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ENUMERATION OF NON-CROSSING PARTITIONS ACCORDING TO SUBWORDS WITH REPEATED LETTERS

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In this paper, we enumerate members of the set NC_n of non-crossing partitions of length n according to the number of occurrences of several infinite families of subword patterns each containing repeated letters. As a consequence of our results, we obtain explicit generating function formulas counting the members of NC_n for $n \geq 0$ according to all subword patterns of length three containing a repeated letter. In particular, we find extensions of the subword equivalences $211 \equiv 221$, $1211 \equiv 1121$ and $112 \equiv 122$ over NC_n.

1. INTRODUCTION

A collection of disjoint nonempty subsets of a set whose union is the set is known as a *partition*, with the constituent subsets referred to as *blocks* of the partition. Let $[n] = \{1, 2, \ldots, n\}$ for $n \geq 1$, with $[0] = \emptyset$. The set of partitions of $[n]$ containing exactly k blocks will be denoted by $\mathcal{P}_{n,k}$, with $\mathcal{P}_n = \cup_{k=0}^{n} \mathcal{P}_{n,k}$ denoting the set of all partitions of $[n]$. A partition $\Pi = B_1/B_2/\cdots/B_k \in \mathcal{P}_{n,k}$ is said to be in *standard form* if its blocks B_i are such that $\min(B_i) < \min(B_{i+1})$ for $1 \leq i \leq k-1$. A partition Π in standard form can be represented sequentially by writing $\pi = \pi_1 \cdots \pi_n$, where $i \in B_{\pi_i}$ for each $i \in [n]$ (see, e.g., [**6**]). The sequence π is referred to as the *canonical sequential form* of the partition Π. Then Π in standard form implies $\pi_1 = 1$ with $\pi_{i+1} \leq \max(\pi_1 \cdots \pi_i) + 1$ for $1 \leq i \leq n-1$, which is known as the *restricted growth* condition (see, e.g., [**11**]).

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A partition Π is said to be *non-crossing* [**4**] if its sequential representation π contains no subsequence of the form a-b-a-b, where $a < b$ (i.e., if π avoids the pattern 1-2-1-2 in the classical sense). Let NC_n denote the set of non-crossing partitions of $[n]$; recall that $|NC_n| = C_n$ for all $n \geq 0$, where $C_n = \frac{1}{n+1}\binom{2n}{n}$ is the n-th Catalan number. We will denote the Catalan number generating function $\sum_{n\geq 0} C_n x^n = \frac{1-\sqrt{1-4x}}{2x}$ by $C(x)$.

Let $\tau = \tau_1 \cdots \tau_m$ be a sequence of positive integers whose set of distinct letters comprise $[\ell]$ for some $1 \leq \ell \leq m$. Then the sequence $\rho = \rho_1 \cdots \rho_n$ is said to *contain* τ as a *subword* (pattern) if some string of consecutive letters of ρ is order-isomorphic to τ. That is, there exists an index $i \in [n-m+1]$ such that $\rho_i \rho_{i+1} \cdots \rho_{i+m-1}$ is isomorphic to τ. If no such index i exists, then ρ *avoids* τ as a subword.

Here, we will be interested in counting the members of NC_n according to the number of occurrences of certain subword patterns, focusing on several infinite families of patterns. Let $\mu_\tau(\pi)$ denote the number of occurrences of the subword τ in the partition π. We compute the generating function $F = F_\tau$ given by

$$F = \sum_{n\geq 0} \left(\sum_{\pi \in NC_n} q^{\mu_\tau(\pi)} \right) x^n$$

in several cases when τ has one or more repeated letters. This extends recent work initiated in [**9**] which focused on subwords where all of the letters in a pattern were distinct. We remark that other finite discrete structures with sequential representations that have been enumerated according to the number of subwords include k-ary words [**1**], set partitions [**10**] and involutions [**7**]. For examples of other types of statistics which have been studied on non-crossing partitions, we refer the reader to [**5, 8, 12, 15, 16**].

This paper is organized as follows. In the next section, we enumerate members of NC_n according to four infinite families of subword patterns and compute the corresponding generating function F in each case. Simple formulas for the total number of occurrences on NC_n for the various patterns are deduced from our formulas for F. Further, an explicit bijection is defined which demonstrates the equivalence of the subwords $(\rho+1)1^a$ and $(\rho'+1)1^{a'}$ of the same length. In the third section, the pattern $12\cdots(m-1)m^a$ is treated using the *kernel method* [**3**] and a formula for the generating function of its joint distribution with an auxiliary parameter on NC_n is found. Finally, a bijection is given which demonstrates the equivalence of $1^a 23\cdots m$ and $12\cdots(m-1)m^a$ as subwords on NC_n for all $a, m \geq 2$.

As special cases of our results, we obtain F_τ for all τ of length three containing a repeated letter. See Table 1 below, where the equation satisfied by F_τ is given for each τ. Note that the case 212 is trivial since any partition π containing a string x of the form $x = bab$ where $a < b$ must contain an occurrence of 1-2-1-2, upon considering the leftmost occurrence of the letter a in π which must precede the first b in x by the restricted growth condition.

Table 1: Generating functions $F = F_\tau$ for subwords τ of length three containing a repeated letter

Subword	Generating function equation	Reference
111	$x(1-qx+(q-1)x^2)F^2 = (1-qx+(q-1)x^3)(F-1)$	Corollary 2
112	$x(1+(q-1)x)F^2 = (1+(q-1)x^2)F-1$	Corollary 2
121	$xF^2 = (1-(q-1)x^2)(F-1)$	Theorem 6
122	$x(1+(q-1)x)F^2 = (1+(q-1)x^2)F-1$	Theorem 10
211	$x(1+(q-1)x)F^2 = (1+2(q-1)x^2)F-1-(q-1)x^2$	Theorem 4
212	$xF^2 = F-1$	Trivial
221	$x(1+(q-1)x)F^2 = (1+2(q-1)x^2)F-1-(q-1)x^2$	Theorem 4

2. DISTRIBUTIONS OF SOME INFINITE FAMILIES OF PATTERNS

We first consider the patterns $\tau = 1^a$ and $\rho = 1^b2$, where $a, b \geq 1$, and treat them together as a joint distribution on NC_n. We shall determine a formula for the generating function of this distribution given by

$$\sum_{n\geq 0}\left(\sum_{\pi\in NC_n} p^{\mu_\tau(\pi)}q^{\mu_\rho(\pi)}\right)x^n,$$

which will be denoted by F. We make use of the *symbolic* enumeration method (see, e.g., [**2**]) in finding F.

Theorem 1. *If $a \geq b \geq 1$, then the generating function F enumerating the members of NC_n for $n \geq 0$ jointly according to the number of occurrences of 1^a and 1^b2 satisfies*

$$(x-px^2+q(p-1)x^a+(q-1)(1-px)x^b)F^2$$
$$= (1-px+q(p-1)x^a+(q-1)(1-px)x^b)F-1+px-(p-1)x^a. \tag{1}$$

If $1 \leq a < b$, then F satisfies

$$(x-px^2+(p-1)x^a+(q-1)(1-x)p^{b-a+1}x^b)F^2$$
$$= (1-px+(p-1)x^a+(q-1)(1-x)p^{b-a+1}x^b)F-1+px-(p-1)x^a. \tag{2}$$

Proof. First assume $a \geq b \geq 1$ and consider the following cases on $\pi \in NC_n$: (i) $\pi = 1^n$ for some $0 \leq n \leq a-1$, (ii) $\pi = 1^n$, where $n \geq a$, (iii) $\pi = 1^r\alpha\beta$, where $1 \leq r \leq b-1$, α is nonempty and contains no 1's and β starts with 1 if nonempty, (iv) π as in (iii), but where $b \leq r \leq a-1$, or (v) π as in (iii), but where $r \geq a$.

Combining cases (i)–(v) implies F is determined by

$$F=\frac{1-x^a}{1-x}+\frac{px^a}{1-px}+\left(\frac{x-x^b}{1-x}+q\frac{x^b-x^a}{1-x}+pq\frac{x^a}{1-px}\right)F(F-1).$$

Note that the sections α and β of π are determined by the factors $F-1$ and F, respectively, in cases (iii)–(v). Further, $r \geq b$ in (iv) and (v) implies that there is an extra occurrence of ρ (accounted for by the lone q factor) arising due to the initial run of 1's within π and the first letter of α. After simplification, the preceding equation for F rearranges to give (1).

If $1 \leq a < b$, then by similar reasoning we have that F satisfies

$$F=\frac{1-x^a}{1-x}+\frac{px^a}{1-px}+\left(\frac{x-x^a}{1-x}+\frac{px^a-p^{b-a+1}x^b}{1-px}+\frac{p^{b-a+1}qx^b}{1-px}\right)F(F-1),$$

which simplifies to gives (2). □

Taking $q=1$ and $p=1$ in Theorem 1, and solving for F (replacing p by q in the resulting formula in the first case), yields the following result.

Corollary 2. *The generating functions counting members of NC_n for $n \geq 0$ according to the number of occurrences of the patterns 1^m and 1^m2 where $m \geq 1$ are given respectively by*

$$\frac{1-qx+(q-1)x^m-\sqrt{(1-qx+(q-1)x^m)((1-4x)(1-qx)-3(q-1)x^m)}}{2x(1-qx+(q-1)x^{m-1})}$$

and

$$\frac{1+(q-1)x^m-\sqrt{(1-(q-1)x^m)^2-4x}}{2x(1+(q-1)x^{m-1})}.$$

Differentiating the formulas in Corollary 2 with respect to q, and extracting the coefficient of x^n, yields simple expressions for the total number of occurrences of the respective subwords on NC_n.

Corollary 3. *The total number of occurrences of 1^m and 1^m2 within all the members of NC_n for $n \geq m \geq 1$ are given by $\binom{2r}{r+1}$ and $\binom{2r-1}{r+1}$, respectively, where $r=n-m+1$.*

Proof. It is also possible to provide a combinatorial explanation of these formulas. For the first, suppose that there is a letter x in the i-th position within a member of NC_r, where $1 \leq i \leq r$. Then insert $m-1$ additional copies of x to directly follow the one already present in position i and mark the occurrence of the subword 1^m in the resulting member of NC_n. Note that all occurrences of 1^m within members of NC_n arise uniquely in this way, which yields $rC_r=\binom{2r}{r+1}$ total occurrences. For the second formula, consider an ascent xy in $\pi \in NC_r$ and insert $m-1$ additional copies of x between x and y. This results in an occurrence of $\tau = 1^m2$ within a member of NC_n in which the role of the '2' is played by y. Thus, counting occurrences of τ in NC_n is equivalent to counting ascents in NC_r. Note that the number of ascents in a non-crossing partition π equals $\mu(\pi)-1$ for all π, where $\mu(\pi)$ denotes the number of blocks of π. Since μ has a Narayana distribution on NC_r (see, e.g., [**13**, A001263]), it follows that the number of blocks over NC_r equals $\sum_{i=1}^{r}\frac{i}{r}\binom{r}{i}\binom{r}{i-1}=\binom{2r-1}{r}$. Since the number of ascents is always one less than the number of blocks, we have that the total number of occurrences of τ in all the members of NC_n is given by $\binom{2r-1}{r}-C_r=\binom{2r-1}{r+1}$. □

Given a sequence ρ and a number x, let $\rho+x$ denote the sequence obtained by adding x to each entry of ρ. Let $\tau=(\rho+1)1^b$, where ρ is a sequential representation of a non-crossing partition of length $a\geq 1$. Assume further that ρ starts with a single 1 if $b\geq 2$ (with no such restriction if $b=1$). Then we have the following general formula for F_τ.

Theorem 4. *Let $\tau=(\rho+1)1^b$, where ρ is of length $a\geq 1$ as described and $b\geq 1$. Then the generating function counting the members of NC_n for $n\geq 0$ according to the number of occurrences of τ is given by*

$$\frac{1+2(q-1)x^{a+b-1}-\sqrt{1-4x-4(q-1)x^{a+b}}}{2x(1+(q-1)x^{a+b-2})}.$$

Proof. Let $G=G_\tau$ enumerate $\pi\in NC_n$ for $n\geq 0$ according to the number of occurrences of τ in $\pi 0^b$ and $F=F_\tau$ denote the usual generating function. We first establish the relation

$$G=F+(q-1)x^{a-1}(F-1). \tag{3}$$

To do so, first note that F and G assign the same q-weights to non-crossing partitions except for those of the form $\pi=\alpha\beta$, where β corresponds to an occurrence of the subword ρ. We now describe how such partitions can be formed. Let x denote the first letter of β. Let $\rho=\rho_1\rho_2\cdots\rho_a$ and $\rho'=\rho_2\cdots\rho_a$. Let ρ^* be the sequence obtained from ρ' by replacing each 1 in ρ' with x and each letter $i>1$ with $i+m-1$, where $m=\max(\alpha\cup\{x\})$. Then appending ρ^* to the partition αx gives π of the form stated above, with αx representing an arbitrary member of NC_n for some $n\geq 1$. Further, since ρ starts with a single 1 if $b>1$, we have that appending ρ^* as described to αx does not introduce an occurrence of τ involving the last letters of αx and the first of ρ^* (as ρ^* must start with $m+1$ if nonempty when $b>1$). Then F and G differ with respect to the assigned q-weight (only) on partitions of the form $\pi=\alpha x\rho^*$, where α and ρ^* are as described. Such π are enumerated by $x^{a-1}(F-1)$, since αx is nonempty and arbitrary and the $a-1$ appended letters comprising ρ^* are determined once α is specified. Subtracting the weight of such π from the count for G, and adding them back with an extra factor of q, implies (3).

We now write a formula for F. To do so, note that $\pi\in NC_n$ for some $n\geq 0$ may be expressed as (i) $\pi=1^n$, (ii) $\pi=1^r\alpha$, where $r\geq 1$ and α is nonempty and does not contain 1, (iii) $\pi = 1^r\alpha 1^s\beta$, where $0 \le s \le b-2$ and β is nonempty and starts with exactly one 1, (iv) $\pi = 1^r\alpha 1^{b-1}\beta$, where β is nonempty but may start with any positive number of 1's in this case. Note that the generating function for all nonempty non-crossing partitions starting with a single 1 according to the number of occurrences of τ is given by $F-1-x(F-1) = (1-x)(F-1)$, by subtraction. Hence, case (iii) is seen to contribute $\frac{x}{1-x}(F-1)(1+x+\cdots+x^{b-2})(1-x)(F-1)$ towards F if $b \ge 2$, with (iii) not applicable (i.e., it is subsumed by (iv)) if $b=1$. In case (iv), one gets a contribution of $\frac{x}{1-x}(G-1)x^{b-1}(F-1)$ towards F for all $b \ge 1$, where the $G-1$ factor accounts for the nonempty section α, as it is followed by (at least) b letters 1. Combining (i)–(iv) then gives

$$F = \frac{1}{1-x} + \frac{x}{1-x}(F-1) + \frac{x-x^b}{1-x}(F-1)^2 + \frac{x^b}{1-x}(F-1)(G-1). \tag{4}$$

To solve (3) and (4), it is easier to consider $U = F-1$. Then (3) implies $G-1 = (1+(q-1)x^{a-1})U$ and thus (4) may be rewritten as

$$U = \frac{x}{1-x}\left(1+U+(1-x^{b-1})U^2 + x^{b-1}(1+(q-1)x^{a-1})U^2\right). \tag{5}$$

Solving for U in (5) gives

$$U = \frac{1-2x-\sqrt{1-4x-4(q-1)x^{a+b}}}{2x(1+(q-1)x^{a+b-2})},$$

which implies the desired formula for $F = U+1$. □

Note that the formula for F_τ in Theorem 4 depends only on the length of the subword τ. A bijective proof showing the equivalence of τ and τ' of the same length is given below. In particular, when $|\tau| = 3$, we have $211 \equiv 221$ as subwords on NC_n for all n, with the common generating function formula given by

$$\frac{1+2(q-1)x^2-\sqrt{1-4x-4(q-1)x^3}}{2x(1+(q-1)x)}.$$

Differentiating the formula in Theorem 4 gives the following.

Corollary 5. *If $n \ge a+b-1$, then the total number of occurrences of $\tau = (\rho+1)1^b$ as described above within all the members of NC_n is given by $\binom{2r-2}{r+1}$, where $r = n-a-b+2$.*

Remark: For each $m \ge 1$, we have from Corollary 3 that the nonzero values in the sequences for the total number of occurrences of 1^m and 1^m2 in NC_n for $n \ge 1$ correspond respectively to A001791 and A002054 in [**13**]. Corollary 5 implies the total number of occurrences of $(\rho+1)1^b$ corresponds to A002694.

Suppose $\tau = (p+1)1^b$ is as described above with $|\rho| = a$ and $\tau' = (\rho'+1)1^{b'}$, where ρ' is of length $a' \ge 1$ and satisfies the same requirements as ρ above, $b' \ge 1$ and $a'+b' = a+b$.

Bijective proof of $\tau \equiv \tau'$ as subwords on NC_n:

Clearly, we may assume $|\tau| = a + b \geq 3$. We first prove the result when $b = b' = 1$. Let $\pi \in NC_n$, represented sequentially. Let $\mathbf{s}$ denote a string of π of the form $\mathbf{s} = u\alpha v$, where $\alpha \neq \emptyset$ and $1 \leq v < u \leq \min(\alpha)$. If $\mathbf{s}$ corresponds to an occurrence τ (τ'), then we will refer to $\mathbf{s}$ as an τ-*string* (τ'-*string*, respectively). We wish to define a bijection f on NC_n in which partitions containing a given number of τ-strings are mapped to those containing the same number of τ'-strings, and vice versa. If no τ- or τ'-strings exist (i.e., if π avoids both τ and τ' as subwords), then let $f(\pi) = \pi$. So let $x_1, x_2, \ldots, x_r$ where $r \geq 1$ denote the complete combined set of τ- and τ'-strings in a left-to-right scan of the sequence π. Note that since $b = b' = 1$, the adjacent strings x_i and x_{i+1} for some $1 \leq i \leq r - 1$ are either disjoint or share a single letter.

We now change each x_i to the other option regarding containment of τ or τ'. We will first change x_1 and then subsequently work on $x_2, x_3, \ldots, x_r$, going from left to right. Suppose first that x_1 is a τ-string. Then we will change x_1 to a τ'-string y_1 as follows. Similar reasoning will apply to the case when x_1 is a τ'-string. Suppose τ has $s + 1$ distinct letters, where $s \geq 1$, and that the τ-string x_1 makes use of the actual letters $v < u = u_1 < u_2 < \cdots < u_s$. Note that ρ a partition and π non-crossing implies $u_2, \ldots, u_s$ represent the leftmost occurrences of the letters of their respective kinds within π and hence $u_\ell = u_2 + \ell - 2$ for $2 \leq \ell \leq s$. Suppose τ' has $t + 1$ distinct letters, where $t \geq 1$. If $s \geq t$, then replace the letters in x_1 with a sequence that is isomorphic to τ' in which the roles of $1, 2, \ldots, t + 1$ are played by $v < u_1 < \cdots < u_t$. Further, if $s > t$, then the letters $u_{t+1} < \cdots < u_s$ are not needed in this replacement, in which case, we reduce each letter of π belonging to $\{u_s + 1, u_s + 2, \ldots\}$, all of which must necessarily occur to the right of x_1 within π, by the amount $s - t$. Note that π non-crossing and τ starting with 2 and ending in 1 implies that the letters $u_{t+1}, \ldots, u_s$ within x_1 do not occur elsewhere in π.

On the other hand, if $t > s$, then we use all of the distinct letters occurring in x_1, together with $u_s + 1, \ldots, u_s + t - s$, when performing the replacement. In this case, we must increase any letters of π greater than or equal to $u_s + 1$, all of which must occur to the right of x_1, by the amount $t - s$ in order to accommodate the new letters used. In all cases, let y_1 denote the τ'-string that results from making the replacement as described and let π_1 be the resulting member of NC_n. Note that the combined set of τ- and τ'-strings in π_1 is given by $y_1, x_2, \ldots, x_r$. We then repeat the process described above on π_1 in replacing x_2 with a string y_2 that represents the other option concerning containment of τ or τ', and let π_2 denote the resulting member of NC_n. Likewise, we continue with $x_3, \ldots, x_r$, and convert them sequentially to $y_3, \ldots, y_r$, letting $\pi_3, \ldots, \pi_r$ denote the corresponding partitions that arise.

Let $f(\pi) = \pi_r$ and we show that f can be reversed. To do so, first note that the positions of the first and last letters of the strings $y_1, \ldots, y_r$ in π_r are the same as the corresponding positions within $x_1, \ldots, x_r$ in π, as they are seen to be invariant in each step of the transition from π to π_r. This follows from the fact that the first and last letters within an occurrence z of τ or τ' are the two smallest letters in z. Therefore, the inverse of f may be found by reversing each of the transitions π_i to π_{i+1} for $0 \le i \le r-1$, where $\pi_0 = \pi$, in reverse order (i.e., starting with the $i = r-1$ transition and ending with $i = 0$). Hence, we have $\mu_\tau(\pi) = \mu_{\tau'}(f(\pi))$ for all $\pi \in NC_n$ when it is assumed $b = b' = 1$.

To complete the proof, it then suffices to show $2\sigma 1^b \equiv 2^b\sigma 1$, where $b \ge 2$ and 2σ is a nonempty non-crossing partition (using the letters in $\{2, 3, \ldots\}$) such that σ starts with 3 if nonempty. To establish this equivalence, let $\pi = \pi_1 \cdots \pi_n \in NC_n$ and we consider (maximal) strings $\mathbf{p}$ within π of the form

$$\mathbf{p} = u_1^{r_1}\sigma_1 u_2^{r_2}\sigma_2 \cdots u_t^{r_t}\sigma_t u_{t+1}^{r_{t+1}},$$

where $t, r_1, \ldots, r_t \ge 1$, $r_{t+1} \ge 0$, $u_1 > u_2 > \cdots > u_t$ (with $u_t > u_{t+1}$ if $r_{t+1} > 0$ and $u_{t+1} = 1$ if $r_{r+1} = \emptyset$) and $u_i\sigma_i$ isomorphic to 2σ for $1 \le i \le t$. Note that if $r_{t+1} = 0$, then either $u_t\sigma_t$ contains the last letter of π or the successor of the final letter of $u_t\sigma_t$ is greater than or equal u_t if σ is nonempty (with the successor being strictly greater if σ is empty). Further, if $r_{t+1} > 0$, then it is understood that σ is nonempty and that the string $u_{t+1}^{r_{t+1}}$ is not directly followed by a sequence of letters α such that $u_{t+1}\alpha$ is isomorphic to 2σ. We replace each such string $\mathbf{p}$ with $\mathbf{p}'$, where

$$\mathbf{p}' = \begin{cases} u_1^{r_{t+1}}\sigma_1 u_2^{r_t}\sigma_2 \cdots u_t^{r_2}\sigma_t u_{t+1}^{r_1}, & \text{if } r_{t+1} > 0, \\ u_1^{r_t}\sigma_1 u_2^{r_{t-1}}\sigma_2 \cdots u_t^{r_1}\sigma_t u_{t+1}^{r_{t+1}}, & \text{if } r_{t+1} = 0. \end{cases}$$

Let $g(\pi)$ denote the member of NC_n that results from replacing each string $\mathbf{p}$ with $\mathbf{p}'$ as described. Then g is an involution on NC_n that replaces each occurrence of the pattern $2\sigma 1^b$ with $2^b\sigma 1$ and vice versa, which implies the desired equivalence and completes the proof. □

Remarks: When $|\tau| = |\tau'| = 3$, then the bijection f above shows $231 \equiv 221$. For example, let $\pi = 1\underline{231}1\underline{451}6\underline{78\overline{6}}\overline{61}9 \in NC_{15}$, where the occurrences of 231 and 221 are underlined and overlined, respectively. Then we have

$$\pi_0 \to \pi_1 = 1\overline{221}1\underline{341}5\underline{67\overline{5}}\overline{51}8 \to \pi_2 = 1\overline{221}1\overline{331}4\underline{56\overline{4}}\overline{41}7 \to \pi_3 = 1\overline{221}1\overline{331}4\overline{55441}6$$
$$\to \pi_4 = 1\overline{221}1\overline{331}4\overline{55\underline{4}}\underline{61}7,$$

and thus $f(\pi) = \pi_4 \in NC_{15}$. Note that π has three occurrences of 231 and one of 221, whereas $f(\pi)$ has three occurrences of 221 and one of 231. When τ and τ' are each of length three, the bijection g shows $221 \equiv 211$. For example, if $n = 12$ and $\pi = 122322114115 \in NC_{12}$, then $g(\pi) = 122332214415$. Note that π and $g(\pi)$ contain one and three and three and one occurrences respectively of 221 and 211. Finally, the mapping g is seen to preserve the number of blocks of a partition, whereas f does not in general.

In the next result, we enumerate members of NC_n with respect to a family of subword patterns generalizing 121.

Theorem 6. *Let $\tau = 1^a(\rho+1)1^b$, where $a, b \ge 1$ and ρ is the sequential representation of a non-crossing partition of length m for some $m \ge 1$. Then the generating* *function counting the members of NC_n for $n \geq 0$ according to the number of occurrences of τ is given by*

$$\frac{(1-x+(1-q)(1-x^s)x^{m+t})\left(1-\sqrt{1-\frac{4x(1-x+(1-q)(1-x^{s-1})x^{m+t})}{1-x+(1-q)(1-x^s)x^{m+t}}}\right)}{2x\,(1-x+(1-q)(1-x^{s-1})x^{m+t})},$$

where $s = \min\{a, b\}$ and $t = \max\{a, b\}$.

Proof. First assume $b \geq a > 1$. To find a formula for $F = F_\tau$ in this case, we refine F by letting F_i for $i \geq 1$ denote the restriction of F to those partitions starting with a sequence of 1's of length exactly i. Then we have $F_1 = x+x(F-1)+x(F-1)^2 = x(F^2-F+1)$, upon considering whether or not a partition enumerated by F_1 contains one or more runs of 1. By the definitions, we have $F_{i+1} = xF_i$ for all $i \neq a-1$, upon considering separately the cases $1 \leq i \leq a-2$ and $i \geq a$, since prepending an extra 1 to a member of NC_n not starting with a run of 1 of length $a-1$ does not introduce an occurrence of τ. We now write a formula for F_a. We consider the following cases on $\pi \in NC_n$ where $n \geq a$: (i) $\pi = 1^a\pi'$, where π' contains no 1's and is possibly empty, (ii) $\pi = 1^a\alpha\beta$, where α is nonempty and contains no 1's with $\alpha \neq \rho+1$ and β is nonempty starting with 1, (iii) $\pi = 1^a\alpha\beta$, where $\alpha = \rho+1$ and β is as before. Note that β in case (ii) is accounted for by $F-1$, whereas in (iii), we need

$$\begin{aligned}\sum_{i=1}^{b-1} F_i + q\sum_{i\geq b} F_i &= \sum_{i=1}^{a-1} x^{i-1}F_1 + \sum_{i=a}^{b-1} x^{i-a}F_a + q\sum_{i\geq b} x^{i-a}F_a \\ &= \frac{1-x^{a-1}}{1-x}F_1 + \frac{1+(q-1)x^{b-a}}{1-x}F_a.\end{aligned}$$

Thus, combining cases (i)–(iii), we have

$$F_a = x^aF + x^a(F-1-x^m)(F-1) + x^{m+a}\left(\frac{1-x^{a-1}}{1-x}F_1 + \frac{1+(q-1)x^{b-a}}{1-x}F_a\right),$$

which implies

$$F_a = \frac{x^a(1+x^m) + x^a(F-1-x^m)F + \frac{x^{m+a}(1-x^{a-1})}{1-x}F_1}{1-\frac{x^{m+a}(1+(q-1)x^{b-a})}{1-x}}. \tag{6}$$

We use the same cases (i)–(iii) in determining F (except that the initial run of 1's can have arbitrary length in (i) and any length $\geq a$ in (ii) and (iii)), along with an additional case where π is of the form $\pi = 1^r\alpha\beta$, wherein $1 \leq r \leq a-1$ and α and

β are nonempty with α not containing 1 and β starting with 1. This yields

$$\begin{aligned}
F &= 1 + \frac{x}{1-x}F + \frac{x-x^a}{1-x}(F-1)^2 + \frac{x^a}{1-x}(F-1-x^m)(F-1) \\
&+ \frac{x^{m+a}}{1-x}\left(\sum_{i=1}^{b-1} F_i + q\sum_{i\geq b} F_i\right) \\
&= 1 + \frac{x}{1-x}F + \frac{x-x^a}{1-x}(F-1)^2 + \frac{x^a}{1-x}(F-1-x^m)(F-1) \\
&+ \frac{x^{m+a}(1-x^{a-1})}{(1-x)^2}F_1 \\
&+ \frac{x^{m+a}(1+(q-1)x^{b-a})}{1-x} \cdot \frac{x^a(1+x^m) + x^a(F-1-x^m)F + \frac{x^{m+a}(1-x^{a-1})}{1-x}F_1}{1-x-x^{m+a}(1+(q-1)x^{b-a})},
\end{aligned} \tag{7}$$

where we have made use of (6).

Note that the F_1 coefficient in (7) may be simplified to give

$$\begin{aligned}
&\frac{x^{m+a}(1-x^{a-1})}{(1-x)^2} + \frac{x^{2(m+a)}(1-x^{a-1})(1+(q-1)x^{b-a})}{(1-x)^2(1-x-x^{m+a}(1+(q-1)x^{b-a}))} \\
&= \frac{x^{m+a}(1-x^{a-1})}{(1-x)^2}\left(1 + \frac{x^{m+a}(1+(q-1)x^{b-a})}{1-x-x^{m+a}(1+(q-1)x^{b-a})}\right) \\
&= \frac{x^{m+a}(1-x^{a-1})}{(1-x)(1-x-x^{m+a}(1+(q-1)x^{b-a}))}.
\end{aligned}$$

Thus, upon clearing fractions in (7), we have

$$\begin{aligned}
(1-x-\ell)(F-1) &= x(1-x-\ell)F^2 - x^{m+a}(1-x-\ell)(F-1) \\
&\quad + \ell x^a(1+x^m+(F-1-x^m)F) + x^{m+a}(1-x^{a-1})F_1,
\end{aligned}$$

where $\ell = x^{m+a}(1+(q-1)x^{b-a})$. By the formula for F_1, the last equation after several algebraic steps yields

$$(1-x+(1-q)(1-x^a)x^{m+b})(F-1) = x(1-x+(1-q)(1-x^{a-1})x^{m+b})F^2,$$

which leads to the stated formula for F in this case.

Now let us consider the case $a = 1$ and $b \geq 1$. By similar reasoning as above, we have

$$\begin{aligned}
F &= 1 + \frac{x}{1-x}F + \frac{x}{1-x}(F-1-x^m)(F-1) + \frac{x^{m+1}(1+(q-1)x^{b-1})}{(1-x)^2}F_1, \\
F_1 &= xF + x(F-1-x^m)(F-1) + \frac{x^{m+1}(1+(q-1)x^{b-1})}{1-x}F_1.
\end{aligned}$$

Solving this system for F gives

$$F = \frac{1 + (1-q)x^{m+b} - \sqrt{(1 + (1-q)x^{m+b})(1 - 4x + (1-q)x^{m+b})}}{2x},$$

which establishes all cases of the formula when $b \geq a \geq 1$.

By a comparable argument, one can establish the stated formula for F when $a > b \geq 1$. Alternatively, note that the formula is symmetric in a and b. Thus, to complete the proof, it suffices to define a bijection on NC_n showing that the μ_τ statistic when $\tau = 1^a(\rho+1)1^b$ has the same distribution as $\mu_{\tau'}$ for $\tau' = 1^b(\rho+1)1^a$ where $a > b \geq 1$. By a *maximal τ-string* within $\pi = \pi_1 \cdots \pi_n \in NC_n$, we mean a sequence $\mathbf{s}$ of consecutive letters of π of the form $\mathbf{s} = x^{i_1}\alpha_1 x^{i_2}\alpha_2 \cdots x^{i_r}\alpha_r x^{i_{r+1}}$, where $r, i_1, \ldots, i_{r+1} \geq 1$, each α_i is isomorphic to ρ and $x < \min\{\alpha_1 \cup \alpha_2 \cup \cdots \cup \alpha_r\}$, that is contained in no other such string of strictly greater length. Identify all maximal τ-strings $\mathbf{s}$ within π; note that the various $\mathbf{s}$ are mutually disjoint, by maximality. Within each string, replace $x^{i_1}, x^{i_2}, \ldots, x^{i_{r+1}}$ with $x^{i_{r+1}}, x^{i_r}, \ldots, x^{i_1}$ (i.e., reverse the order of the x-runs), leaving the α_i unchanged. Let $\pi' \in NC_n$ denote the partition that results from performing this operation on all maximal τ-strings $\mathbf{s}$; note that $\pi \mapsto \pi'$ is an involution and hence bijective. Since any occurrence of τ must lie within some $\mathbf{s}$, the mapping $\pi \mapsto \pi'$ implies the desired equivalence of distributions and completes the proof. □

Theorem 6 yields the following formula for the total number of occurrences of $1^a(\rho+1)1^b$.

Corollary 7. *If $n \geq m + a + b - 1$, then the total number of occurrences of $\tau = 1^a(\rho+1)1^b$ as described above within all the members of NC_n is given by $\binom{2r}{r+1}$, where $r = n - m - a - b + 1$.*

Remarks: When $s = 1$ in Theorem 6, the formula for $F = F_\tau$ may be simplified further to give

$$F = \frac{1 + (1-q)x^{a+m} - \sqrt{(1 + (1-q)x^{a+m})(1 - 4x + (1-q)x^{a+m})}}{2x},$$

where $\tau = 1^a(\rho+1)1$ or $1(\rho+1)1^a$ and $a \geq 1$. Note that there is really no loss of generality in assuming ρ is a sequential representation of some (non-crossing) partition in the hypotheses for Theorem 6 above. This is because if the first occurrence of some letter c in ρ precedes the first occurrence of d with $c > d$, then containment of $\tau = 1^a(\rho+1)1^b$ by a partition π would imply an occurrence of 1-2-1-2 of the form y-z-y-z, where z corresponds to the $d+1$ in $\rho+1$ and y to the 1 of τ. In addition to implying the symmetry in a and b of the pattern τ, the formula in Theorem 6 shows that $\tau = 1^a(\rho+1)1^b$ is equivalent to $\tau' = 1^a(\rho'+1)1^{b'}$ of the same length, where ρ' denotes a nonempty non-crossing partition and $a \leq \min\{b, b'\}$. For example, when $|\tau| = 4$, we have $1121 \equiv 1211 \equiv 1221 \equiv 1231$ as subwords on NC_n. A bijective proof of $1^a(\rho+1)1^b \equiv 1^a(\rho'+1)1^{b'}$ can be obtained by modifying somewhat the mapping f described above, the details of which we leave to the interested reader.

3. THE SUBWORDS $12\cdots(m-1)m^a$ AND $1^a23\cdots m$

Let $\tau = 12\cdots(m-1)m^a$, where $a, m \geq 2$. To aid in enumerating the members of NC_n with respect to occurrences of τ, we consider the joint distribution with a further parameter on NC_n that was introduced in [**9**]. Given $\pi = \pi_1\cdots\pi_n \in NC_n$, excluding the increasing partition $12\cdots n$, let $\text{rep}(\pi)$ denote the smallest repeated letter of π. Below, we will find, more generally, the generating function for the joint distribution $\sum_{\pi\in NC_n} v^{\text{rep}(\pi)}q^{\mu_\tau(\pi)}$, where $\text{rep}(12\cdots n)$ is defined to be zero.

Let $NC_{n,i}$ for $1 \leq i \leq n-1$ denote the subset of NC_n whose members have smallest repeated letter i. Define $a(n,i) = \sum_{\pi\in NC_{n,i}} q^{\mu_\tau(\pi)}$ for $n \geq 2$ and $1 \leq i \leq n-1$ and $a(n) = \sum_{\pi\in NC_n} q^{\mu_\tau(\pi)}$ for $n \geq 1$, with $a(0) = 1$.

To aid in finding recurrences for $a(n)$ and $a(n,i)$, we consider a generalization of μ_τ as follows. Given $\ell \geq 0$ and a partition π, let $\mu_\tau^{(\ell)}(\pi)$ denote the number of occurrences of τ in the sequence $12\cdots\ell(\pi+\ell)$. Define

$$a^{(\ell)}(n,i) = \sum_{\pi\in NC_{n,i}} q^{\mu_\tau^{(\ell)}(\pi)}, \qquad n \geq 2 \text{ and } 1 \leq i \leq n-1,$$

and

$$a^{(\ell)}(n) = \sum_{\pi\in NC_n} q^{\mu_\tau^{(\ell)}(\pi)}, \qquad n \geq 1,$$

with $a^{(\ell)}(0) = 1$. Note that $a^{(0)}(n,i) = a(n,i)$ and $a^{(0)}(n) = a(n)$ for all n and i.

We have the following system of recurrences satisfied by the $a^{(\ell)}(n,i)$ and $a^{(\ell)}(n)$.

Lemma 8. *If $n \geq a$ and $1 \leq i \leq n-a+1$, then*

$$\begin{aligned} a^{(\ell)}(n,i) = {} & \sum_{j=i+1}^{n} a^{(\ell+i)}(j-i-1)a^{(0)}(n-j+1) \\ & + \begin{cases} 0, & \text{if } i+\ell \leq m-1, \\ (q-1)a^{(0)}(n-i-a+2), & \text{if } i+\ell \geq m, \end{cases} \end{aligned} \tag{8}$$

for all $\ell \geq 0$. Furthermore, we have

$$a^{(\ell)}(n) = C_{a-1} + \sum_{i=1}^{n-a+1} a^{(\ell)}(n,i), \qquad n \geq a, \tag{9}$$

with $a^{(\ell)}(n) = C_n$ for $0 \leq n \leq a-1$.

Proof. Since $a^{(\ell)}(0) = 1$ for all $\ell \geq 0$, formula (8) is equivalent to

$$\begin{aligned} a^{\ell}(n,i) = {} & \sum_{j=i+2}^{n} a^{(\ell+i)}(j-i-1)a^{(0)}(n-j+1) \\ & + \begin{cases} a^{(0)}(n-i), & \text{if } i+\ell \leq m-1, \\ a^{(0)}(n-i) + (q-1)a^{(0)}(n-i-a+2), & \text{if } i+\ell \geq m, \end{cases} \end{aligned} \tag{10}$$

which we will now show. To do so, first consider the position j of the second occurrence of i within $\pi \in NC_{n,i}$. If $j \geq i+2$, such π are expressible as $\pi = 12\cdots i\alpha i\beta$, where α is nonempty and contains no letters i and β is possibly empty. Then we get $a^{(\ell+i)}(j-i-1)a^{(0)}(n-j+1)$ possibilities and summing over all $j \geq i+2$ yields the first part of (10) in either case. So assume $j = i+1$ and first suppose $i+\ell \leq m-1$. Then there is no occurrence of τ in π involving any of its first $i-1$ letters, regardless of the length of the leftmost run of i's, which implies a contribution of $a^{(0)}(n-i)$ and hence the first case of (10). If $i+\ell \geq m$, then we consider cases based on the length of the leftmost run of i's as follows. Suppose first that π is expressible as $\pi = 12\cdots(i-1)i^r\pi'$, where π' does not start with i and $2 \leq r \leq a-1$, assuming for now $a \geq 3$. Then, by subtraction, there are $a^{(0)}(n-i-r+2) - a^{(0)}(n-i-r+1)$ possibilities and summing over all r gives

$$\sum_{r=2}^{a-1}(a^{(0)}(n-i-r+2) - a^{(0)}(n-i-r+1)) = a^{(0)}(n-i) - a^{(0)}(n-i-a+2).$$

On the other hand, if $\pi = 12\cdots(i-1)i^r\pi'$, where $r \geq a$, then $i+\ell \geq m$ implies that there is an occurrence of τ involving the first $i+a-1$ letters of π (when taken together with the understood suffix $12\cdots\ell$ consisting of strictly smaller letters). Then the sequence $i^{r-a+1}\pi'$ corresponds to a partition enumerated by $a^{(0)}(n-i-a+2)$, as it is directly preceded by at least one i, and hence the contribution towards the overall weight in this case is given by $qa^{(0)}(n-i-a+2)$. Combining this case with the previous yields the second part of (10) when $i+\ell \geq m$ and completes the proof of (10).

For (9), first note that the initial conditions when $0 \leq n \leq a-1$ are apparent since no occurrence of τ is possible for such n for all ℓ. Suppose k is the smallest repeated letter in $\pi \in NC_n$. If $1 \leq k \leq n-a+1$, then π is accounted for by the sum in (9), by the definitions. Otherwise, π can be represented as $\pi = 12\cdots(n-a+1)\pi'$, where π' contains no letters in $[n-a+1]$, for which there are C_{a-1} possibilities since no such π can contain an occurrence of τ (as the m^a part of τ cannot be achieved by any letter in π'). Combining this with the prior case yields (9). □

Define

$$A(x,u) = \sum_{n\geq 0}\sum_{\ell\geq 0} a^{(\ell)}(n)u^\ell x^n$$

and

$$A(x,u,v) = \sum_{n\geq a}\sum_{\ell\geq 0}\sum_{i=1}^{n-a+1} a^{(\ell)}(n,i)u^\ell v^{i-1}x^n.$$

Rewriting the recurrences in Lemma 8 in terms of generating functions yields the following system of functional equations.

Lemma 9. *We have*

$$A(x,u) = A(x,u,1) + \frac{x^aC_{a-1}}{(1-u)(1-x)} + L(x,u), \tag{11}$$

$$A(x,u,v) = \frac{x(A(x,0)-1)(A(x,vx)-A(x,u))}{vx-u} - M(x,u,v)$$

$$+\frac{(q-1)x^{a-1}(A(x,0)-1)(u^m(1-vx)-(vx)^m(1-u))}{(1-u)(1-vx)(u-vx)}, \tag{12}$$

where $L(x,u)=\frac{1}{1-u}\sum_{j=0}^{a-1}C_jx^j$ *and*

$$M(x,u,v)=\begin{cases}0, & \textit{if } a=2,\\ \frac{x}{(1-u)(1-vx)}\sum_{j=0}^{a-3}\sum_{i=1}^{a-2-j}C_iC_jx^{i+j}, & \textit{if } a\geq 3.\end{cases}$$

Proof. Multiplying both sides of (9) by $u^\ell x^n$, and summing over $n\geq a$ and $\ell\geq 0$, yields

$$\begin{aligned}A(x,u) &= \sum_{n\geq a}\sum_{\ell\geq 0}\sum_{i=1}^{n-a+1}a^{(\ell)}(n,i)u^\ell x^n + \sum_{n\geq a}\sum_{\ell\geq 0}C_{a-1}u^\ell x^n + \sum_{n=0}^{a-1}\sum_{\ell\geq 0}a^{(\ell)}(n)u^\ell x^n\\ &= A(x,u,1)+\frac{x^aC_{a-1}}{(1-u)(1-x)}+\frac{1}{1-u}\sum_{j=0}^{a-1}C_jx^j,\end{aligned}$$

which gives (11), by the initial values for $a^{(\ell)}(n)$.

To rewrite (8) in terms of generating functions, we first must find

$$\sum_{n\geq a}\sum_{\ell\geq 0}\sum_{i=1}^{n-a+1}\sum_{j=i+1}^{n}a^{(\ell+i)}(j-i-1)a^{(0)}(n-j+1)u^\ell v^{i-1}x^n.$$

First observe the following manipulation of sums:

$$\begin{aligned}\sum_{n\geq a}\sum_{\ell\geq 0}\sum_{i=1}^{n-a+1}\sum_{j=i+1}^{n}(\ldots) &= \sum_{\ell\geq 0}\sum_{i\geq 1}\sum_{j\geq i+1}\sum_{n\geq \max\{j,i+a-1\}}(\ldots)\\ &= \sum_{\ell\geq 0}\sum_{i\geq 1}\sum_{j=i+1}^{i+a-1}\sum_{n\geq i+a-1}(\ldots)+\sum_{\ell\geq 0}\sum_{i\geq 1}\sum_{j\geq i+a}\sum_{n\geq j}(\ldots),\end{aligned}$$

where $(\ldots)$ denotes the original summand above. Replacing j with $j+i+1$ in both sums in the last expression implies

$$\begin{aligned}&\sum_{n\geq a}\sum_{\ell\geq 0}\sum_{i=1}^{n-a+1}\sum_{j=i+1}^{n}a^{(\ell+i)}(j-i-1)a^{(0)}(n-j+1)u^\ell v^{i-1}x^n\\ &=\sum_{\ell\geq 0}\sum_{i\geq 1}\sum_{j=0}^{a-2}\sum_{n\geq i+a-1}a^{(\ell+i)}(j)a^{(0)}(n-j-i)u^\ell v^{i-1}x^n\\ &\quad+\sum_{\ell\geq 0}\sum_{i\geq 1}\sum_{j\geq a-1}\sum_{n\geq j+i+1}a^{(\ell+i)}(j)a^{(0)}(n-j-i)u^\ell v^{i-1}x^n\end{aligned}$$

$$
\begin{aligned}
&= \sum_{\ell\ge 0}\sum_{i\ge 1}\sum_{j=0}^{a-2}\sum_{n\ge a-1-j} a^{(\ell+i)}(j)a^{(0)}(n)u^{\ell}v^{i-1}x^{n+i+j} \\
&\quad + \sum_{\ell\ge 0}\sum_{i\ge 1}\sum_{j\ge a-1}\sum_{n\ge 1} a^{(\ell+i)}(j)a^{(0)}(n)u^{\ell}v^{i-1}x^{n+i+j} \\
&= \sum_{\ell\ge 0}\sum_{i\ge 1}\sum_{j=0}^{a-2} a^{(\ell+i)}(j)u^{\ell}v^{i-1}x^{i+j}\left(\sum_{n\ge 1} a^{(0)}(n)x^{n} - \sum_{n=1}^{a-2-j} a^{(0)}(n)x^{n}\right) \\
&\quad + \sum_{\ell\ge 0}\sum_{i\ge 1}\sum_{j\ge a-1} a^{(\ell+i)}(j)u^{\ell}v^{i-1}x^{i+j}\sum_{n\ge 1} a^{(0)}(n)x^{n} \\
&= \sum_{\ell\ge 0}\sum_{i\ge 1}\sum_{j\ge 0} a^{(\ell+i)}(j)u^{\ell}v^{i-1}x^{i+j}\sum_{n\ge 1} a^{(0)}(n)x^{n} \\
&\quad - \sum_{\ell\ge 0}\sum_{i\ge 1}\sum_{j=0}^{a-2} a^{(\ell+i)}(j)u^{\ell}v^{i-1}x^{i+j}\sum_{n=1}^{a-2-j} a^{(0)}(n)x^{n} \\
&= (A(x,0)-1)\sum_{i\ge 1}\sum_{\ell\ge i}\sum_{j\ge 0} a^{(\ell)}(j)u^{\ell-i}v^{i-1}x^{i+j} \\
&\quad - \frac{x}{(1-u)(1-vx)}\sum_{j=0}^{a-3} C_j x^{j}\sum_{n=1}^{a-2-j} C_n x^{n} \\
&= \frac{x(A(x,0)-1)}{u-vx}\sum_{\ell\ge 1}\sum_{j\ge 0} a^{(\ell)}(j)x^{j}(u^{\ell}-(vx)^{\ell}) - M(x,u,v) \\
&= \frac{x(A(x,0)-1)(A(x,vx)-A(x,u))}{vx-u} - M(x,u,v).
\end{aligned}
$$

For converting the second part of formula (8), we consider cases on i. Omitting the factor $q-1$, this yields

$$
\begin{aligned}
&\sum_{i=1}^{m-1}\sum_{\ell\ge m-i}\sum_{n\ge i+a-1} a^{(0)}(n-i-a+2)u^{\ell}v^{i-1}x^{n} \\
&\quad + \sum_{i\ge m}\sum_{\ell\ge 0}\sum_{n\ge i+a-1} a^{(0)}(n-i-a+2)u^{\ell}v^{i-1}x^{n} \\
&= (A(x,0)-1)\sum_{i=1}^{m-1}\sum_{\ell\ge m-i} u^{\ell}v^{i-1}x^{i+a-2} + (A(x,0)-1)\sum_{i\ge m}\sum_{\ell\ge 0} u^{\ell}v^{i-1}x^{i+a-2} \\
&= (A(x,0)-1)\left(\frac{ux^{a-1}(u^{m-1}-(vx)^{m-1})}{(1-u)(u-vx)} + \frac{v^{m-1}x^{a+m-2}}{(1-u)(1-vx)}\right) \\
&= \frac{x^{a-1}(A(x,0)-1)(u^{m}(1-vx)-(vx)^{m}(1-u))}{(1-u)(1-vx)(u-vx)}.
\end{aligned}
$$

Combining the two contributions to the generating function above yields (12). □

Theorem 10. *Let $y = A(x,0)$ denote the generating function counting members of NC_n for $n \geq 0$ according to the number of occurrences of $\tau = 12\cdots(m-1)m^a$, where $a, m \geq 2$. Then y satisfies the polynomial equation*

$$xy^2 - y + 1 + (q-1)x^{a+m-2}y^{m-1}(y-1) = 0. \tag{13}$$

More generally, the generating function counting members of NC_n jointly according to the smallest repeated letter and number of occurrences of τ (marked by v and q, respectively) is given by $vA(x,0,v) + \frac{1}{1-x}$ if $a = 2$ and by

$$vA(x,0,v) + \frac{1}{1-x} + \frac{v^2x^aC_{a-1} - v^2x^2}{1-vx} + v\sum_{i=2}^{a-1} C_ix^i + (v-1)\sum_{n\geq 2}\sum_{j=r}^{n-1} C_{n-j}x^j,$$

if $a \geq 3$, where $r = \max\{1, n-a+2\}$ and

$$A(x,0,v) = \frac{(y-1)(A(x,vx)-y)}{v} - M(x,0,v) + \frac{(q-1)v^{m-1}x^{a+m-2}(y-1)}{1-vx}, \tag{14}$$

with $A(x,u)$ given by (17).

Proof. We first find an equation satisfied by y. Note that (12) at $u = 0$ and $v = 1$, taken together with (11), gives

$$y^2 = (y-1)A(x,x) - M(x,0,1) + \frac{(q-1)x^{a+m-2}(y-1)}{1-x} + \frac{x^aC_{a-1}}{1-x} + L(x,0). \tag{15}$$

We apply the kernel method to (12) to obtain an expression for $A(x,x)$. Taking $u = xA(x,0) = xy$ and $v = 1$ in (12) implies

$$\begin{aligned} A(x,x) = {} & \frac{(q-1)x^{a-2}((xy)^m(1-x) - x^m(1-xy))}{(1-x)(1-xy)} - M(x,xy,1) \\ & + \frac{x^aC_{a-1}}{(1-x)(1-xy)} + L(x,xy). \end{aligned} \tag{16}$$

Now observe

$$\sum_{j=0}^{a-3}\sum_{i=1}^{a-2-j} C_iC_jx^{i+j} = \sum_{i=1}^{a-2} x^i \sum_{j=0}^{i-1} C_{i-j}C_j = \sum_{i=0}^{a-2} x^i(C_{i+1} - C_i),$$

by the recurrence for Catalan numbers. Thus, the right-hand side of (15) may be written as

$$\frac{(q-1)x^{a-2}((xy)^m(1-x)-x^m(1-xy))(y-1)}{(1-x)(1-xy)}+\frac{y-1}{1-xy}\sum_{j=0}^{a-1}C_jx^j$$
$$+\frac{x(y-1)}{(1-x)(1-xy)}\left(x^{a-1}C_{a-1}-\sum_{i=0}^{a-2}x^i(C_{i+1}-C_i)\right)$$
$$-\frac{x}{1-x}\sum_{i=0}^{a-2}x^i(C_{i+1}-C_i)+\frac{(q-1)x^{a+m-2}(y-1)}{1-x}+\frac{x^aC_{a-1}}{1-x}+\sum_{j=0}^{a-1}C_jx^j$$
$$=\frac{(q-1)x^{a+m-2}y^m(y-1)}{1-xy}-\frac{xy}{1-xy}\sum_{i=0}^{a-2}x^i(C_{i+1}-C_i)+\frac{x^ayC_{a-1}}{1-xy}$$
$$+\frac{y(1-x)}{1-xy}\sum_{j=0}^{a-1}C_jx^j$$
$$=\frac{y}{1-xy}\Bigg((q-1)x^{a+m-2}y^{m-1}(y-1)-\sum_{i=1}^{a-1}C_ix^i+x\sum_{i=0}^{a-2}C_ix^i+x^aC_{a-1}$$
$$+(1-x)\sum_{j=0}^{a-1}C_jx^j\Bigg)$$
$$=\frac{(q-1)x^{a+m-2}y^m(y-1)+y}{1-xy}.$$

Equating this last expression with y^2 then leads to (13). Solving for $A(x,u)$ in (12) at $v=1$, making use of (11), gives

$$A(x,u)=\frac{x-u}{xy-u}\Bigg(\frac{x^aC_{a-1}}{(1-u)(1-x)}+\frac{(1-q)x^{a-1}(u^m(1-x)-x^m(1-u))(y-1)}{(1-u)(1-x)(x-u)}$$
$$+\frac{x(y-1)}{x-u}A(x,x)-M(x,u,1)+L(x,u)\Bigg), \tag{17}$$

where $A(x,x)$ is given by (16). Letting $u=0$ in (12) now leads to (14). Finally, taking into account the v-weights of members of NC_n having smallest repeated letter i where $r\le i\le n-1$, along with the increasing partition (which has weight 1 for all $n\ge 0$), implies the generating function enumerating members of NC_n for $n\ge 0$ jointly according to the rep value and number of occurrences of τ is given by $vA(x,0,v)+\frac{1}{1-x}$ if $a=2$ and by

$$vA(x,0,v)+\frac{1}{1-x}+\sum_{n=2}^{a-1}x^n\sum_{j=1}^{n}v^j(C_{n-j+1}-C_{n-j})$$
$$+\sum_{n\ge a}x^n\sum_{k=n-a+2}^{n}v^k(C_{n-k+1}-C_{n-k}),$$

if $a \geq 3$. Rewriting the last expression somewhat yields the stated formula for the joint generating function and completes the proof. □

Corollary 11. *If $n \geq a+m-1$, then the total number of occurrences of $\tau = 1\cdots(m-1)m^a$ within all the members of NC_n is given by $\frac{r}{2r+m}\binom{2r+m}{r}$, where $r = n-a-m+2$.*

Proof. Let $C = C(x)$, $F = A(x,0)$ and $D = \frac{\partial F}{\partial q}\mid_{q=1}$. Differentiating both sides of (13) with respect to q, and noting $F\mid_{q=1} = C$, yields

$$2xCD - D + x^{a+m-2}C^{m-1}(C-1) = 0,$$

i.e.,

$$D = \frac{x^{a+m-2}C^{m-1}(C-1)}{1-2xC} = \frac{x^{a+m-2}C^{m-1}(C-1)}{\sqrt{1-4x}}.$$

Extracting the coefficient of x^n for $n \geq a+m-1$, and making use of [**14**, Eqn. 2.5.15], then gives

$$\begin{aligned}[x^n]D &= \binom{2r+m}{r} - \binom{2r+m-1}{r} = \left(1 - \frac{r+m}{2r+m}\right)\binom{2r+m}{r}\\ &= \frac{r}{2r+m}\binom{2r+m}{r}.\end{aligned}$$

□

Remark: The $m=2$ and $m=3$ cases of the formula $\frac{r}{2r+m}\binom{2r+m}{r}$ from Corollary 11 coincide respectively with sequences A002054 and A002694 in [**13**].

We conclude with the following equivalence between τ and $1^a 23\cdots m$.

Theorem 12. *We have $1^a 23\cdots m \equiv 12\cdots(m-1)m^a$ as subwords on NC_n for all $a, m \geq 2$.*

Proof. We provide a bijective proof of this result. Suppose that the descents from left to right within $\pi = \pi_1\cdots\pi_n \in NC_n$ correspond to the letters $a_i > b_i$ for $1 \leq i \leq r$ and some $r \geq 0$. Let ρ_1 denote the section of π to the left of and including a_1 and ρ_{r+1} the section to the right of and including b_r (if $r = 0$, then ρ_1 comprises all of π). If $r \geq 2$, then let ρ_i for $2 \leq i \leq r$ denote the subsequence of π starting with b_{i-1} and ending with a_i. Note that ρ_i for each i is weakly increasing, as it consists of the letters between consecutive descents of π (or occurring prior to the first or after the last descent of π).

Suppose that the descent bottom letters $b_1,\ldots,b_r$ within π are given, with $b_0 = 1$. Let section ρ_i of π for $1 \leq i \leq r+1$ be represented sequentially as $\rho_i = s_0^{(i)}s_1^{(i)}\cdots s_{t_i}^{(i)}$, where $s_0^{(i)} = b_{i-1}$. Define the binary sequence $\mathbf{d^{(i)}} = d_1^{(i)}d_2^{(i)}\cdots d_{t_i}^{(i)}$, where $d_k^{(i)} = 1$ if $s_k^{(i)} > s_{k-1}^{(i)}$ and $d_k^{(i)} = 0$ if $s_k^{(i)} = s_{k-1}^{(i)}$ for $1 \leq k \leq t_i$. Note that π non-crossing implies it is uniquely determined by its descent bottoms $b_1,\ldots,b_r$, taken together with its complete set of associated binary sequences $\mathbf{d^{(1)}},\ldots,\mathbf{d^{(r+1)}}$.

Let π' be the uniquely determined member of NC_n whose descent bottoms are the same as those of π (i.e., are given by $b_1, \ldots, b_r$) and whose associated binary sequences are given by $\mathrm{rev}(\mathbf{d^{(1)}}), \ldots, \mathrm{rev}(\mathbf{d^{(r+1)}})$, where $\mathrm{rev}(s)$ denotes the reversal of a sequence s. Note that the section ρ_i' of π' corresponding to ρ_i in π for $1 \le i \le r+1$ will have the same set of distinct letters for all i, and thus π' will have the same ascent tops as π. Also, an occurrence of $1^a 23 \cdots m$ or $12 \cdots (m-1)m^a$ within some section ρ_i of π will result in an occurrence of the other pattern within ρ_i' of π', and vice versa. Further, an occurrence of either pattern must lie completely within a section ρ_i of π or ρ_i' of π', as neither contains a descent. Since the mapping $\pi \mapsto \pi'$ is an involution on NC_n, and hence bijective, the desired equivalence of patterns follows. □

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