Applicable Analysis and Discrete Mathematics
available online at http://pefmath.etf.rs

Appl. Anal. Discrete Math. **18** (2024), 551–571.
https://doi.org/10.2298/AADM221203013W

SHARP BOUNDS ON HANKEL DETERMINANTS OF BOUNDED TURNING FUNCTIONS INVOLVING THE HYPERBOLIC TANGENT FUNCTION

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In this paper, we determine the sharp upper bounds of Hankel determinants for logarithmic and inverse functions of bounded turning functions associated with the hyperbolic tangent function.

1. INTRODUCTION

Let $\mathcal{H}(\mathbb{D})$ denote the class of analytic functions in $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$. Suppose that $\mathcal{A}$ is the normalized subclass of $\mathcal{H}(\mathbb{D})$:

$$\mathcal{A} := \left\{ f \in \mathcal{H}(\mathbb{D}) : f(z) = z + \sum_{n=2}^{\infty} a_n z^n \right\}. \tag{1}$$

The set $\mathcal{S}$ of all normalized univalent functions in $\mathbb{D}$ is contained in the set $\mathcal{A}$.

The coefficient hypothesis presented by Bieberbach [**6**] in 1916 led to the field's emergence as a promising area of new research. This conjecture was proved by de Branges [**10**] in 1985. Between 1916 and 1985, many scholars of the day sought to confirm or disprove this conjecture. As a consequence, they discovered numerous subfamilies of the class $\mathcal{S}$ of univalent functions connected to distinct image domains. The families of starlike and convex functions, respectively, denoted

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2020 Mathematics Subject Classification. 30C45, 30C50.
Keywords and Phrases. Bounded turning function, hyperbolic tangent function, Hankel determinant.

by $\mathcal{S}^*$ and $\mathcal{K}$, are the most fundamental and significant subclasses of the set $\mathcal{S}$. In 1992, Ma-Minda [**25**] considered the general form of the family $\mathcal{S}^*$ as follows:

$$\mathcal{S}^*(\phi) := \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec \phi(z) \right\},$$

where ϕ is a holomorphic function with $\phi'(0) > 0$ and has positive real part. Also, the function ϕ maps $\mathbb{D}$ onto a star-shaped region with respect to $\phi(0) = 1$ and is symmetric with the real axis. They addressed some specific results such as distortion, growth, and covering theorems. In recent years, several subfamilies of the set $\mathcal{A}$ were studied as special cases of the class $\mathcal{S}^*(\phi)$:

(i) If we choose $\phi(z) = \frac{1+Az}{1+Bz}$ with $-1 \leq B < A \leq 1$, then we get the class $\mathcal{S}^*(A,B) := \mathcal{S}^*\left(\frac{1+Az}{1+Bz}\right)$, which was investigated in [**12**]. Furthermore, $\mathcal{S}^*(\alpha) := \mathcal{S}^*(1-2\alpha,-1)$ is the familiar starlike functions of order α $(0 \leq \alpha < 1)$.

(ii) The family $\mathcal{S}^*_{\mathcal{L}} := \mathcal{S}^*(\phi(z))$ with $\phi(z) = \sqrt{1+z}$ was discussed by Sokól-Stankiewicz [**38**]. The function $\phi(z) = \sqrt{1+z}$ maps the region $\mathbb{D}$ onto the image domain which is bounded by $|w^2 - 1| < 1$.

(iii) The class $\mathcal{S}^*_{car} := \mathcal{S}^*(\phi(z))$ with

$$\phi(z) = 1 + \frac{4}{3}z + \frac{2}{3}z^2$$

was examined by Sharma-Jain-Ravichandran [**34**], which consists of functions $f \in \mathcal{A}$ in such a manner that $\frac{zf'(z)}{f(z)}$ is located in the region bounded by the cardioid given by

$$\left(9x^2 + 9y^2 - 18x + 5\right)^2 - 16\left(9x^2 + 9y^2 - 6x + 1\right) = 0.$$

(iv) By selecting $\phi(z) = 1 + \sin z$, the class $\mathcal{S}^*(\phi(z))$ leads to the family $\mathcal{S}^*_{\sin}$ which was explored in [**9**] (see also [**49**]), while $\mathcal{S}^*_e := \mathcal{S}^*(e^z)$ was introduced in [**26**], and later studied in [**35**]. This class was recently generalized by [**44**], the authors determined upper bound of the third-order Hankel determinant.

(v) The families $\mathcal{S}^*_{\cos} := \mathcal{S}^*(\cos(z))$ and $\mathcal{S}^*_{\cosh} := \mathcal{S}^*(\cosh(z))$ were investigated, respectively, by Raza-Bano [**4**], and Alotaibi-Arif-Alghamdi-Hussain [**1**].

(vi) For $\phi(z) = 1 + \sinh^{-1}(z)$, we get the family $\mathcal{S}^*_\rho := \mathcal{S}^*\left(1 + \sinh^{-1}(z)\right)$ which was introduced by Arora-Kumar [**2**] (see also [**48**]). They discussed several relationships between this class and the known function classes.

(vii) The class $\mathcal{S}^*(1+\tanh(z))$ of starlike functions associated with tangent hyperbolic function was discussed in [**45, 46**].

In 2021, Barukab-Arif-Abbas-Khan [**5**] investigated sharp Hankel determinant of third-order for the following function class:

$$\mathcal{R}_s = \left\{ f \in \mathcal{A} : f'(z) \prec 1 + \sinh^{-1}(z) \quad (z \in \mathbb{D}) \right\}.$$

In the present paper, we consider a trigonometric function $\varphi_1(z) = 1+\tanh(z)$ with $\varphi_1(0) = 1$. Also, one can easily seen that $\Re(\varphi_1(z)) > 0$. By using this function, a subclass of analytic functions can be defined as follows:

$$\mathcal{BT}_{\tanh} := \{ f \in \mathcal{S} : f'(z) \prec 1 + \tanh(z) \quad (z \in \mathbb{D}) \}. \tag{2}$$

In other words, a function $f \in \mathcal{BT}_{\tanh}$ if and only if there exists a holomorphic function q, fulfilling $q(z) \prec 1 + \tanh(z)$, such that

$$f(z) = \int_0^z q(t)dt.$$

The Hankel determinant $\mathcal{H}_{q,n}(f)$ $(q, n \in \mathbb{N} := \{1, 2, 3, \ldots\})$ for a function $f \in \mathcal{S}$ of the series form (1) was given by Pommerenke [**28, 29**] as follows:

$$\mathcal{H}_{q,n}(f) := \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q-2} \end{vmatrix}.$$

In particular, the following determinants are known as the first-order, second-order and third-order Hankel determinants of f, respectively,

$$\mathcal{H}_{2,1}(f) = \begin{vmatrix} 1 & a_2 \\ a_2 & a_3 \end{vmatrix} = a_3 - a_2^2,$$

$$\mathcal{H}_{2,2}(f) = \begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix} = a_2 a_4 - a_3^2,$$

and

$$\mathcal{H}_{3,1}(f) = \begin{vmatrix} 1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \\ a_3 & a_4 & a_5 \end{vmatrix} = a_3\left(a_2 a_4 - a_3^2\right) - a_4\left(a_4 - a_2 a_3\right) + a_5\left(a_3 - a_2^2\right).$$

There are comparatively few observations in literature in relation to the Hankel determinant for the function f belongs to the general family $\mathcal{S}$. For the function $f \in \mathcal{S}$, the best established sharp inequality is $|\mathcal{H}_{2,n}(f)| \leq \lambda\sqrt{n}$, where λ is an absolute constant, which is due to Hayman [**11**]. Furthermore, for the same class $\mathcal{S}$, it was obtained in [**27**] that

$$|\mathcal{H}_{2,2}(f)| \leq \lambda \quad \left(1 \leq \lambda \leq \frac{11}{3}\right),$$

and

$$|\mathcal{H}_{3,1}(f)| \leq \mu \quad \left(\frac{4}{9} \leq \mu \leq \frac{32+\sqrt{285}}{15}\right).$$

For more recent developments on Hankel determinants of analytic functions, we refer the reader to [**3, 7, 8, 13, 14, 18, 21, 22, 23, 31, 36, 39, 40, 41, 42, 43, 48, 50, 51, 52**].

The logarithmic power series of f is defined by

$$F_f(z) := \log\left(\frac{f(z)}{z}\right) = 2\sum_{m=1}^{\infty} \gamma_m z^m.$$

For $f \in \mathcal{S}$, the first four logarithmic coefficients are given by

$$\gamma_1 = \frac{1}{2}a_2, \tag{3}$$

$$\gamma_2 = \frac{1}{2}\left(a_3 - a_2^2\right), \tag{4}$$

$$\gamma_3 = \frac{1}{2}\left(a_4 - a_2a_3 + \frac{1}{3}a_2^3\right), \tag{5}$$

and

$$\gamma_4 = \frac{1}{2}\left(a_5 - a_2a_4 + a_2^2a_3 - \frac{1}{2}a_3^2 - \frac{1}{4}a_2^4\right). \tag{6}$$

Recently, Kowalczyk-Lecko [**16, 17**] proposed the study of Hankel determinant $\mathcal{H}_{q,n}(F_f/2)$, whose elements are logarithmic coefficients of f, i.e.,

$$\mathcal{H}_{q,n}(F_f/2) := \begin{vmatrix} \gamma_n & \gamma_{n+1} & \cdots & \gamma_{n+q-1} \\ \gamma_{n+1} & \gamma_{n+2} & \cdots & \gamma_{n+q} \\ \vdots & \vdots & \vdots & \vdots \\ \gamma_{n+q-1} & \gamma_{n+q} & \cdots & \gamma_{n+2q-2} \end{vmatrix}.$$

The expressions of $\mathcal{H}_{q,n}(F_f/2)$ are given as follows:

$$\mathcal{H}_{2,1}(F_f/2) = \gamma_1\gamma_3 - \gamma_2^2,$$

and

$$\mathcal{H}_{2,2}(F_f/2) = \gamma_2\gamma_4 - \gamma_3^2.$$

We observe that $\mathcal{H}_{2,1}(F_f/2)$ is just corresponding to the well-known functional

$$\mathcal{H}_{2,1}(f) = a_3 - a_2^2$$

over the class $\mathcal{S}$ or its subclasses.

By noting that study on the inverse functions for the functions in different subclasses of $\mathcal{S}$ is also an important topic. For every univalent function f defined in $\mathbb{D}$, the famous Koebe one-quarter covering theorem guaranties that there exists the inverse f^{-1} at least on the disk with radius $1/4$. Assume that

$$f^{-1}(w) = w + A_2 w^2 + A_3 w^3 + \cdots, \tag{7}$$

we introduce the Hankel determinant of f^{-1} which is given by

$$\mathcal{H}_{q,n}\left(f^{-1}\right) := \begin{vmatrix} A_n & A_{n+1} & \dots & A_{n+q-1} \\ A_{n+1} & A_{n+2} & \dots & A_{n+q} \\ \vdots & \vdots & \vdots & \vdots \\ A_{n+q-1} & A_{n+q} & \dots & A_{n+2q-2} \end{vmatrix}.$$

In particular, the following determinants are defined, respectively, as the second-order and third-order Hankel determinants of f^{-1}:

$$\mathcal{H}_{2,2}\left(f^{-1}\right) = \begin{vmatrix} A_2 & A_3 \\ A_3 & A_4 \end{vmatrix} = A_2 A_4 - A_3^2,$$

and

$$\mathcal{H}_{3,1}\left(f^{-1}\right) = \begin{vmatrix} 1 & A_2 & A_3 \\ A_2 & A_3 & A_4 \\ A_3 & A_4 & A_5 \end{vmatrix} = A_3\left(A_2 A_4 - A_3^2\right) - A_4\left(A_4 - A_2 A_3\right) + A_5\left(A_3 - A_2^2\right).$$

We note that f^{-1} is not necessary to be univalent, this concept is also a natural generalization of Hankel determinant for $f \in \mathcal{S}$.

In view of (1) and (7), bearing in mind that $f\left(f^{-1}(w)\right) = w$, we see that

$$A_2 = -a_2, \tag{8}$$

$$A_3 = 2a_2^2 - a_3, \tag{9}$$

$$A_4 = 5a_2 a_3 - 5a_2^3 - a_4, \tag{10}$$

and

$$A_5 = 14a_2^4 + 3a_3^2 - 21a_2^2 a_3 + 6a_2 a_4 - a_5. \tag{11}$$

For recent study on the bounds of Hankel determinants for logarithmic and inverse functions of analytic functions, we refer the reader to [**19, 32, 33, 37, 47**].

We observe that several sharp coefficient functionals for the class $\mathcal{BT}_{\tanh}$ are derived in [**15**]. In this paper, our aim is to derive the sharp bounds of several coefficient problems for the class $\mathcal{BT}_{\tanh}$ of bounded turning functions. These problems include the bounds $|\mathcal{H}_{2,2}(F_f/2)|$, $\left|\mathcal{H}_{2,2}\left(f^{-1}\right)\right|$ and $\left|\mathcal{H}_{3,1}\left(f^{-1}\right)\right|$ for the class $\mathcal{BT}_{\tanh}$.

2. PRELIMINARY RESULTS

To prove our main results, we need the following definition and lemma.

Definition 1. A function $p \in \mathcal{P}$ if it has the series expansion

$$p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n \quad (z \in \mathbb{D})$$

along with $\Re(p(z)) > 0$.

Lemma 2. *Let $p \in \mathcal{P}$ with $c_1 \geq 0$. Then for x, δ, $\rho \in \overline{\mathbb{D}} := \{z \in \mathbb{C} : |z| \leq 1\}$, we have*

$$2c_2 = c_1^2 + \left(4 - c_1^2\right) x, \tag{12}$$

$$4c_3 = c_1^3 + 2c_1 x \left(4 - c_1^2\right) - x^2 c_1 \left(4 - c_1^2\right) + 2\left(1 - |x|^2\right)\left(4 - c_1^2\right)\delta, \tag{13}$$

and

$$\begin{aligned} 8c_4 =& c_1^4 + (4 - c_1^2)x\left[c_1^2\left(x^2 - 3x + 3\right) + 4x\right] \\ & - 4(4 - c_1^2)(1 - |x|^2)\left[c_1\delta(x - 1) + \overline{x}\delta^2 - \left(1 - |\delta|^2\right)\rho\right]. \end{aligned} \tag{14}$$

The formula for c_2 is given by [**30**], the formula c_3 is due to Libera-Zlotkiewicz [**24**], and the formula for c_4 was proved in [**20**].

3. HANKEL DETERMINANTS OF LOGARITHMIC FUNCTIONS FOR THE CLASS $\mathcal{BT}_{\text{TANH}}$

In this section, we discuss Hankel determinants of logarithmic functions for the class $\mathcal{BT}_{\tanh}$.

Theorem 3. *Let $f \in \mathcal{BT}_{\tanh}$. Then*

$$|\mathcal{H}_{2,2}(F_f/2)| \leq \frac{1}{64}.$$

The inequality is sharp.

Proof. Let $f \in \mathcal{BT}_{\tanh}$. Then, we find that (2) can be rewritten as

$$f'(z) = 1 + \tanh(\omega(z)),$$

where w is a schwarz function. If $p \in \mathcal{P}$, in term of the schwarz function w, we get

$$p(z) = \frac{1+w(z)}{1-w(z)} = 1 + c_1 z + c_2 z^2 + c_3 z^3 + c_4 z^4 + \cdots ,$$

or equivalently,

$$\begin{aligned} w(z) =& \frac{p(z)-1}{p(z)+1} \\ =& \frac{c_1 z + c_2 z^2 + c_3 z^3 + c_4 z^4 + \cdots}{2 + c_1 z + c_2 z^2 + c_3 z^3 + c_4 z^4 + \cdots} \\ =& \frac{1}{2}c_1 z + \left(\frac{1}{2}c_2 - \frac{1}{4}c_1^2\right) z^2 + \left(\frac{1}{8}c_1^3 - \frac{1}{2}c_1 c_2 + \frac{1}{2}c_3\right) z^3 \\ &+ \left(\frac{1}{2}c_4 - \frac{1}{2}c_1 c_3 - \frac{1}{4}c_2^2 - \frac{1}{16}c_1^4 + \frac{3}{8}c_1^2 c_2\right) z^4 + \cdots . \end{aligned} \tag{15}$$

Now, we find from (1) that

$$f'(z) = 1 + 2a_2 z + 3a_3 z^2 + 4a_4 z^3 + 5a_5 z^4 + \cdots . \tag{16}$$

By using the series expansion of (15), we get

$$\begin{aligned} 1 + \tanh(w(z)) =& 1 + \frac{1}{2}c_1 z + \left(\frac{1}{2}c_2 - \frac{1}{4}c_1^2\right) z^2 + \left(\frac{1}{12}c_1^3 - \frac{1}{2}c_1 c_2 + \frac{1}{2}c_3\right) z^3 \\ &+ \left(\frac{1}{2}c_4 + \frac{1}{4}c_1^2 c_2 - \frac{1}{2}c_1 c_3 - \frac{1}{4}c_2^2\right) z^4 + \cdots . \end{aligned} \tag{17}$$

In what follows, by comparing (16) with (17), we obtain

$$a_2 = \frac{1}{4}c_1, \tag{18}$$

$$a_3 = \frac{1}{6}c_2 - \frac{1}{12}c_1^2, \tag{19}$$

$$a_4 = \frac{1}{48}c_1^3 - \frac{1}{8}c_1 c_2 + \frac{1}{8}c_3, \tag{20}$$

and

$$a_5 = \frac{1}{10}c_4 - \frac{1}{20}c_2^2 - \frac{1}{10}c_1 c_3 + \frac{1}{20}c_1^2 c_2. \tag{21}$$

By substituting (18), (19), (20), (21) into (3), (4), (5) and (6), we have

$$\gamma_1 = \frac{1}{8}c_1,$$

(22)
$$\gamma_2 = \frac{1}{12}c_2 - \frac{11}{192}c_1^2,$$

(23)
$$\gamma_3 = \frac{3}{128}c_1^3 - \frac{1}{12}c_1c_2 + \frac{1}{16}c_3,$$

and

(24)
$$\gamma_4 = \frac{19}{360}c_2c_1^2 - \frac{137}{18432}c_1^4 - \frac{21}{320}c_1c_3 + \frac{1}{20}c_4 - \frac{23}{720}c_2^2.$$

We observe that $\mathcal{H}_{2,2}(F_f/2)$ can be rewritten as

$$\mathcal{H}_{2,2}(F_f/2) = \gamma_2\gamma_4 - \gamma_3^2.$$

By using (22), (23) and (24) with $c_1 = c$, we have

(25)
$$\begin{aligned}\mathcal{H}_{2,2}(F_f/2) =& \frac{1}{3538944}\Bigg(-437c^6 + \frac{4656}{5}c^4c_2 + \frac{14688}{5}c^3c_3 - \frac{12672}{5}c^2c_2^2 \\ &+ \frac{87552}{5}cc_2c_3 - \frac{47104}{5}c_2^3 + \frac{73728}{5}c_2c_4 - 13824c_3^2 - \frac{50688}{5}c^2c_4\Bigg).\end{aligned}$$

To simplify the computation, we assume that $r = 4 - c_1^2$ with $c_1 = c$ in (12), (13) and (14). By using the simplified forms of these formulae, we get

$$2c_2 = 4 - r + rx,$$

$$4c^3c_3 = c^6 - c^4rx^2 + 2c^3r\left(1-|x|^2\right)\delta + 2c^4rx,$$

$$4c^2c_2^2 = c^2(4-r)^2 + 2c^2(4-r)rx + c^2r^2x^2,$$

$$\begin{aligned}8c^2c_4 =& c^4rx^3 - 4c^2r\bar{x}\left(1-|x|^2\right)\delta^2 - 4c^3rx\left(1-|x|^2\right)\delta - 3c^4rx^2 \\ &+ 4c^2r\left(1-|x|^2\right)\left(1-|\delta|^2\right)\rho + 4c^3r\left(1-|x|^2\right)\delta + 3c^4rx + c^6 + 4c^2rx^2,\end{aligned}$$

$$8cc_2c_3 = -c^2r^2x^3 - c^4rx^2 + 2cxr^2\left(1-|x|^2\right)\delta + 2c^2r^2x^2 + 2c^3r\left(1-|x|^2\right)\delta + 3c^4rx + c^6,$$

$$8c_2^3 = r^3x^3 + 3c^2r^2x^2 + 3c^4rx + c^6,$$

$$\begin{aligned}16c_2c_4 =& 4c^2rx^2 + 4r^2x^3 + c^6 + 4c^2r\left(1-|x|^2\right)\left(1-|\delta|^2\right)\rho \\ &+ 4cr^2x\left(1-|x|^2\right)\delta + 4r^2x\left(1-|x|^2\right)\left(1-|\delta|^2\right)\rho + c^2r^2x^4 \\ &- 4c^3rx\left(1-|x|^2\right)\delta - 4c^2r\bar{x}\left(1-|x|^2\right)\delta^2 - 4cr^2x^2\left(1-|x|^2\right)\delta \\ &+ 4c^4rx + 4c^3r\left(1-|x|^2\right)\delta + c^4tx^3 - 4r^2x\bar{x}\left(1-|x|^2\right)\delta^2 \\ &+ 3r^2x^2c^2 - 3c^4rx^2 - 3c^2r^2x^3,\end{aligned}$$

and

$$\begin{aligned}16c_3^2 =& c^2r^2x^4 - 4r^2x^2\left(1-|x|^2\right)\delta - 4c^2r^2x^3 - 2c^4rx^2 + 4r^2\left(1-|x|^2\right)^2\delta^2\\ &+ 8cr^2x\left(1-|x|^2\right)\delta + 4c^3r\left(1-|x|^2\right)\delta + 4c^2r^2x^2 + 4c^4rx + c^6.\end{aligned}$$

By substituting these expressions into (25), we get

$$\begin{aligned}&\mathcal{H}_{2,2}\left(F_f/2\right)\\ &= \frac{1}{3538944}\Big[-69c^6 + \frac{648}{5}c^4xr - \frac{6912}{5}c^2rx^2 - \frac{7488}{5}c^2r^2x^3 + \frac{18432}{5}x^3t^2\\ &\quad - \frac{792}{5}c^4x^2r - 3456r^2\left(1-|x|^2\right)^2\delta^2 + \frac{6912}{5}c^2r\bar{x}\left(1-|x|^2\right)\delta^2 - \frac{1728}{5}c^4tx^3\\ &\quad - \frac{6912}{5}c^2r\left(1-|x|^2\right)\left(1-|\delta|^2\right)\rho - \frac{1152}{5}cr^2x^2\left(1-|x|^2\right)\delta + \frac{288}{5}x^4t^2c^2\\ &\quad + \frac{6912}{5}c^3rx\left(1-|x|^2\right)\delta + 1008c^3r\left(1-|x|^2\right)\delta\\ &\quad + 1152cr^2x\left(1-|x|^2\right)\delta - 480x^2r^2c^2 - \frac{18432}{5}r^2x\bar{x}\left(1-|x|^2\right)\delta^2\\ &\quad - \frac{5888}{5}r^3x^3 + \frac{18432}{5}r^2x\left(1-|x|^2\right)\left(1-|\delta|^2\right)\rho\Big].\end{aligned}$$

Since $r = 4 - c^2$, we see that

$$\mathcal{H}_{2,2}\left(F_f/2\right) = \frac{1}{3538944}\left[\tau_1(c,x) + \tau_2(c,x)\delta + \tau_3(c,x)\delta^2 + \tau_4(c,x,\delta)\rho\right],$$

where

$$\begin{aligned}\tau_1(c,x) =& -\frac{8}{5}\left(4-c^2\right)x\Big[4\left(4-c^2\right)x\left(-9x^2c^2 + 50xc^2 + 75c^2 + 160x\right)\\ &+ 216c^4x^2 + 99c^4x - 81c^4 + 864xc^2\Big] - 69c^6,\end{aligned}$$

$$\tau_2(c,x) = -\frac{144}{5}\left(4-c^2\right)\left(1-|x|^2\right)c\left[\left(8x^2-40x\right)\left(4-c^2\right) - 48xc^2 - 35c^2\right],$$

$$\tau_3(c,x) = -\frac{1152}{5}\left(4-c^2\right)\left(1-|x|^2\right)\left[\left(x^2+15\right)\left(4-c^2\right) - 6\bar{x}c^2\right],$$

and

$$\tau_4(c,x,\delta) = \frac{2304}{5}\left(4-c^2\right)\left(1-|x|^2\right)\left(1-|\delta|^2\right)\left[8x\left(4-c^2\right) - 3c^2\right].$$

Now, by utilizing $|\delta| = y$, $|x| = x$ and taking $|\rho| \le 1$, we have

(26)

$$\begin{aligned}\left|\mathcal{H}_{2,2}\left(F_f/2\right)\right| \le& \frac{1}{3538944}\left(|\tau_1(c,x)| + |\tau_2(c,x)|\,y + |\tau_3(c,x)|\,y^2 + |\tau_4(c,x,\delta)|\right)\\ \le& \frac{1}{3538944}U(c,x,y),\end{aligned}$$

where

$$U(c,x,y) = \phi_1(c,x) + \phi_2(c,x)\,y + \phi_3(c,x)\,y^2 + \phi_4(c,x)\left(1-y^2\right)$$

along with

$$\begin{aligned}\phi_1(c,x) =& \frac{8}{5}\left(4-c^2\right)\Big[4\left(4-c^2\right)\left(9x^2c^2+50xc^2+75c^2+160x\right)x \\ &+216c^4x^2+99c^4x+81c^4+864xc^2\Big]x+69c^6,\end{aligned}$$

$$\phi_2(c,x) = \frac{144}{5}\left(4-c^2\right)\left(1-x^2\right)\left[\left(8x^2+40x\right)\left(4-c^2\right)+48xc^2+35c^2\right]c,$$

$$\phi_3(c,x) = \frac{1152}{5}\left(4-c^2\right)\left(1-x^2\right)\left[\left(x^2+15\right)\left(4-c^2\right)+6xc^2\right],$$

and

$$\phi_4(c,x) = \frac{2304}{5}\left(4-c^2\right)\left(1-x^2\right)\left[8x\left(4-c^2\right)+3c^2\right].$$

Suppose that $\varrho_j(c,x) := \phi_j(c,x)$ $(j=1,2,4)$. We also define

$$\varrho_3(c,x) := \frac{1152}{5}\left(4-c^2\right)\left(1-x^2\right)\left[\left(x^2+15\right)\left(4-c^2\right)+6c^2\right],$$

and

$$V(c,x,y) := \varrho_1(c,x) + \varrho_2(c,x)\,y + \varrho_3(c,x)\,y^2 + \varrho_4(c,x)\left(1-y^2\right).$$

It is not hard to check that $\phi_j(c,x) \leq \varrho_j(c,x)$ for $j=1,2,3,4$, and thus $U(c,x,y) \leq V(c,x,y)$ on the closed cuboid Δ.

Now, we shall find the maximum value of V on Δ. By taking partial derivative of V with respect to y, we have

$$\frac{\partial V}{\partial y} = \varrho_2(c,x) + 2\left[\varrho_3(c,x) - \varrho_4(c,x)\right]y.$$

In view of $\varrho_2(c,x) \geq 0$ and

$$\varrho_3(c,x) - \varrho_4(c,x) = \frac{1152}{5}(4-c^2)^2(1-x^2)(x^2-16x+15) \geq 0$$

on $[0,2]\times[0,1]$, we know that $\frac{\partial V}{\partial y} \geq 0$ for all $y \in [0,1]$. Thus, we obtain $V(c,x,y) \leq V(c,x,1)$ and this allows us to consider the maximum value of V on the face of $y=1$ of Δ.

When $y=1$, $V(c,x,y)$ reduces to

$$V(c,x,1) = 69c^6 + \left(4-c^2\right)\left[v_4(c)x^4 + v_3(c)x^3 + v_2(c)x^2 + v_1(c)x + v_0(c)\right],$$

where

$$\begin{aligned}
v_0(c) &= 1008c^3 - \frac{10368}{5}c^2 + 13824,\\
v_1(c) &= \frac{72}{5}(9c^3 + 16c^2 + 320)c,\\
v_2(c) &= -\frac{24}{5}(67c^4 + 258c^3 - 1072c^2 - 192c + 2688),\\
v_3(c) &= \frac{128}{5}(c^4 - 9c^3 + 10c^2 - 180c + 160),\\
v_4(c) &= \frac{288}{5}(4 - c^2)(c^2 - 4c - 4).
\end{aligned}$$

By noting that $v_4(c) \le 0$ for $c \in [0, 2]$, we get

$$V(c, x, 1) \le 69c^6 + \left(4 - c^2\right)\left[v_3(c)x^3 + v_2(c)x^2 + v_1(c)x + v_0(c)\right] =: \vartheta(c, x).$$

For $c = 0$ in $\vartheta(c, x)$, we see that

$$\vartheta(0, x) = 16384x^3 - \frac{258048}{5}x^2 + 55296 \le 55296 \quad (x \in [0, 1]).$$

If we put $c = 2$ in ϑ, it implies that $\vartheta(2, x) \equiv 4416$. Then we only need to discuss the case $(c, x) \in (0, 2) \times (0, 1)$. By solving the system of equations $\frac{\partial\vartheta}{\partial c} = 0$ and $\frac{\partial\vartheta}{\partial x} = 0$, the real critical points of ϑ are about at $(0, 0)$, $(2, -1.7500)$, $(-2, 0.7391)$, $(2, -0.7219)$, $(2, 1.4719)$, $(-1.9981, -0.6873)$, $(13.1434, 0.1943)$, $(-1.8245, 0.3101)$, $(-0.8216, 0.9498)$ and $(12.8915, 35.2906)$. Since there exist solutions lie in $(0, 2) \times (0, 1)$, we conclude that

$$V(c, x, y) \le 55296 \quad ((c, x, y) \in [0, 2] \times [0, 1] \times [0, 1]).$$

By virtue of (26), we find that

$$|\mathcal{H}_{2,2}(F_f/2)| \le \frac{1}{3538944}U(c, x, y) \le \frac{1}{3538944}V(c, x, y) \le \frac{1}{64}.$$

For the function f_1 given by

$$f_1(z) = \int_0^z \left[1 + \tanh\left(t^3\right)\right] dt = z + \frac{1}{4}z^4 - \frac{1}{30}z^{10} + \cdots \in \mathcal{BT}_{\tanh},$$

we see that

$$|\mathcal{H}_{2,2}(F_f/2)| = \frac{1}{64},$$

which implies that the inequality in Theorem 3 is sharp. □

4. HANKEL DETERMINANTS OF INVERSE FUNCTIONS FOR THE CLASS $\mathcal{BT}_{\text{TANH}}$

In this section, we determine the Hankel determinants of inverse functions for the class $\mathcal{BT}_{\tanh}$.

Theorem 4. *Let* $f \in \mathcal{BT}_{\tanh}$. *Then*

$$\left|\mathcal{H}_{2,2}\left(f^{-1}\right)\right| \leq \frac{1}{9}.$$

This result is sharp.

Proof. By putting (18), (19), (20), (21) into (8), (9), (10) and (11), we find that

$$A_2 = -\frac{1}{4}c_1, \tag{27}$$

$$A_3 = -\frac{1}{6}c_2 + \frac{5}{24}c_1^2, \tag{28}$$

$$A_4 = -\frac{13}{64}c_1^3 + \frac{1}{3}c_1c_2 - \frac{1}{8}c_3, \tag{29}$$

and

$$A_5 = -\frac{259}{480}c_2c_1^2 + \frac{83}{384}c_1^4 + \frac{23}{80}c_1c_3 - \frac{1}{10}c_4 + \frac{2}{15}c_2^2. \tag{30}$$

In view of (27), (28) and (29), we have

$$\mathcal{H}_{2,2}\left(f^{-1}\right) = \left|A_2A_4 - A_3^2\right| = \frac{1}{2304}\left|17c_1^4 - 32c_1^2c_2 + 72c_1c_3 - 64c_2^2\right|.$$

By using (12) and (13) to express c_2 and c_3 in terms of c_1, suppose also that $c_1 = c \in [0,2]$, we obtain

$$\begin{aligned}\mathcal{H}_{2,2}\left(f^{-1}\right) =& \frac{1}{2304}\Big|3c^4 - 18c^2x^2\left(4-c^2\right) - 16x^2\left(4-c^2\right)^2 \\ &+ 36c\delta\left(4-c^2\right)\left(1-|x|^2\right) - 12c^2x\left(4-c^2\right)\Big|.\end{aligned}$$

By replacing $|\delta| \leq 1$ and $|x| = \eta$ with $\eta \leq 1$, and by using triangle inequality and taking $c \in [0,2]$, we know that

$$\begin{aligned}\left|\mathcal{H}_{2,2}\left(f^{-1}\right)\right| \leq& \frac{1}{2304}\Big[3c^4 + 18c^2\eta^2\left(4-c^2\right) + 16\eta^2\left(4-c^2\right)^2 \\ &+ 36c\left(4-c^2\right)\left(1-\eta^2\right) + 12\eta c^2\left(4-c^2\right)\Big] =: \omega(c,\eta).\end{aligned}$$

Differentiating $\omega(c,\eta)$ with respect to η, we get

$$\frac{\partial\omega(c,\eta)}{\partial\eta} = -\frac{1}{576}(c-2)(c+2)\left(\eta c^2 - 18\eta c + 3c^2 + 32\eta\right).$$

It is easy to show that $\omega'(c,\eta) \geq 0$ for $\eta \in [0,1]$, which implies that $\omega(c,\eta) \leq \omega(c,1)$.

If we set $\eta = 1$, it follows that

$$\left|\mathcal{H}_{2,2}\left(f^{-1}\right)\right| \leq \frac{1}{2304}\left[3c^4 + 30c^2\left(4-c^2\right) + 16\left(4-c^2\right)^2\right] =: \Delta(c).$$

By noting that $\Delta'(c) \leq 0$ for $c \in [0,2]$, we know that $\Delta(c)$ is a decreasing function, it attains maximum value at $c = 0$, Therefore, we have

$$\left|\mathcal{H}_{2,2}\left(f^{-1}\right)\right| \leq |\Delta(0)| \leq \frac{1}{9}.$$

For the function f_2 given by

$$f_2(z) = \int_0^z \left[1 + \tanh\left(t^2\right)\right] dt = z + \frac{1}{3}z^3 - \frac{1}{21}z^7 + \cdots, \tag{31}$$

we know that

$$\left|\mathcal{H}_{2,2}\left(f^{-1}\right)\right| = \frac{1}{9},$$

which shows that the desired inequality of Theorem 4 is sharp. □

Theorem 5. *If $f \in \mathcal{BT}_{\tanh}$, then*

$$\left|\mathcal{H}_{3,1}\left(f^{-1}\right)\right| \leq \frac{2}{27}.$$

The result is sharp for the function f_2 given by (31).

Proof. By the definition of $\mathcal{H}_{3,1}\left(f^{-1}\right)$, we find that

$$\begin{aligned}\mathcal{H}_{3,1}\left(f^{-1}\right) =& 2A_2A_3A_4 - A_4^2 - A_2^2A_5 - A_3^3 + A_3A_5\\ =& 3a_2^2a_3^2 + a_2^6 - a_2^2a_5 - 3a_2^4a_3 - 2a_3^3 + 2a_2a_3a_4 - a_4^2 + a_3a_5.\end{aligned}$$

By using (27), (28), (29) and (30) with $c_1 = c$, we know that

(32)

$$\begin{aligned}\mathcal{H}_{3,1}\left(f^{-1}\right) =& \frac{1}{110592}\left(263c^6 - \frac{5112}{5}c^4c_2 + \frac{2304}{5}c^3c_3 + 960c^2c_2^2 + \frac{13824}{5}cc_2c_3\right.\\ &\left.- \frac{9728}{5}c_2^3 + \frac{9216}{5}c_2c_4 - 1728c_3^2 - \frac{8064}{5}c^2c_4\right).\end{aligned}$$

To simplify the computation, we assume that $v = 4 - c_1^2$ with $c_1 = c$ in (12), (13) and (14), by using the simplified forms of these formulae, we have

$$2c_2 = 4 - v + vx,$$

$$4c^3c_3 = c^6 - c^4vx^2 + 2c^3v\left(1 - |x|^2\right)\delta + 2c^4vx,$$

$$4c^2c_2^2 = c^2(4 - v)^2 + 2c^2(4 - v)vx + c^2v^2x^2,$$

$$\begin{aligned}8c^2c_4 =& c^4vx^3 - 4c^2v\bar{x}\left(1 - |x|^2\right)\delta^2 - 4c^3vx\left(1 - |x|^2\right)\delta - 3c^4vx^2 \\ &+ 4c^2v\left(1 - |x|^2\right)\left(1 - |\delta|^2\right)\rho + 4c^3v\left(1 - |x|^2\right)\delta + 3c^4vx + c^6 + 4c^2vx^2,\end{aligned}$$

$$\begin{aligned}8cc_2c_3 =& - c^2v^2x^3 - c^4vx^2 + 2cxv^2\left(1 - |x|^2\right)\delta + 2c^2v^2x^2 + 2c^3v\left(1 - |x|^2\right)\delta \\ &+ c^6 + 3c^4vx,\end{aligned}$$

$$8c_2^3 = v^3x^3 + 3c^2v^2x^2 + 3c^4vx + c^6,$$

$$\begin{aligned}16c_2c_4 =& 4c^2vx^2 + 4v^2x^3 + c^6 + 4c^4vx + 4c^2v\left(1 - |x|^2\right)\left(1 - |\delta|^2\right)\rho \\ &+ 4cv^2x\left(1 - |x|^2\right)\delta + 4v^2x\left(1 - |x|^2\right)\left(1 - |\delta|^2\right)\rho + c^4vx^3 + c^2v^2x^4 \\ &- 4c^3vx\left(1 - |x|^2\right)\delta - 4c^2v\bar{x}\left(1 - |x|^2\right)\delta^2 - 3c^2v^2x^3 \\ &- 4cv^2x^2\left(1 - |x|^2\right)\delta - 4v^2x\bar{x}\left(1 - |x|^2\right)\delta^2 \\ &+ 3v^2x^2c^2 + 4c^3v\left(1 - |x|^2\right)\delta - 3c^4vx^2,\end{aligned}$$

and

$$\begin{aligned}16c_3^2 =& c^2v^2x^4 - 4cv^2x^2\left(1 - |x|^2\right)\delta - 4c^2v^2x^3 - 2c^4vx^2 + 4v^2\left(1 - |x|^2\right)^2\delta^2 \\ &+ 8cv^2x\left(1 - |x|^2\right)\delta + 4c^3v\left(1 - |x|^2\right)\delta + 4c^2v^2x^2 + 4c^4vx + c^6.\end{aligned}$$

By substituting these expressions into (32), we get

$$\begin{aligned}
&\mathcal{H}_{3,1}\left(f^{-1}\right) \\
&= \frac{1}{110592}\Big[15c^6 + 144c^3v\left(1-|x|^2\right)\delta - \frac{384}{5}c^4xv - \frac{1728}{5}c^2vx^2 \\
&\quad - \frac{1296}{5}c^2v^2x^3 + \frac{2304}{5}x^3v^2 + \frac{72}{5}c^4x^2v + \frac{576}{5}x^2v^2c^2 - 432v^2\left(1-|x|^2\right)^2\delta^2 \\
&\quad + \frac{1728}{5}c^2v\bar{x}\left(1-|x|^2\right)\delta^2 - \frac{432}{5}c^4vx^3 + \frac{1728}{5}c^3vx\left(1-|x|^2\right)\delta \\
&\quad - \frac{1728}{5}c^2v\left(1-|x|^2\right)\left(1-|\delta|^2\right)\rho - \frac{144}{5}cv^2x^2\left(1-|x|^2\right)\delta + \frac{36}{5}x^4v^2c^2 \\
&\quad - \frac{2304}{5}v^2x\bar{x}\left(1-|x|^2\right)\delta^2 - \frac{1216}{5}v^3x^3 + 288cv^2x\left(1-|x|^2\right)\delta \\
&\quad + \frac{2304}{5}v^2x\left(1-|x|^2\right)\left(1-|\delta|^2\right)\rho\Big].
\end{aligned}$$

Since $v = 4 - c^2$, we see that

$$\mathcal{H}_{3,1}\left(f^{-1}\right) = \frac{1}{110592}\left[\mu_1(c,x) + \mu_2(c,x)\delta + \mu_3(c,x)\delta^2 + \mu_4(c,x,\delta)\rho\right],$$

where

$$\begin{aligned}
\mu_1(c,x) = &-\frac{4}{5}x\left(4-c^2\right)\Big[\left(4-c^2\right)x\left(-9x^2c^2 + 20xc^2 - 144c^2 + 640x\right) \\
&+ 108c^4x^2 - 18c^4x + 87c^4 + 432xc^2\Big] + 15c^6,
\end{aligned}$$

$$\mu_2(c,x) = -\frac{144}{5}c\left(4-c^2\right)\left(1-|x|^2\right)\left[\left(x^2-10x\right)\left(4-c^2\right) - 12xc^2 - 5c^2\right],$$

$$\mu_3(c,x) = -\frac{144}{5}\left(4-c^2\right)\left(1-|x|^2\right)\left[\left(x^2+15\right)\left(4-c^2\right) - 12\bar{x}c^2\right],$$

and

$$\mu_4(c,x,\delta) = \frac{576}{5}\left(4-c^2\right)\left(1-|x|^2\right)\left(1-|\delta|^2\right)\left[4x\left(4-c^2\right) - 3c^2\right].$$

Now, by utilizing $|\delta| = y$, $|x| = x$ and taking $|\rho| \le 1$, we obtain

$$\begin{aligned}
\left|\mathcal{H}_{3,1}\left(f^{-1}\right)\right| &= \frac{1}{110592}\left(|\mu_1(c,x)| + |\mu_2(c,x)|y + |\mu_3(c,x)|y^2 + |\mu_4(c,x,\delta)|\right) \\
&\le \frac{1}{110592}T(c,x,y),
\end{aligned} \tag{33}$$

where

$$T(c,x,y) := \zeta_1(c,x) + \zeta_2(c,x)y + \zeta_3(c,x)y^2 + \zeta_4(c,x)\left(1-y^2\right)$$

along with

$$\zeta_1(c,x) = \frac{4}{5}x\left(4-c^2\right)\Big[x\left(4-c^2\right)\left(9x^2c^2+20xc^2+144c^2+640x\right) + 108c^4x^2+18c^4x+87c^4+432xc^2\Big]+15c^6,$$

$$\zeta_2(c,x) = \frac{144}{5}c\left(4-c^2\right)\left(1-x^2\right)\left[\left(x^2+10x\right)\left(4-c^2\right)+12xc^2+5c^2\right],$$

$$\zeta_3(c,x) = \frac{144}{5}\left(4-c^2\right)\left(1-x^2\right)\left[\left(x^2+15\right)\left(4-c^2\right)+12xc^2\right],$$

and

$$\zeta_4(c,x) = \frac{576}{5}\left(4-c^2\right)\left(1-x^2\right)\left[4x\left(4-c^2\right)+3c^2\right].$$

Suppose that $\eta_j(c,x) = \zeta_j(c,x)$ $(j=1,2,4)$, and take

$$\eta_3(c,x) = \frac{144}{5}\left(4-c^2\right)\left(1-x^2\right)\left[\left(x^2+15\right)\left(4-c^2\right)+12c^2\right],$$

and

$$R(c,x,y) = \eta_1(c,x)+\eta_2(c,x)y+\eta_3(c,x)y^2+\eta_4(c,x)\left(1-y^2\right).$$

It is noted that $T(c,x,y) \leq R(c,x,y)$, since $\zeta_j(c,x) \leq \eta_j(c,x)$ for $j=1,2,3,4$.

We now derive the maximum value of R in the cuboid Δ. Taking partial derivative of R with respect to y, we obtain

$$\frac{\partial R}{\partial y} = \eta_2(c,x)+2\left[\eta_3(c,x)-\eta_4(c,x)\right]y.$$

Since $\eta_2(c,x) \geq 0$ and

$$\eta_3(c,x)-\eta_4(c,x) = \frac{144}{5}\left(4-c^2\right)^2\left(1-x^2\right)\left(x^2-16x+15\right) \geq 0$$

on $[0,2]\times[0,1]$, we see that $\frac{\partial R}{\partial y} \geq 0$ for all $y \in [0,1]$. Hence, we have $R(c,x,y) \leq R(c,x,1)$ on $[0,2]\times[0,1]$.

It is left for us to find the maximum value of R on the face of $y=1$. When $y=1$, we know that $R(c,x,y)$ reduces to

$$R(c,x,1) = 15c^6+(4-c^2)\left[(k_4(c)x^4+k_3(c)x^3+k_2(c)x^2+k_1(c)x+k_0(c)\right] =: K(c,x),$$

where

$$k_0(c) = 144c^3-\frac{432}{5}c^2+1728,$$

$$k_1(c) = \frac{12}{5}(29c^3+24c^2+480)c,$$

$$k_2(c) = -\frac{72}{5}(7c^4+12c^3-60c^2-8c+112),$$

$$k_3(c) = -\frac{32}{5}(-11c^4+9c^3+70c^2+180c-320),$$

$$k_4(c) = \frac{36}{5}(4-c^2)(c^2-4c-4).$$

If we set $c = 0$ in K, then

$$K(0, x) = -\frac{2304}{5}x^4 + 8192x^3 - \frac{32256}{5}x^2 + 6912 =: \Lambda(x).$$

Since the unique solution of $\Lambda'(x) = 0$ lies in $(0, 1)$ is $x_0 = \frac{20-\sqrt{337}}{3} \approx 0.5475$, it is not hard to find that Λ attains its maximum value 8192 at $x = 1$. By setting $c = 2$ in $K(c, x)$, we obtain $K(2, x) \equiv 960$ for all $x \in [0, 1]$.

We are now left to discuss the case $(c, x) \in (0, 2) \times (0, 1)$. Consider the system of equations $\frac{\partial K}{\partial c} = 0$ and $\frac{\partial K}{\partial x} = 0$. Note that the real solutions of $K(c, x)$ are about $(-7.2937, -46.6616)$, $(-2.0289, -0.4917)$, $(0.4033, 0.2926)$, $(-1.8318, 0.3055)$, $(0, 0)$, $(-3.3233, 8.5550)$, $(2, 1.5998)$, $(2, -1.8143)$, $(2, -0.4522)$, $(2, -0.6651)$, $(2, -0.2869)$, $(-2, 0.6187)$ and $(6.6181, 16.7155)$, we know that the unique critical point of K lies in $(0, 2) \times (0, 1)$ is $(0.4033, 0.2926)$ and $K(0.4033, 0.2926) \approx 6809.0525$.

In view of the above various cases, we deduce that $R(c, x, y) \leq 8192$ on $[0, 2] \times [0, 1] \times [0, 1]$. By virtue of (33), we see that

$$\left|\mathcal{H}_{3,1}\left(f^{-1}\right)\right| \leq \frac{1}{110592}T(c, x, y) \leq \frac{1}{110592}R(c, x, y) \leq \frac{2}{27}.$$

For the function f_2 given by (31), we find that

$$\left|\mathcal{H}_{3,1}\left(f^{-1}\right)\right| = \frac{2}{27},$$

which shows that the desired inequality of Theorem 5 is sharp. □

Acknowledgements. The present investigation was supported by the *Natural Science Foundation of Hunan Province* under Grant no. 2022JJ30185 and the *National Natural Science Foundation* under Grant no. 11961013 of the P. R. China. The authors would like to thank Lei Shi and the referees for their valuable comments and suggestions, which was essential to improve the quality of this paper.

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(Received 03. 12. 2022.)
(Revised 21. 04. 2024.)

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