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SCALED EVOLUTION OF THE HANIN INEQUALITY

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The Hanin inequality is a kind of a reverse of the non-weighted arithmetic-quadratic mean inequality defined for non-negative real n-tuples, involving maxima of the corresponding n-tuple. Our goal is to extend the Hanin inequality in several directions. We first give two-parametric extension of the basic inequality, as well as its refinement. Then, we discuss the corresponding weighted forms of the established results. Finally, we derive several complex extensions of the Hanin inequality.

1. INTRODUCTION

The Jensen inequality for convex functions is certainly one of the most interesting inequalities which is extensively used in almost all branches of mathematics, especially in mathematical analysis and statistics.

Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function, where I is an interval, $\mathbf{a} = (a_1, a_2, \dots, a_n) \in I^n$ and $\mathbf{w} = (w_1, w_2, \dots, w_n) \in \mathbb{R}_+^n$, where $\sum_{j=1}^n w_j = 1$. In 2012, Krnić et al. [**5**] (see also [**4**]), proved that

$$(1)\qquad n w_{\max} \mathcal{I}_n(f, \mathbf{a}) \geq \sum_{j=1}^n w_j f(a_j) - f\Big(\sum_{j=1}^n w_j a_j\Big) \geq n w_{\min} \mathcal{I}_n(f, \mathbf{a}),$$

where $w_{\min} = \min_{1\leq j\leq m} w_j$, $w_{\max} = \max_{1\leq j\leq m} w_j$, and

$$\mathcal{I}_n(f, \mathbf{x}) = \frac{1}{n}\sum_{j=1}^m f(x_j) - f\Big(\frac{1}{n}\sum_{j=1}^m x_j\Big).$$

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Clearly, the first inequality in (1) represents the reverse, while the second is refinement of the Jensen inequality. Based on (1), numerous inequalities such as the Young inequality, the Hölder inequality, power mean inequalities, etc. have been refined (see, e.g. [**2, 4, 5, 9, 11, 12, 13**] and the references cited therein). In addition, for a comprehensive overview of the old and new results in connection to the Jensen inequality, the reader is referred to monographs [**3, 8, 10**] and the references cited therein.

In particular, if $f(x) = x^2$ relation (1) provides the refinement and the reverse of the arithmetic-quadratic mean inequality. Recall that the weighted arithmetic and quadratic means are respectively defined by

$$A_n(\mathbf{a};\mathbf{w}) = \sum_{j=1}^n w_j a_j, \quad Q_n(\mathbf{a};\mathbf{w}) = \sqrt{\sum_{j=1}^n w_j a_j^2},$$

where $\mathbf{a} = (a_1, a_2, \ldots, a_n) \in \mathbb{R}_+^n$ and $\mathbf{w} = (w_1, w_2, \ldots, w_n) \in \mathbb{R}_+^n$, and where $\sum_{j=1}^n w_j = 1$. If $\mathbf{w} = (\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n})$, we obtain the corresponding non-weighted arithmetic and quadratic mean $A_n(\mathbf{a}) = \frac{1}{n}\sum_{j=1}^n a_j$ and $Q_n(\mathbf{a}) = \sqrt{\frac{1}{n}\sum_{j=1}^n a_j^2}$. Hence, in this setting relation (1) reduces to

$$nw_{\max}\big(Q_n^2(\mathbf{a}) - A_n^2(\mathbf{a})\big) \geq Q_n^2(\mathbf{a};\mathbf{w}) - A_n^2(\mathbf{a};\mathbf{w}) \geq nw_{\min}\big(Q_n^2(\mathbf{a}) - A_n^2(\mathbf{a})\big), \tag{2}$$

which provides the reverse and refinement of the weighted arithmetic-quadratic mean inequality in terms of the corresponding non-weighted inequality. However, relation (2) does not improve the non-weighted inequality $A_n(\mathbf{a}) \leq Q_n(\mathbf{a})$.

On the other hand, a kind of a reverse of the non-weighted arithmetic-quadratic mean inequality, established by Hanin [**1**], asserts that

$$\frac{M^2}{4} + A_n^2(\mathbf{a}) \geq Q_n^2(\mathbf{a}), \tag{3}$$

where $M = \max_{1\leq j\leq n} a_j$. The equality in (3) holds if either $a_1 = a_2 = \ldots = a_n = 0$, or, for n even, $n/2$ of a_j's are equal to zero, while the remaining half of a_j's are equal to $a > 0$. The original source for (3) is [**1**], where one can find the corresponding simple proof:

$$\left(\frac{M}{2} - A_n(\mathbf{a})\right)^2 + MA_n(\mathbf{a}) - Q_n^2(\mathbf{a}) = \left(\frac{M}{2} - A_n(\mathbf{a})\right)^2 + \frac{1}{n}\sum_{j=1}^n a_j(M - a_j) \geq 0.$$

The reader is also referred to the second hand information from [**7**, p. 288], where (3) is just listed without additional comments.

The main objective of the present paper is to extend the Hanin inequality (3) in several directions. The outline of the paper is as follows: after this Introduction, in Section 2 we first give an extension of the basic Hanin inequality which involves two parameters. Then we discuss efficiency of the established inequality, i.e. we impose some general conditions under which the obtained inequality is more accurate than the initial Hanin inequality. Further, we also give a refinement of the Hanin inequality based on the fact that a sample variance is invariant with respect to the translation. In Section 3 we discuss weighted forms of the Hanin-type inequalities. In addition, our results are compared with some previously known results from the literature. Finally, Section 4 is devoted to several complex extensions of the Hanin inequality.

2. MAIN RESULTS

Our first intention is to extend the original Hanin inequality by introducing some new parameters. More precisely, the first extension of (3) is the following two-parametric Hanin inequality:

Theorem 1. *If $0 < \beta \leq 2\alpha$, then holds the inequality*

$$M^2 + \alpha^2 A_n^2(\mathbf{a}) \geq \beta Q_n^2(\mathbf{a}), \tag{4}$$

where $M = \max_{1\leq j\leq n} a_j$. The equality in (4) holds if either $M = 0$ or $\alpha = \frac{n}{k}$, $\beta = \frac{2n}{k}$, $1 \leq k \leq n$, and $\mathbf{a}$ is an n-tuple consisting of $n-k$ zeroes, while the remaining entries have the same positive value.

Proof. In order to establish the corresponding proof, we start with the following identity:

$$\begin{aligned} &(M - \alpha A_n(\mathbf{a}))^2 + 2\alpha M A_n(\mathbf{a}) - \beta Q_n^2(\mathbf{a}) \\ &= (M - \alpha A_n(\mathbf{a}))^2 + \frac{1}{n}\sum_{j=1}^{n} a_j(2\alpha M - \beta a_j). \end{aligned} \tag{5}$$

Clearly, since $\frac{1}{n}\sum_{j=1}^{n} a_j(2\alpha M - \beta a_j) \geq (2\alpha - \beta)Q_n^2(\mathbf{a})$, it turns out that the right-hand side of (5) is non-negative, due to the condition $2\alpha - \beta \geq 0$. Therefore, (4) holds.

It remains to consider the equality conditions. Clearly, if $M = 0$, (4) is trivial equality. On the other hand, equality in (4) holds if and only if both terms on the right-hand side of identity (5) are equal to zero. If there exists an index j' such that $M > a_{j'} > 0$, then $2\alpha M - \beta a_{j'} > (2\alpha - \beta)a_{j'} \geq 0$, which means that inequality (4) is strict. Hence, for the equality in (4) to hold, each a_j must be equal to either zero or M, and $\beta = 2\alpha$. Now, suppose that n-tuple $\mathbf{a}$ consists of $n-k$ zeroes, while the remaining k of a_j's are equal to M, $1 \leq k \leq n$. Then, $A_n(\mathbf{a}) = \frac{kM}{n}$ and $M - \alpha A_n(\mathbf{a}) = (1 - \frac{\alpha k}{n})M$, which implies that $\alpha = \frac{n}{k}$ and $\beta = \frac{2n}{k}$. This proves our assertion. □

Remark 2. Let $\alpha^2 = \beta$. Combining the latter condition with $0 < \beta \leq 2\alpha$, it follows that $0 < \alpha \leq 2$. In this setting (4) reduces to

$$\frac{M^2}{\alpha^2} + A_n^2(\mathbf{a}) \geq Q_n^2(\mathbf{a}), \quad 0 < \alpha \leq 2.$$

Of course, the above parametric inequality is the best possible for $\alpha = 2$, that is, in the case of the original Hanin inequality (3). Note also that in this setting equality conditions from the statement of Theorem 1 reduce to the equality conditions for the original Hanin inequality.

It should be noticed here that, using the relationship between $A_n(\mathbf{a})$ and $Q_n(\mathbf{a})$, the above theorem provides mutual bounds for the quadratic mean in terms of the arithmetic mean.

Corollary 3. *If $0 < \beta \le 2\alpha$, then it holds*

$$A_n(\mathbf{a}) \le Q_n(\mathbf{a}) \le \frac{M + \alpha A_n(\mathbf{a})}{\sqrt{\beta}}. \tag{6}$$

Proof. It follows from (4) since $\sqrt{x^2 + y^2} \le x + y$ for all $x, y \ge 0$. □

Our next intention is efficiency discussion for the two-parametric Hanin inequality (4). In other words, our goal now is to impose some general conditions under which the scaled inequality (4) is sharper than the original Hanin inequality (3). In order to do this, we define the quantity

$$c = \frac{4A_n^2(\mathbf{a})}{4A_n^2(\mathbf{a}) + M^2}, \tag{7}$$

where $A_n(\mathbf{a})$ is the arithmetic mean of a non-negative n-tuple $\mathbf{a} = (a_1, a_2, \ldots, a_n)$, $\mathbf{a} \ne (0, 0, \ldots, 0)$ and $M = \max_{1 \le j \le n} a_j$.

Remark 4. It should be noticed here that the parameter c defined by (7) is always positive, although it can be sufficiently small. To see this, consider the n-tuple $\mathbf{a} = (\frac{1}{2}, 0, 0, \ldots, 0)$. In this setting, we have that $c = \frac{4}{n^2+4}$, so c tends to zero as n tends to infinity. On the other hand, since $A_n(\mathbf{a}) \le M$, it follows that $4A_n^2(\mathbf{a}) + M^2 \ge 5A_n^2(\mathbf{a})$, and consequently, $c \le \frac{4}{5}$. In conclusion, we have that $c \in (0, \frac{4}{5})$.

Theorem 5. *Let $D(c) = \{(u, v);\ v \le 2u,\ v \ge cu^2 + 4(1 - c)\}$, where c is defined by (7). If $(\alpha, \beta) \in D(c)$, then inequality (4) is sharper than (3).*

Proof. If $(\alpha, \beta) \in D(c)$, then $\beta \le 2\alpha$ and

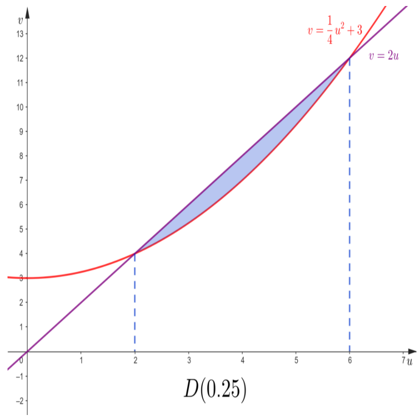
$$\beta \ge c\alpha^2 + 4(1 - c) = \frac{4M^2 + 4\alpha^2 A_n^2(\mathbf{a})}{M^2 + 4A_n^2(\mathbf{a})},$$

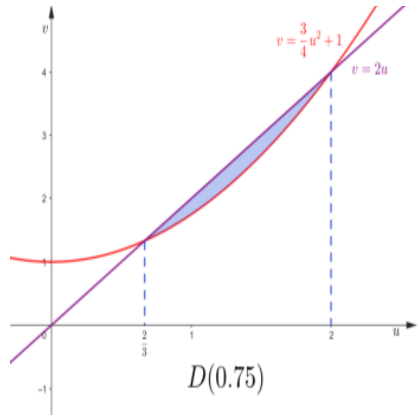
which is equivalent to

$$\frac{M^2}{4} + A_n^2(\mathbf{a}) \ge \frac{M^2 + \alpha^2 A_n^2(\mathbf{a})}{\beta}.$$

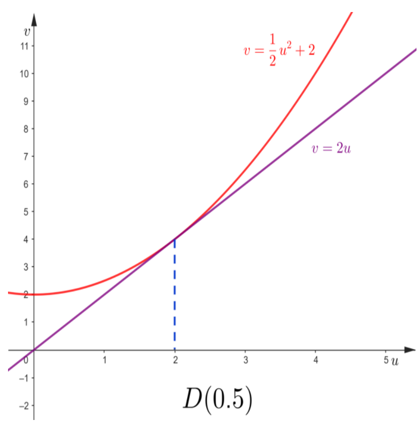
Clearly, the latter inequality implies that (4) is sharper than (3) in this setting. □

Remark 6. It should be noticed here that the set $D(c)$, defined in the previous theorem, is a region in uv-plane bounded by the line $v = 2u$ and parabola $v = cu^2 + 4(1-c)$. By a straightforward computation, it follows that this region extends along the u-axis with coordinates $u \in [2, \frac{2}{c} - 2]$ when $c \in (0, \frac{1}{2})$, that is, with coordinates $u \in [\frac{2}{c} - 2, 2]$ when $c \in (\frac{1}{2}, \frac{4}{5})$.





Finally, if $c = \frac{1}{2}$, then the line $v = 2u$ is tangent to the parabola $v = \frac{1}{2}u^2 + 2$.



In this case, $D(c)$ consists of only one point, i.e. inequality (4) reduces to the original Hanin inequality (3).

We aim now to establish the refinement of the Hanin inequality. The corresponding refinement can be expressed by including the minimum of the non-negative n-tuple $\mathbf{a} = (a_1, a_2, \ldots, a_n)$, i.e. $m = \min_{1 \le j \le n} a_j$. More generally, we obtain the refinement of the two-parametric inequality (4) established in Theorem 1. The key step in the following proof is the fact that a sample variance is invariant with respect to a translation.

Theorem 7. *Let $\mathbf{a} = (a_1, a_2, \dots, a_n)$ be a non-negative real n-tuple and let $0 < \beta \le 2\alpha$. Then holds the inequality*

$$\frac{(M-m)^2}{\beta} + \left(\frac{\alpha^2}{\beta} - 1\right)\big(A_n(\mathbf{a}) - m\big)^2 + A_n^2(\mathbf{a}) \ge Q_n^2(\mathbf{a}). \tag{8}$$

The equality in (8) *holds under the same conditions as in the statement of Theorem* 1.

Proof. The initial step is to rewrite (4) taking the usual statistical notation for the unadjusted (biased) sample variance

$$S_n^2(\mathbf{a}) = \frac{1}{n}\sum_{j=1}^{n}\left(a_j - \frac{1}{n}\sum_{j=1}^{n} a_j\right)^2 = Q_n^2(\mathbf{a}) - A_n^2(\mathbf{a}).$$

Therefore (4) is equivalent to

$$M^2 + (\alpha^2 - \beta)A_n^2(\mathbf{a}) \ge \beta S_n^2(\mathbf{a}). \tag{9}$$

Since S_n^2 is invariant with respect to the translation $\mathbf{a} \mapsto (a_1 - t, \dots, a_n - t)$, consider the quadratic function

$$f(t) = (M-t)^2 + (\alpha^2 - \beta)A_n^2(\mathbf{a} - t) = (M-t)^2 + (\alpha^2-\beta)\big(A_n(\mathbf{a}) - t\big)^2,$$

which possesses the minimum value $f(t_0) = \frac{(\alpha^2-\beta)\,(M - A_n(\mathbf{a}))^2}{\alpha^2-\beta+1}$ at the point

$$t_0 = A_n(\mathbf{a}) + \frac{M - A_n(\mathbf{a})}{\alpha^2 - \beta + 1}.$$

On the other hand, the inequality

$$(M-t)^2 + (\alpha^2-\beta)\big(A_n(\mathbf{a}) - t\big)^2 \ge \beta S_n^2(\mathbf{a}) \tag{10}$$

holds for a non-negative n-tuple $(a_1 - t, \dots, a_n - t)$, that is, when $t \le m = \min_{1\le j\le n} a_j$.

Now, it should be noticed that, since $m \le A_n(\mathbf{a}) \le M$ and $\alpha^2 - \beta + 1 \ge \alpha^2 - 2\alpha + 1 = (\alpha - 1)^2 \ge 0$, it follows that $t_0 \ge m$, so $(a_1 - t_0, \dots, a_n - t_0)$ is not a non-negative n-tuple. This means that (10) does not hold at the point t_0, in general.

On the other hand, relation (10) is valid for all $t \le m$. Moreover, since the above quadratic function is decreasing on the interval $(-\infty, m]$, the best possible estimate in (10) appears when $t = m$. This yields (8) and the proof is completed. □

Remark 8. If $\alpha^2 = \beta$, then, similarly to Remark 2, inequality (8) reduces to

$$\frac{(M-m)^2}{\alpha^2} + A_n^2(\mathbf{a}) \ge Q_n^2(\mathbf{a}), \quad 0 < \alpha \le 2.$$

Clearly, this inequality is the best possible for $\alpha = 2$, i.e.

$$\frac{(M-m)^2}{4} + A_n^2(\mathbf{a}) \geq Q_n^2(\mathbf{a}),$$

which is refinement of the original Hanin inequality (3).

3. THE WEIGHTED HANIN-TYPE INEQUALITIES

In the previous section we have studied non-weighted Hanin-type inequalities. The corresponding results are easily modified to hold for the case of the weighted arithmetic and quadratic means. We start our discussion with the weighted form of Theorem 1.

Theorem 9. *If $0 < \beta \leq 2\alpha$, then holds the inequality*

$$M^2 + \alpha^2 A_n^2(\mathbf{a};\mathbf{w}) \geq \beta Q_n^2(\mathbf{a};\mathbf{w}). \tag{11}$$

The equality in (11) holds if either $M = 0$ or $\alpha = \frac{1}{w^+}$, $\beta = \frac{2}{w^+}$, where

$$w^+ = \sum_{j=1, w_j a_j \neq 0}^{n} w_j,$$

and $\mathbf{a}$ is an n-tuple with entries equal to either 0 or M.

Proof. In this setting, identity (5) can be rewritten as

$$\begin{aligned}&(M - \alpha A_n(\mathbf{a};\mathbf{w}))^2 + 2\alpha M A_n(\mathbf{a};\mathbf{w}) - \beta Q_n^2(\mathbf{a};\mathbf{w}) \\ &= (M - \alpha A_n(\mathbf{a};\mathbf{w}))^2 + \sum_{j=1}^{n} w_j a_j (2\alpha M - \beta a_j).\end{aligned}$$

Now, we aim to establish equality conditions in (11). If $M \neq 0$, then equality in (11) holds if and only if both terms of the right-hand side of the above identity are equal to zero. If there exists an index j' such that $w_j' > 0$ and $M > a_{j'} > 0$, then $w_j' a_j' (2\alpha M - \beta a_{j'}) > w_j'(2\alpha - \beta) a_{j'}^2 \geq 0$, which means that inequality (11) is strict. Hence, equality in (11) holds if $w_j = 0$ or a_j is equal to either zero or M, and $\beta = 2\alpha$. This means that the entries in the n-tuple $\mathbf{a}$ are either 0 or M, that is,

$$A_n(\mathbf{a};\mathbf{w}) = \sum_{j=1}^{n} w_j a_j = M \sum_{j=1, w_j a_j \neq 0}^{n} w_j = M w^+.$$

Consequently, $M - \alpha A_n(\mathbf{a};\mathbf{w}) = (1 - \alpha w^+) M$, which implies that $\alpha = \frac{1}{w^+}$ and $\beta = \frac{2}{w^+}$, as claimed. □

Remark 10. Clearly, Theorem 9 is an extension of Theorem 1. It should be also noticed here that if $w = \left(\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n}\right)$, then $w^+ = \frac{k}{n}$, where k is a number of entries not equal to zero.

Similarly to the previous result, the proof of Theorem 7 can be easily modified to a new framework, so we leave it to the interested reader. As a result, we obtain improved form of Theorem 9.

Theorem 11. *Let* $\mathbf{a} = (a_1, a_2, \ldots, a_n)$ *be a non-negative real* n*-tuple and let* $0 < \beta \leq 2\alpha$. *Then holds the inequality*

$$\frac{(M-m)^2}{\beta} + \left(\frac{\alpha^2}{\beta} - 1\right)\left(A_n(\mathbf{a};\mathbf{w}) - m\right)^2 + A_n^2(\mathbf{a};\mathbf{w}) \geq Q_n^2(\mathbf{a};\mathbf{w}). \tag{12}$$

The equality in (12) *holds under the same conditions as in the statement of Theorem* 9.

Remark 12. Having in mind Remark 2, relation (12) implies the inequality

$$Q_n^2(\mathbf{a};\mathbf{w}) - A_n^2(\mathbf{a};\mathbf{w}) \leq \frac{(M-m)^2}{4}, \tag{13}$$

which represents direct weighted extension of the initial Hanin inequality (3).

The advantage of the particular method developed in the previous section, compared to relation (2), is that this method does not depend on the weights. However, generally speaking, the two approaches are not comparable, as we will see in the next remark.

Remark 13. In order to conclude this section, we are going to compare the above inequality (13) with the first inequality in (2), that is, with the relation

$$Q_n^2(\mathbf{a};\mathbf{w}) - A_n^2(\mathbf{a};\mathbf{w}) \leq n w_{\max}\left(Q_n^2(\mathbf{a}) - A_n^2(\mathbf{a})\right). \tag{14}$$

It turns out that these two inequalities are not comparable in general. To see this, let $n = 3$, $\mathbf{a} = (9, 2, 1)$ and $\mathbf{w} = (\frac{6}{19}, \frac{6}{19}, \frac{7}{19})$. Then,

$$n w_{\max}\left(Q_n^2(\mathbf{a}) - A_n^2(\mathbf{a})\right) = 14 < 16 = \frac{(M-m)^2}{4},$$

which means that inequality (14) is more accurate in this case. On the other hand, for $\mathbf{a} = (9, 2, 1)$ and $\mathbf{w} = (\frac{5}{19}, \frac{5}{19}, \frac{9}{19})$, we have that

$$n w_{\max}\left(Q_n^2(\mathbf{a}) - A_n^2(\mathbf{a})\right) = 18 > 16 = \frac{(M-m)^2}{4},$$

which makes inequality (13) more precise in this setting.

4. THE HANIN INEQUALITY FOR COMPLEX NUMBERS

If $\mathbf{a} = (a_1, a_2, \ldots, a_n)$ is a complex n-tuple, then $|\mathbf{a}| = (|a_1|, |a_2| \ldots, |a_n|)$ is a non-negative real n-tuple. By putting $|\mathbf{a}|$ instead of $\mathbf{a}$ in (3), we arrive at the inequality

$$\frac{R^2}{4} + A_n^2(|\mathbf{a}|) \geq Q_n^2(|\mathbf{a}|), \tag{15}$$

where $R = \max_{1 \leq j \leq n} |a_j|$. Now, our intention is to establish a sharper form of this inequality. Here, we mean the fact that

$$A_n(|\mathbf{a}|) = \frac{1}{n}\sum_{j=1}^{n} |a_j| \geq \frac{1}{n}\Big|\sum_{j=1}^{n} a_j\Big| = |A_n(\mathbf{a})|,$$

by a triangle inequality.

We aim now to extend the above inequality in two different directions. In order to derive the first one, we are going to use a trigonometric form of a complex number.

Theorem 14. *Let $\mathbf{a} = (a_1, a_2, \ldots, a_n)$ be a complex n-tuple such that $a_j = \rho_j e^{i\phi_j}$, $j = 1, 2, \ldots, n$. Further, let $R = \rho_l = \max_{1 \leq j \leq n} \rho_j$ and $M^* = a_l = Re^{i\phi}$. If*

$$\cos(\phi - \phi_j) \geq \frac{\beta}{2\alpha}, \ j = 1, 2, \ldots, n, \tag{16}$$

where $\alpha, \beta > 0$, then holds the inequality

$$R^2 + \alpha^2 |A_n(\mathbf{a})|^2 \geq \beta Q_n^2(|\mathbf{a}|). \tag{17}$$

The equality in (17) holds if either $M^* = 0$ or $\alpha = \frac{n}{k}$, $\beta = \frac{2n}{k}$, $1 \leq k \leq n$, and $\mathbf{a}$ is an n-tuple consisting of $n - k$ zeroes, while the remaining entries have the same value.

Proof. Since $|x - y|^2 = |x|^2 - 2\mathrm{Re}(x\overline{y}) + |y|^2$, $x, y \in \mathbb{C}$, we have that

$$\begin{aligned}
&R^2 + \alpha^2 |A_n(\mathbf{a})|^2 - \beta Q_n^2(|\mathbf{a}|) \\
&= |M^* - \alpha A_n(\mathbf{a})|^2 + 2\alpha \mathrm{Re}\big(M^* \overline{A_n(\mathbf{a})}\big) - \beta Q_n^2(|\mathbf{a}|) \\
&= |M^* - \alpha A_n(\mathbf{a})|^2 + \frac{2\alpha}{n}\mathrm{Re}\Big(Re^{i\phi}\sum_{j=1}^{n} \rho_j e^{-i\phi_j}\Big) - \frac{\beta}{n}\sum_{j=1}^{n} \rho_j^2 \\
&= |M^* - \alpha A_n(\mathbf{a})|^2 + \frac{1}{n}\sum_{j=1}^{n} \rho_j\big(2\alpha R\cos(\phi - \phi_j) - \beta\rho_j\big) \\
&\geq |M^* - \alpha A_n(\mathbf{a})|^2 + \frac{1}{n}\sum_{j=1}^{n} \rho_j^2\big(2\alpha\cos(\phi - \phi_j) - \beta\big) \geq 0,
\end{aligned}$$

due to condition (16).

It remains to consider equality conditions in (17). Clearly, if $M^*=0$, then the equality sign holds trivially. Otherwise, if there exists an index j' such that $\rho_{j'}>0$ and $\cos(\phi-\phi'_j)>\frac{\beta}{2\alpha}$, then $2\alpha\cos(\phi-\phi'_j)-\beta>0$, so from the above relation we conclude that (17) is strict. Therefore, equality in (17) hold if either $\rho_j=0$, or $2\alpha\cos(\phi-\phi_j)-\beta=0$ and $R=\rho_j$, for every $j=1,2,\ldots,n$. In particular, $\phi=\phi_l$, which implies that $\beta=2\alpha$. Consequently, the n-tuple $\mathbf{a}$ consists of $n-k$ zeroes, while the remaining k of a_j's are equal to $M^*=Re^{i\phi}$, $1\le k\le n$. Then, $A_n(\mathbf{a})=\frac{kM^*}{n}$ and $|M^*-\alpha A_n(\mathbf{a})|=|1-\frac{\alpha k}{n}||M^*|$, so $\alpha=\frac{n}{k}$, $\beta=\frac{2n}{k}$. The proof is now completed. □

Remark 15. If $\mathbf{a}=(a_1,a_2,\ldots,a_n)$ is a non-negative real n-tuple, then Theorem 14 reduces to Theorem 1. Namely, then, $\phi_1=\phi_2=\ldots=\phi_n=0$, so condition (16) reduces to $0<\beta\le 2\alpha$, that is, inequality (17) becomes (4). It should be also noticed here that condition (16) is stronger than $0<\beta\le 2\alpha$.

Remark 16. If $\alpha=2$ and $\beta=4$, then inequality (17) reduces to the basic complex Hanin inequality

$$\frac{R^2}{4}+|A_n(\mathbf{a})|^2\ge Q_n^2(|\mathbf{a}|).$$

In this case, condition (16) reduces to $\cos(\phi-\phi_j)\ge 1$, that is $\cos(\phi-\phi_j)=1$, i.e. $\phi_1=\phi_2=\ldots=\phi_n$. This means that n-tuple $\mathbf{a}$ consists of complex numbers with the same arguments. A similar conclusion can be drawn for a more general case, that is, when $\beta=2\alpha$.

Our next intention is to derive a complex analogue of Theorem 7. We will again use invariance of a sample variance with respect to the translation. Unfortunately, we will not be able to extend Theorem 14 since condition (16) is difficult to verify. Hence, we will restrict ourselves to complex n-tuples with both non-negative real and imaginary parts. More precisely, we aim now to derive another complex version of the Hanin inequality which is independent of condition (16).

For a complex n-tuple $\mathbf{a}=(a_1,a_2,\ldots,a_n)$, we define $M_{\mathrm{Re}}=\max_{1\le j\le n}\mathrm{Re}(a_j)$, $M_{\mathrm{Im}}=\max_{1\le j\le n}\mathrm{Im}(a_j)$, and $M^+=M_{\mathrm{Re}}+iM_{\mathrm{Im}}$. Then holds the following version of the complex Hanin inequality.

Theorem 17. *Let $\mathbf{a}=(a_1,a_2,\ldots,a_n)$ be a complex n-tuple such that $\mathrm{Re}(a_j)\ge 0$, $\mathrm{Im}(a_j)\ge 0$, $j=1,2,\ldots,n$. If $0<\beta\le 2\alpha$, then holds the inequality*

$$|M^+|^2+\alpha^2|A_n(\mathbf{a})|^2\ge \beta Q_n^2(|\mathbf{a}|). \tag{18}$$

The equality in (18) holds if either $M^+=0$ or $\alpha=\frac{n}{k}$, $\beta=\frac{2n}{k}$, $1\le k\le n$, and $\mathbf{a}$ is an n-tuple whose components consist of exactly k real and imaginary parts equal to M_{Re} and M_{Im} respectively, $1\le k\le n$, while the remaining real and imaginary parts are equal to zero.

Proof. Let $x_j = \mathrm{Re}(a_j)$, $y_j = \mathrm{Im}(a_j)$. Then, we have that

$$\begin{aligned}
&|M^+|^2 + \alpha^2 |A_n(\mathbf{a})|^2 - \beta Q_n^2(|\mathbf{a}|)\\
&= |M^+ - \alpha A_n(\mathbf{a})|^2 + 2\alpha \mathrm{Re}\big(M^+ \overline{A_n(\mathbf{a})}\big) - \beta Q_n^2(|\mathbf{a}|)\\
&= |M^+ - \alpha A_n(\mathbf{a})|^2 + \frac{2\alpha}{n}\big(M_{\mathrm{Re}}\big(\sum_{j=1}^n x_j\big) + M_{\mathrm{Im}}\big(\sum_{j=1}^n y_j\big)\big) - \frac{\beta}{n}\sum_{j=1}^n \rho_j^2\\
&\geq |M^+ - \alpha A_n(\mathbf{a})|^2 + \frac{2\alpha}{n}\big(\sum_{j=1}^n x_j^2 + \sum_{j=1}^n y_j^2\big) - \frac{\beta}{n}\sum_{j=1}^n \rho_j^2\\
&= |M^+ - \alpha A_n(\mathbf{a})|^2 + \frac{2\alpha - \beta}{n}\sum_{j=1}^n \rho_j^2 \geq 0,
\end{aligned}$$

so (18) holds.

It remains to consider equality conditions in (18). Clearly, if $M^+ = 0$ equality holds trivially. Otherwise, the first inequality sign in the above relation implies that x_j is equal to either zero or M_{Re}, and y_j is equal to either zero or M_{Im}, for every $j = 1, 2, \ldots, n$. Further, the second inequality sign implies that n-tuple $\mathbf{a}$ consists of entries with exactly k real and imaginary parts equal to M_{Re} and M_{Im}, $1 \leq k \leq n$, while the remaining real and imaginary parts are equal to zero. In addition, it follows that $2\alpha = \beta$ and so $\alpha = \frac{n}{k}$, $\beta = \frac{2n}{k}$. This completes the proof. □

The previous theorem can be adapted for the case of complex numbers from the third quadrant, that is, with both negative real and imaginary parts. For that sake, we define $m_{\mathrm{Re}} = \min_{1\leq j\leq n} \mathrm{Re}(a_j)$, $m_{\mathrm{Im}} = \min_{1\leq j\leq n} \mathrm{Im}(a_j)$, and $m^- = m_{\mathrm{Re}} + i m_{\mathrm{Im}}$.

Corollary 18. *Let $\mathbf{a} = (a_1, a_2, \ldots, a_n)$ be a complex n-tuple such that $\mathrm{Re}(a_j) \leq 0$, $\mathrm{Im}(a_j) \leq 0$, $j = 1, 2, \ldots, n$. If $0 < \beta \leq 2\alpha$, then holds the inequality*

$$|m^-|^2 + \alpha^2 |A_n(\mathbf{a})|^2 \geq \beta Q_n^2(|\mathbf{a}|). \tag{19}$$

The equality in (19) holds if either $m^- = 0$ or $\alpha = \frac{n}{k}$, $\beta = \frac{2n}{k}$, $1 \leq k \leq n$, and $\mathbf{a}$ is an n-tuple whose components consist of exactly k real and imaginary parts equal to m_{Re} and m_{Im} respectively, $1 \leq k \leq n$, while the remaining real and imaginary parts are equal to zero.

Proof. It follows by putting the n-tuple $-\mathbf{a} = (-a_1, -a_2, \ldots, -a_n)$ in (17) instead of $\mathbf{a}$. □

Remark 19. If $\mathbf{a} = (a_1, a_2, \ldots, a_n)$ is a non-negative real n-tuple then both Theorems 14 and 17 coincide with Theorem 1. In other words, both relations (17) and (18) reduce to (4).

Now, similarly to Theorem 7, we give strengthened form of Theorem 17. In order to establish the corresponding proof, we will once again use the invariance of the sample variance with respect to the translation, of course adjusted to the complex setting.

Theorem 20. *Let* $\mathbf{a} = (a_1, a_2, \ldots, a_n)$ *be a complex* n*-tuple such that* $\mathrm{Re}(a_j) \geq 0$, $\mathrm{Im}(a_j) \geq 0$, $j = 1, 2, \ldots, n$. *If* $0 < \beta \leq 2\alpha$, *then holds the inequality*

$$\frac{|M^+ - m_{\mathrm{Re}}|^2}{\beta} + \left(\frac{\alpha^2}{\beta} - 1\right) |A_n(\mathbf{a}) - m_{\mathrm{Re}}|^2 + |A_n(\mathbf{a})|^2 \geq Q_n^2(|\mathbf{a}|). \tag{20}$$

The equality in (20) *holds under the same conditions as in the statement of Theorem* 17.

Proof. We follow the procedure as in the proof of Theorem 7. Let $(S_n^*)^2(\mathbf{a})$ stands for a modified sample variance, that is, $(S_n^*)^2(\mathbf{a}) = Q_n^2(|\mathbf{a}|) - |A_n(\mathbf{a})|^2$, defined for a complex n-tuple $\mathbf{a}$. We have that

$$\begin{aligned}
(S_n^*)^2(\mathbf{a}) &= \frac{1}{n}\sum_{j=1}^{n} |a_j - t|^2 - |A_n(\mathbf{a} - t)|^2 \\
&= \frac{1}{n}\sum_{j=1}^{n} \big(|a_j|^2 - 2t\mathrm{Re}(a_j) + t^2\big) - \big(|A_n(\mathbf{a})|^2 - 2t\mathrm{Re}(A_n(\mathbf{a})) + t^2\big) \\
&= Q_n^2(|\mathbf{a}|) - |A_n(\mathbf{a})|^2,
\end{aligned}$$

which means that $(S_n^*)^2(\mathbf{a})$ is also invariant with respect to the translation $\mathbf{a} \mapsto (a_1 - t, \ldots, a_n - t)$. Rewriting (18) with $(S_n^*)^2(\mathbf{a})$ instead of $Q_n^2(|\mathbf{a}|)$, we arrive at the inequality

$$|M^+|^2 + (\alpha^2 - \beta)|A_n(\mathbf{a})|^2 \geq \beta (S_n^*)^2(\mathbf{a}). \tag{21}$$

Similarly to the proof of Theorem 7, we aim now to minimize the left-hand side of (21), where we will use invariance of $(S_n^*)^2(\mathbf{a})$ with respect to the translation. Hence, consider the quadratic function

$$h(t) = |M^+ - t|^2 + (\alpha^2 - \beta)|A_n(\mathbf{a} - t)|^2 = |M^+ - t|^2 + (\alpha^2 - \beta)|A_n(\mathbf{a}) - t|^2.$$

By a straightforward computation, it follows that h is a convex function with minimum value at the point

$$t_0 = \frac{\mathrm{Re}\big(M^+ + (\alpha^2 - \beta)A_n(\mathbf{a})\big)}{\alpha^2 - \beta + 1} = \mathrm{Re}\big(A_n(\mathbf{a})\big) + \frac{\mathrm{Re}\big(M^+ - A_n(\mathbf{a})\big)}{\alpha^2 - \beta + 1}.$$

On the other hand, the inequality

$$|M^+ - t|^2 + (\alpha^2 - \beta)|A_n(\mathbf{a}) - t|^2 \geq \beta (S_n^*)^2(\mathbf{a}) \tag{22}$$

holds for a complex n-tuple $(a_1 - t, a_2 - t, \ldots, a_n - t)$, that is, when $t \leq m_{\mathrm{Re}}$. Now, since $m_{\mathrm{Re}} \leq \mathrm{Re}\big(A_n(\mathbf{a})\big) \leq M_{\mathrm{Re}}$ and $\alpha^2 - \beta + 1 \geq 0$, it follows that $t_0 \geq m_{\mathrm{Re}}$, so the n-tuple $(a_1 - t_0, a_2 - t_0, \dots, a_n - t_0)$ has at least one coordinate with negative real part. In other words, inequality (22) is not valid at the point t_0, in general. Of course, relation (22) holds for all $t \le m_{\mathrm{Re}}$. Furthermore, since $h(t)$ is decreasing function on the interval $(-\infty, m]$, the most precise form of (22) appears when $t = m_{\mathrm{Re}}$. This provides (20), as claimed. □

Remark 21. If $a = (a_1, a_2, \dots, a_n)$ is a non-negative real n-tuple, then Theorem 20 reduces to Theorem 7, i.e. inequality (20) becomes (8).

Remark 22. For the simplicity, the results in this section are given in a non-weighted forms. According to Section 3 they can be easily modified to hold in the non-weighted setting. Here, they are omitted.

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