi.org/10.1007/s13398-020-00978-1`

8. C.-P. Chen: *Sharp Wilker- and Huygens-type inequalities for inverse trigonometric and inverse hyperbolic functions*. Integral Transforms Spec. Funct. **23** (12) (2012), 865–873.

9. C.-P. Chen: *Wilker and Huygens type inequalities for the lemniscate functions*. J. Math. Inequal. **6** (2012), 673–684.

10. C.-P. Chen: *Wilker and Huygens type inequalities for the lemniscate functions, II*. Math. Inequal. Appl. **16** (2013), 577–586.

11. C.-P. Chen, W.-S. Cheung: *Wilker- and Huygens-type inequalities and solution to Oppenheim's problem*. Integral Transforms Spec. Funct. **23** (2012), 325–336.

12. C.-P. Chen, W.-S Cheung: *Sharp Cusa and Becker-Stark inequalities*. J. Ineq. Appl. **2011** (2011), Article 136. `http://www.journalofinequalitiesandapplications.com/content/2011/1/136`

13. C.-P. Chen, W.-S. Cheung: *Sharpness of Wilker and Huygens type inequalities*. J. Inequal. Appl. (2012) 72. `https://link.springer.com/article/10.1186/1029-242X-2012-72`

14. A. M. Fink: *Two Inequalities*. Publikacije Elektrotehničkog Fakulteta Univerzitet u Beogradu. Serija Matematika **6** (1995), 48–49.

15. G. Gasper: *Positive integrals of Bessel functions*. SIAM J. Math. Anal. **6** (5) (1975), 868–881.

16. B.-N. Guo, Q.-M. Luo, F. Qi: *Sharpening and generalizations of Shafer-Fink's double inequality for the arc sine function*. Filomat **27** (2) (2013), 261–265. `http://dx.doi.org/10.2298/FIL1302261G`

17. B.-N. Guo, B.-M. Qiao, F. Qi, W. Li: *On new proofs of Wilker inequalities involving trigonometric functions*. Math. Inequal. Appl. **6** (2003), 19–22.

18. C. Huygens: *Oeuvres Completes 1888-1940*. Société Hollondaise des Science, Haga.

19. D. Jackson: *Über eine trigonometrische Summe*. Rend. Circ. Mat. Palermo **32** (1911), 257–262.

20. G. Jameson, N. Lord, J. McKee: *99.05 An inequality for* $\mathrm{Si}(x)$. The Mathematical Gazette. **99** (544) (2015), 133–139.

21. B. J. Malešević: *Application of λ-method on Shafer-Fink's inequality*. Univerzitet u Beogradu. Publikacije Elektrotehničkog Fakulteta. Serija Matematika, **8** (1997), 103–105.

22. B. J. Malešević: *One method for proving inequalities by computer*. J. Inequal. Appl. **2007** (2007), Article ID 78691. https://link.springer.com/content/pdf/10.1155/2007/78691.pdf?pdf=core

23. B. J. Malešević:*An application of λ-method on inequalities of Shafer-Fink's type*. Math. Inequal. Appl. **10** (2007), 529–534.

24. B.J. Malešević, B. Banjac, I. Jovović: *A proof of two conjectures of Chao-Ping Chen for inverse trigonometric functions*. J. Math. Inequal. **11** (1) (2017), 151–162.

25. B. Malešević, M. Rašajski, T. Lutovac: *Refinements and generalizations of some inequalities of Shafer-Fink's type for the inverse sine function*. J. Inequal. Appl. **2017** (2017) , Article 275.
https://journalofinequalitiesandapplications.springeropen.com/articles/10.1186/s13660-017-1554-1

26. D. S. Mitrinović: *Analytic Inequalities*. Springer-Verlag, Berlin, 1970.

27. D. S. Mitrinović, P. M. Vasić: *Analytic Inequalities*. Springer, New York, 1970.

28. C. Mortitci: *The natural approach of Wilker-Cusa-Huygens inequalities*. Math. Inequal. Appl. **14** (2011), 535–541.

29. E. Neuman: *One- and two-sided inequalities for Jacobian elliptic functions and related results*. Integral Transforms Spec. Funct. **21** (2010), 399–407.

30. E. Neuman: *On Wilker and Huygens type inequalities*. Math. Inequal. Appl. 15 (2) (2012), 271–279.

31. E. Neuman: *Wilker and Huygens type inequalities for Jacobian elliptic and theta functions*. Integral Transforms Spec. Funct. **25** (3) (2014), 240–248.

32. E. Neuman: *On lemniscate functions*. Integral Transforms Spec. Funct. **24** (2013), 164–171.

33. E. Neuman, J. Sándor: *On some inequalities involving trigonometric and hyperbolic functions with emphasis on the Cusa-Huygens, Wilker, and Huygens inequalities*. Math. Inequal. Appl. **13** (2010), 715–723.

34. A. Oppenheim, *E1277*. Amer. Math. Monthly **64** (1957), 504.

35. W. Pan, L. Zhu: *Generalizations of Shafer-Fink-Type inequalities for the arc sine function*. J. Inequal. Appl. **2009** (2009), Article ID 705317.
https://link.springer.com/content/pdf/10.1155/2009/705317.pdf?pdf=core

36. I. Pinelis: *L'Hospital rules of monotonicity and Wilker-Anglesio inequality*. Amer. Math. Monthly, **111** (2004), 905–909.

37. F. Qi, B.-N. Guo: *Sharpening and generalizations of Shafer's inequality for the arc sine function*. Integral Transforms Spec. Funct. **23** (2012), 129–134.

38. J. Sándor, M. Bencze: *On Huygens' trigonometric inequality*. RGMIA Res. Rep. Collection **8** (3) (2005), Article 14.
https://rgmia.org/papers/v8n3/Huygens.pdf

39. J. S. Sumner, A. A. Jagers, M. Vowe, J. Anglesio: *Inequalities involving trigonometric functions*. Amer. Math. Monthly **98** (1991), 264–267.

40. J. B. Wilker: *Problem E 3306*. Amer. Math. Monthly **96** (1989), 55.

41. S.-H. Wu: *On extension and refinement of Wilker's inequality*. Rocky Mountain J. Math. **39** (2009), 683–687.

42. S.-H. Wu, A. Baricz: *Generalizations of Mitrinović, Adamović and Lazarevic's inequalities and their applications*. Publ. Math. Debrecen **75** (2009), 447–458.

43. S.-H. Wu, L. Debnath: *Wilker-type inequalities for hyperbolic functions*. Appl. Math. Lett. **25** (5) (2012) 837–842.

44. S.-H. Wu, H. M. Srivastava: *A weighted and exponential generalization of Wilker's inequality and its applications*. Integral Transforms and Spec. Funct. **18** (2007), 529–535.

45. S.-H. Wu, H. M. Srivastava: *A further refinement of Wilker's inequality*. Integral Transforms and Spec. Funct. **19** (2008), 757–765.

46. B. Zhang, C.-P. Chen: *Sharp Wilker and Huygens type inequalities for trigonometric and inverse trigonometric functions*. J. Math. Inequal. **14** (3) (2020), 673–684.

47. L. Zhang, L. Zhu: *A new elementary proof of Wilker's inequalities*. Math. Inequal. Appl. **11** (2008), 149-151.

48. L. Zhu: *On Shafer-Fink inequalities*. Math. Inequal. Appl. **8** (2005), 571–574.

49. L. Zhu: *On Shafer-Fink-type inequality*. J. Inequal. Appl. **2007** (2007), Article ID 67430.
`https://link.springer.com/content/pdf/10.1155/2007/67430.pdf?pdf=core`

50. L. Zhu: *A solution of a problem of Oppenheim*. Math. Inequal. Appl. **10** (2007), 57–61.

51. L. Zhu: *Some new inequalities of the Huygens type*. Comput. Math. Appl. **58** (2009), 1180–1182.

52. L. Zhu: *Sharp inequalities of Mitrinovic-Adamovic type*. RACSAM Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. **113** (2019), 957–968. `https://doi.org/10.1007/s13398-018-0521-0`

53. L. Zhu: *An unity of Mitrinovic-Adamovic and Cusa-Huygens inequalities and the analogue for hyperbolic functions*. RACSAM Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. **113** (2019), 3399–3412.
`https://doi.org/10.1007/s13398-019-00706-4`

54. L. Zhu: *A new simple proof of Wilker's inequality*. Math. Inequal. Appl. **8** (2005), 749–750.

55. L. Zhu: *Some new Wilker-type inequalities for circular and hyperbolic functions*. Abstr. Appl. Anal. 2009, Article ID 485842.

56. L. Zhu: *A source of inequalities for circular functions*. Comput. Math. Appl. **58** (2009), 1998–2004.

57. L. Zhu: *Inequalities for Hyperbolic functions and their applications*. J. Ineq. Appl. Vol. (2010), Article ID 130821.

58. L. Zhu: *On Wilker-type inequalities*. Math. Inequal. Appl. **10** (2007), 727–731.

Jun-Ling Sun
School of Mathematics and Information Science,
Henan Polytechnic University,
Jiaozuo City 454003, Henan Province,
People's Republic of China.
E-Mail: *sunjunling1979@126.com*

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Chao-Ping Chen
School of Mathematics and Information Science,
Henan Polytechnic University,
Jiaozuo City 454003, Henan Province,
People's Republic of China.
E-Mail: *chenchaoping@sohu.com*