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REFINEMENTS OF SOME INEQUALITIES INVOLVING MEANS

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The aim of this work is to give some sharp refinements of some inequalities involving the arithmetic, geometric and harmonic means. We use the Equal Variable Theorem (EV Theorem). Our method can be applied to a wide class of inequalities.

1. INTRODUCTION

The aim of this work is to improve some inequalities involving the arithmetic, geometric and harmonic means. To be concrete, we prove, then refine the following inequalities:

$$\sqrt[n]{\left(\prod_{i=1}^{n} a_i\right)^2} \leq \frac{1}{\binom{n}{2}} \sum_{i<j} a_i a_j$$

and

$$\left(\frac{n}{\frac{1}{a_1}+...+\frac{1}{a_n}}\right)^2 \leq \frac{1}{\binom{n}{2}} \sum_{i<j} a_i a_j .$$

More precisely, we find the largest constants C_n, R_n for which

$$\sqrt[n]{\left(\prod_{i=1}^{n} a_i\right)^2} + C_n \left(a_{n-1}-a_n\right)^2 \leq \frac{1}{\binom{n}{2}} \sum_{i<j} a_i a_j ,$$

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and

$$\left(\frac{n}{\frac{1}{a_1}+...+\frac{1}{a_n}}\right)^2+R_n\left(a_{n-1}-a_n\right)^2\leq\frac{1}{\binom{n}{2}}\sum_{i<j}a_ia_j$$

for every $a_1\geq a_2\geq...\geq a_n>0$, namely

$$C_n=\frac{n-2}{n^2}\ ,\quad R_n=\frac{2n-3}{n^2}.$$

In the sequel, we need the following results:

Proposition 1 ([**1**]). *Let $n\geq 3$, let the fixed sums $S_1=a_1+a_2+...+a_{n-1}$ and $S_2=a_1^2+a_2^2+...+a_{n-1}^2$, where $a_1\geq a_2\geq...\geq a_{n-1}>0$. Then the number a_{n-1} has the maximum value for $a_1\geq a_2=...=a_{n-1}$.*

Proposition 2 ([**1**, Cor. 1.5]). *Let $n\geq 3$, let the fixed sums $S_1=a_1+a_2+...+a_n$ and $S_2=a_1^2+a_2^2+...+a_n^2$, where $a_1\geq a_2\geq...\geq a_n\geq 0$ and let f be a differentiable function on $(0,\infty)$ such that f' is strictly convex on $(0,\infty)$. Moreover, either $f(x)$ is continuous at $x=0$ or $\lim_{x\to 0}f(x)=-\infty$. Then,*

$$F_n=f(a_1)+f(a_2)+...+f(a_n)$$

has the maximum value for $a_1\geq a_2=...=a_n\geq 0$ and the minimum value for either $a_n=0$ or $a_1=a_2=...=a_{n-1}\geq a_n>0$.

The above result is also known as Equal Variable Theorem (EV Theorem).

Proposition 3 ([**1**, Cor. 1.8]). *Let $n\geq 3$, let the fixed sums $S_1=a_1+a_2+...+a_n$ and $S_2=a_1^2+a_2^2+...+a_n^2$, where $a_1\geq a_2\geq...\geq a_n\geq 0$. Then the product $P=a_1a_2...a_n$ has the maximum value for $a_1\geq a_2=...=a_n\geq 0$ and the minimum value for either $a_n=0$ or $a_1=a_2=...=a_{n-1}\geq a_n>0$.*

Proposition 4 ([**1**, Cor. 1.9]). *Let $n\geq 3$, let the fixed sums $S_1=a_1+a_2+...+a_n$ and $S_2=a_1^2+a_2^2+...+a_n^2$, where $a_1\geq a_2\geq...\geq a_n\geq 0$. Then the sum*

$$S=\frac{1}{a_1}+\frac{1}{a_2}+...+\frac{1}{a_n}$$

has the minimum value for $a_1\geq a_2=...=a_n\geq 0$.

2. SHARP CONSTANT IN GM-AM INEQUALITY

Let a_1, ..., $a_n>0$. The $AM-GM$ inequality applied to $\binom{n}{2}$ terms a_ia_j $(1\leq i<j\leq n)$ gives:

$$\frac{1}{\binom{n}{2}}\sum_{i<j}a_ia_j \geq \sqrt[\binom{n}{2}]{\prod_{i<j}a_ia_j} = \sqrt[\binom{n}{2}]{\left(\prod_{i=1}^{n}a_i\right)^{n-1}} = \sqrt[n]{\left(\prod_{i=1}^{n}a_i\right)^{2}},$$

or

$$\sqrt[n]{\left(\prod_{i=1}^{n}a_i\right)^{2}} \leq \frac{1}{\binom{n}{2}}\sum_{i<j}a_ia_j. \tag{1}$$

Remark that (1) is stated in an interesting form, as the right-hand side of (1) is the arithmetic mean of $\binom{n}{2}$ terms (product of any two a_i's), while the left-hand side is the square of the geometric mean of the n given numbers a_i).

We refine (1) in the next form:

Theorem 5. *Let $n \geq 4$. Then the following inequality holds true,*

$$\sqrt[n]{\left(\prod_{i=1}^{n}a_i\right)^{2}} + C_n\left(a_{n-1}-a_n\right)^2 \leq \frac{1}{\binom{n}{2}}\sum_{i<j}a_ia_j, \tag{2}$$

for every $a_1 \geq a_2 \geq ... \geq a_n > 0$, where the constant

$$C_n = \frac{n-2}{n^2}$$

is the best possible.

Proof. By multiplying by $n(n-1)$, we rewrite (2) in the equivalent form:

$$n(n-1)\sqrt[n]{\left(\prod_{i=1}^{n}a_i\right)^{2}} + K_n\left(a_{n-1}-a_n\right)^2 \leq 2\sum_{i<j}a_ia_j \tag{3}$$

and

$$K_n = n(n-1)C_n = \frac{(n-1)(n-2)}{n}.$$

In order to find the best possible constant K_n satisfying (3), note that $K_n \geq 0$, since (3) holds true for $K_n = 0$.

As (3) is homogeneous, we can assume that $a_n = 1$, then $a_1 > 1$ (the case $a_n = 1$ and $a_1 = 1$ is trivial). We fix

$$\sum_{i=1}^{n-1}a_i = S > n-1 \quad \text{and} \quad \sum_{i=1}^{n-1}a_i^2 = Q.$$

According to the Lemma 2, there exist $y \geq x \geq 1$ such that

$$S = y + (n-2)x \quad \text{and} \quad Q = y^2 + (n-2)x^2.$$

Moreover, by using the EV Theorem, we obtain:

$$\max \prod_{i=1}^{n} a_i^2 \leq x^{2n-4} y^2.$$

Now we have to deal with the following form of (3):

$$\begin{aligned} n(n-1)\sqrt[n]{x^{2n-4}y^2} + K_n(x-1)^2 &\leq (n-2)(n-3)x^2 \\ &\quad +2(n-2)xy + 2(n-2)x + 2y, \end{aligned}$$

with $y \geq x$. Further, with the notations $u = \sqrt[n]{x}$, $v = \sqrt[n]{y}$, the inequality becomes:

$$\begin{aligned} & (n-2)(n-3)u^{2n} + 2(n-2)u^n v^n \\ & +2(n-2)u^n + 2v^n \\ -n(n-1)u^{2n-4}v^2 \quad \geq \quad & K_n(u^n-1)^2, \end{aligned} \tag{4}$$

with $v \geq u$. Let us consider the function $f : [v, \infty) \to \mathbb{R}$,

$$f(v) = (n-2)(n-3)u^{2n} + 2(n-2)u + 2(n-2)u^n + 2v^n - n(n-1)u^{2n-4}v^2.$$

We have $f'(v) = 2nvg(v)$, where

$$g(v) = (n-2)u^n v^{n-2} + v^{n-2} - (n-1)u^{2n-4}.$$

The function g is strictly increasing with respect to v, so

$$g(v) \geq g(u) = u^{n-2}h(u),$$

where $h(u) = (n-2)u^n - (n-1)u^{n-2} + 1$. As $u \geq 1$, to get

$$\begin{aligned} h(u) &\geq (n-2)u^{n-1} - (n-1)u^{n-2} + 1 \\ &= (u-1)^2\left(1 + 2u + \dots + (n-2)u^{n-3}\right) \\ &\geq 0. \end{aligned}$$

In consequence, $g \geq 0$, then $f' \geq 0$, so f is strictly increasing on (u, ∞).

The inequality (4) reduces to

$$f(u) \geq K_n(u^n-1)^2. \tag{5}$$

This inequality (5) is true if $u = 1$, for every K_n. If $u > 1$, we rewrite (5) as

$$K_n \leq \frac{f(u)}{(u^n-1)^2}.$$

By taking the limit, we obtain:

$$K_n \leq \lim_{u \searrow 1} \frac{f(u)}{(u^n-1)^2} = \frac{(n-1)(n-2)}{n}$$

(the last limit follows easily by using l'Hospital rule twice).

Now it remains to prove that

$$(n-1)\cdot\frac{(n-2)\,u^{2n}+2u^{n}-nu^{2n-2}}{\left(u^{n}-1\right)^{2}}>\frac{(n-1)\,(n-2)}{n},$$

or $\phi\,(u)>0$, for $u>1$, where

$$\phi\,(u)=n\left[(n-2)\,u^{2n}+2u^{n}-nu^{2n-2}\right]-(n-2)\left(u^{n}-1\right)^{2}.$$

We have $\phi'\,(u)=2n\,(n-1)\,u^{n-1}\left[(n-2)\,u^{n}-nu^{n-2}+2\right]>0$, for $u>1$, since

$$\begin{aligned}\frac{(n-2)\,u^{n}-nu^{n-2}+2}{\left(u-1\right)^{2}} &= 2+4u+6u^{2}+...+(2n-4)u^{n-3}+(n-2)\,u^{n-2}\\ &> 0,\end{aligned}$$

so $\phi\,(u)$ is strictly increasing on $(1,\infty)$. Thus $\phi\,(u)\geq\phi\,(1)=0$, then

$$\inf_{u>1}\left\{(n-1)\cdot\frac{(n-2)\,u^{n}-nu^{n-2}+2}{\left(u^{n}-1\right)^{2}}\right\}=\frac{(n-1)\,(n-2)}{n}.$$

Finally, the largest value K_n is

$$K_n=\frac{(n-1)\,(n-2)}{n}.$$

□

3. SHARP CONSTANT IN HM-AM INEQUALITY

The following inequality holds true, for every a_1, ..., $a_n>0$:

(6) $$\left(\frac{n}{\frac{1}{a_1}+...+\frac{1}{a_n}}\right)^{2}\leq\frac{1}{\binom{n}{2}}\sum_{i<j}a_ia_j.$$

This is weaker than (1), by $HM-GM$ inequality.

Remark that in (6), the right-hand side is the arithmetic mean of $\binom{n}{2}$ terms (product of any two a_i's), while the left-hand side is the square of the harmonic mean of the given numbers a_i.

The result of this section is the following

Theorem 6. *Let $n \geq 4$. Then the following inequality holds true,*

$$\left(\frac{n}{\frac{1}{a_1}+...+\frac{1}{a_n}}\right)^2 + R_n\left(a_{n-1}-a_n\right)^2 \leq \frac{1}{\binom{n}{2}}\sum_{i<j} a_i a_j, \tag{7}$$

for every $a_1 \geq a_2 \geq ... \geq a_n > 0$, where the constant

$$R_n = \frac{2n-3}{n^2}$$

is the best possible.

Proof. By multiplying by $n(n-1)$, we rewrite (7) in the equivalent form:

$$n(n-1)\left(\frac{n}{\frac{1}{a_1}+...+\frac{1}{a_n}}\right)^2 + S_n\left(a_{n-1}-a_n\right)^2 \leq 2\sum_{i<j} a_i a_j \tag{8}$$

and

$$S_n = n(n-1)R_n = \frac{(n-1)(2n-3)}{n}.$$

In order to find the best possible constant S_n satisfying (8), note that $S_n \geq 0$, since (8) holds true for $S_n = 0$.

As (8) is homogeneous, we can assume that $a_n = 1$, then $a_1 > 1$ (the case $a_n = 1$ and $a_1 = 1$ is trivial). We fix

$$\sum_{i=1}^{n-1} a_i = S > n-1 \quad \text{and} \quad \sum_{i=1}^{n-1} a_i^2 = Q.$$

According to the Lemma 1, there exist $y \geq x \geq 1$ such that

$$S = y + (n-2)x \quad \text{and} \quad Q = y^2 + (n-2)x^2.$$

Further, $a_{n-1} \leq x$. By using the EV Theorem for the function $t \mapsto 1/t$ on $(0,\infty)$, the sum

$$\frac{1}{a_1} + \frac{1}{a_2} + ... + \frac{1}{a_{n-1}}$$

attains its minimum value for $a_1 = y$ and $a_2 = ... = a_{n-1} = x$. As a consequence, (8) can be reduced to the study of the following

$$\begin{aligned} n(n-1)\left(\frac{n}{\frac{1}{y}+\frac{n-2}{x}+1}\right)^2 + S_n(x-1)^2 \quad &\leq \quad (n-2)(n-3)x^2 \\ &+2(n-2)xy + 2(n-2)x + 2y, \end{aligned}$$

with $y \geq x$. Let us consider the function

$$\begin{aligned} f(y) \quad &= \quad (n-2)(n-3)x^2 + 2(n-2)xy + 2(n-2)x + 2y \\ &-n(n-1)\left(\frac{n}{\frac{1}{y}+\frac{n-2}{x}+1}\right)^2. \end{aligned}$$

We have
$$\frac{1}{2}f'(y)=(n-2)\,x+1-(n-1)\,y\left(\frac{n}{y+\frac{n-2}{x}\cdot y+1}\right)^{3}.$$
We study the sign of f' by replacing y by z^3. We define
$$g(z)=\frac{z^3+\frac{n-2}{x}\cdot z^3+1}{z},$$
with $z\geq\sqrt[3]{x}$. Then
$$g'(z)=2z+\frac{n-2}{x}\cdot 2z-\frac{1}{z^2}>0,\quad z\geq\sqrt[3]{x}.$$
Thus g is strictly increasing together to f'. It follows that $f'(y)\geq f'(x)=2h(x)$, where
$$h(x)=(n-2)\,x+1-(n-1)\,x\left(\frac{n}{x+n-1}\right)^{3},\qquad x\geq 1.$$
We have
$$h'(x)=(n-2)\left(1-\left(\frac{n}{x+n-1}\right)^{3}\right)+\left(\frac{n}{x+n-1}\right)^{3}\cdot\frac{(3n-4)\,x-(n-1)}{x+n-1},$$
so $h'>0$, since
$$1-\left(\frac{n}{x+n-1}\right)^{3}>0\ \text{ and }\ \frac{(3n-4)\,x-(n-1)}{x+n-1}>0.$$
Now,
$$f'(y)\geq f'(x)=2h(x)\geq 2h(1)=0,\quad y\geq x.$$
Furthermore, f is increasing,
$$f(y)\geq f(x)=(n-1)(n-2)\,x^2+2(n-1)\,x-n(n-1)\left(\frac{nx}{x+n-1}\right)^{2}.$$
If we take $a_1=...=a_{n-1}=x$ and $a_n=1$ in (8) we get
$$S_n(x-1)^2\leq f(x),$$
or $S_n\leq P(x)$, where
$$P(x)=\frac{(n-1)\,x\left((n-2)\,x+2(n-1)^2\right)}{(x+n-1)^2},\quad x>1.$$
The function P is strictly increasing on $(1,(n-1)^2]$ and strictly decreasing on $[(n-1)^2,\infty)$, since
$$P'(x)=\frac{2(n-1)^2\left((n-1)^2-x\right)}{(x+n-1)^3}.$$

Thus

$$\begin{aligned}\inf_{x>1} P(x) &= \min\{P(1+0),\ P(\infty)\} \\ &= \min\left\{\frac{(n-1)(2n-3)}{n},\ (n-2)(n-1)\right\} \\ &= \frac{(n-1)(2n-3)}{n}.\end{aligned}$$

In conclusion, the largest constant is

$$S_n = \frac{(n-1)(2n-3)}{n}.$$

□

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