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SHARP SHAFER-FINK, LAZAREVIĆ AND CUSA TYPE INEQUALITIES FOR THE GUDERMANNIAN AND INVERSE GUDERMANNIAN FUNCTIONS

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In this paper, we present sharp Shafer-Fink, Lazarević and Cusa type inequalities for the Gudermannian and inverse Gudermannian functions.

1. INTRODUCTION

For $0 \le x \le 1$, the following double inequality holds:

$$\frac{3x}{2+\sqrt{1-x^2}} \le \arcsin x \le \frac{\pi x}{2+\sqrt{1-x^2}}. \tag{1}$$

The left-hand side inequality was presented by Shafer (see, *e.g.*, [22, p. 247]), while the right-hand side inequality was established by Fink [10]. Shafer-Fink's inequalities have attracted much interest of many mathematicians and have motivated a large number of papers involving various generalizations and improvements [11, 15–18, 31, 32, 41, 42]. Zhu [43] provided a solution to an open problem posed by Oppenheim in [30], and deduced some Shafer-Fink inequalities from the solution of Oppenheim's problem. Chen and Cheung [8] gave a concise proof to Oppenheim's problem.

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The double inequality

$$\frac{3x}{1+2\sqrt{1+x^2}} < \arctan x < \frac{2x}{1+\sqrt{1+x^2}} \tag{2}$$

holds for $x > 0$. The left-hand side inequality is due to Shafer [35, 36], while the right-hand side inequality can be found in, *e.g.*, [13, p. 288]. Shafer's inequality (2) was improved and generalized in [25, 28, 33, 37–41].

Shafer [37–39] proved that for $0 < x < \sqrt{15}/4$,

$$\operatorname{arctanh} x < \frac{8x}{3+\sqrt{25-\frac{80}{3}x^2}}. \tag{3}$$

Zhu [44] provided an alternative proof of (3) in a concise way. The proof in [44] contains a small mistake. Bagul and Dhaigude [6] corrected this mistake and gave alternative proofs of (3).

The Gudermann function $\operatorname{gd}(x)$ and its inverse function $\operatorname{gd}^{-1}(x)$ are defined by (see [29, 4.23 (viii)])

$$\begin{aligned}\operatorname{gd}(x) &= \int_0^x \operatorname{sech} t\,\mathrm{d}t = 2\arctan(e^x) - \frac{\pi}{2}\\ &= \arcsin(\tanh x) = \operatorname{arccsc}(\coth x) = \arccos(\operatorname{sech} x)\\ &= \operatorname{arcsec}(\cosh x) = \arctan(\sinh x) = \operatorname{arccot}(\operatorname{csch} x), \qquad x \in \mathbb{R}\end{aligned}$$

and

$$\begin{aligned}\operatorname{gd}^{-1}(x) &= \int_0^x \sec t\,\mathrm{d}t = \ln\tan\left(\frac{1}{2}x + \frac{1}{4}\pi\right) = \ln(\sec x + \tan x)\\ &= \operatorname{arcsinh}(\tan x) = \operatorname{arccsch}(\cot x) = \operatorname{arccosh}(\sec x)\\ &= \operatorname{arcsech}(\cos x) = \operatorname{arctanh}(\sin x) = \operatorname{arccoth}(\csc x), \qquad |x| < \frac{\pi}{2},\end{aligned}$$

respectively.

Neuman [26] obtained five Wilker and Huygens-type inequalities involving the Gudermannian and inverse Gudermannian functions. For example, Neuman [26, Theorem 5] proved the following result:

Let $\alpha, \beta > 0$. Then the inequality

$$\alpha + \beta < \alpha\left(\frac{\operatorname{gd}(x)}{x}\right)^p + \beta\left(\frac{\operatorname{gd}^{-1}(x)}{x}\right)^q, \qquad 0 < |x| < \frac{\pi}{2} \tag{4}$$

holds true provided

$$q > 0 \quad \text{and} \quad p\alpha \le q\beta.$$

In this paper, we present sharp Shafer-Fink type inequalities for the Gudermannian and inverse Gudermannian functions (Theorems 4 and 6). Also, we establish sharp Lazarević and Cusa type inequalities for the Gudermannian and inverse Gudermannian functions (Theorems 8, 12 and 13).

2. LEMMAS

The following lemmas are needed in our present investigation.

Lemma 1 ([2–4]). *Let $-\infty < a < b < \infty$, and $f, g : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable in (a, b). Suppose $g' \neq 0$ on (a, b). If $f'(x)/g'(x)$ is increasing (decreasing) on (a, b), then so are*

$$\frac{f(x) - f(a)}{g(x) - g(a)} \quad \text{and} \quad \frac{f(x) - f(b)}{g(x) - g(b)}.$$

If $f'(x)/g'(x)$ is strictly monotone, then the monotonicity in the conclusion is also strict.

In what follows, $\mathbb{N}$ represents the set of positive integers and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. The Euler numbers E_n $(n \in \mathbb{N}_0)$ are defined by the generating function:

$$\frac{2e^z}{e^{2z} + 1} = \operatorname{sech} z = \sum_{n=0}^{\infty} E_n \frac{z^n}{n!}, \qquad |z| < \frac{\pi}{2}. \tag{5}$$

First eight Euler numbers with even indices are

$$E_0 = 1, \quad E_2 = -1, \quad E_4 = 5, \quad E_6 = -61, \quad E_8 = 1385,$$
$$E_{10} = -50521, \quad E_{12} = 2702765, \quad E_{14} = -199360981,$$

while those with odd index are zero:

$$E_{2n+1} = 0, \qquad n \in \mathbb{N}_0.$$

Noting that $(-1)^n E_{2n} > 0$ $(n \in \mathbb{N}_0)$, (5) can be written as

$$\operatorname{sech} z = \frac{2e^z}{e^{2z} + 1} = \sum_{n=0}^{\infty} E_{2n} \frac{z^{2n}}{(2n)!} = \sum_{n=0}^{\infty} \frac{(-1)^n |E_{2n}|}{(2n)!} z^{2n}, \qquad |z| < \frac{\pi}{2}. \tag{6}$$

We then obtain from (6) that

$$\operatorname{gd}(x) = \int_0^x \operatorname{sech} t \, \mathrm{d}t = \sum_{n=0}^{\infty} \frac{(-1)^n |E_{2n}|}{(2n+1)!} x^{2n+1}, \qquad |x| < \frac{\pi}{2}. \tag{7}$$

Lemma 2. *The secant function has the power series expansion (see [29, p. 117]):*

$$\sec z = \sum_{n=0}^{\infty} \frac{|E_{2n}|}{(2n)!} z^{2n}, \qquad |z| < \frac{\pi}{2}. \tag{8}$$

We obtain from (8) that

$$\operatorname{gd}^{-1}(x) = \int_0^x \sec t \, \mathrm{d}t = \sum_{n=0}^{\infty} \frac{|E_{2n}|}{(2n+1)!} x^{2n+1}, \qquad |x| < \frac{\pi}{2}. \tag{9}$$

Lemma 3. *For all* $n \in \mathbb{N}_0$,

$$\frac{4^{n+1}}{\pi^{2n+1}}\left(\frac{1}{1+3^{-1-2n}}\right) < \frac{|E_{2n}|}{(2n)!} < \frac{4^{n+1}}{\pi^{2n+1}}. \tag{10}$$

The inequality (10) can be found in [1, p. 805].

The numerical values given in this paper have been calculated by using the computer program MAPLE 11.

3. SHAFER-FINK TYPE INEQUALITIES

In view of (1), (2) and (3), we introduce the approximations family

$$\frac{\operatorname{gd}(x)}{x} \approx \frac{p}{q+\sqrt{1+rx^2}}, \qquad x \to 0, \tag{11}$$

and $p, q, r \in \mathbb{R}$ are parameters. We are interested in finding fixed parameters p, q and r such that the function

$$R(x) = \frac{\operatorname{gd}(x)}{x} - \frac{p}{q+\sqrt{1+rx^2}}$$

converges as fast as possible to zero, which produces the best approximation among all approximations given by (11). By using the computer program MAPLE 11, we write $R(x)$ as power series

$$\begin{aligned} R(x) =& \frac{q-p+1}{q+1} + \frac{-q^2-2q-1+3pr}{6(q+1)^2}x^2 \\ &+ \frac{q^3+3q^2+3q-3pr^2q-9pr^2+1}{24(q+1)^3}x^4 + O(x^6). \end{aligned} \tag{12}$$

This produces the best approximation from (12):

$$\begin{cases} q-p+1=0 \\ -q^2-2q-1+3pr=0 \\ q^3+3q^2+3q-3pr^2q-9pr^2+1, \end{cases}$$

that is,

$$p=1, \quad q=0, \quad r=\frac{1}{3}.$$

We then find the best approximation near the origin:

$$\frac{\operatorname{gd}(x)}{x} \approx \frac{1}{\sqrt{1+\frac{1}{3}x^2}}. \tag{13}$$

In view of (13) it is natural to ask: What is the smallest number λ and what is the largest number μ such that the inequality

$$\frac{1}{\sqrt{1+\lambda x^2}} < \frac{\mathrm{gd}(x)}{x} < \frac{1}{\sqrt{1+\mu x^2}},$$

holds for $x > 0$? Theorem 4 answers this question.

Theorem 4. *For $x > 0$,*

$$\frac{1}{\sqrt{1+\left(\frac{2}{\pi}\right)^2 x^2}} < \frac{\mathrm{gd}(x)}{x} < \frac{1}{\sqrt{1+\frac{1}{3}x^2}}, \tag{14}$$

where the constants $\left(\frac{2}{\pi}\right)^2$ and $\frac{1}{3}$ are the best possible.

Proof. Inequality (14) can be written as

$$\frac{1}{3} < f(x) < \left(\frac{2}{\pi}\right)^2, \qquad x > 0,$$

where

$$f(x) = \frac{\left(\frac{x}{\mathrm{gd}(x)}\right)^2 - 1}{x^2} = \frac{1}{[\mathrm{gd}(x)]^2} - \frac{1}{x^2} = \frac{1}{[\arctan(\sinh x)]^2} - \frac{1}{x^2}.$$

Direct computation gives

$$\lim_{x\to 0} f(x) = \frac{1}{3} \quad \text{and} \quad \lim_{x\to\infty} f(x) = \left(\frac{2}{\pi}\right)^2 = 0.40528473\ldots.$$

In order to prove Theorems 4, it suffices to show that $f(x)$ is strictly increasing for $x > 0$. Differentiating $f(x)$ with respect to x yields

$$\frac{1}{2}x^3\,[\arctan(\sinh x)]^3\, f'(x) = [\arctan(\sinh x)]^3 - \frac{x^3}{\cosh x}.$$

We now prove that $f'(x) > 0$ for $x > 0$, it suffices to show that

$$g(x) = \arctan(\sinh x) - \frac{x}{(\cosh x)^{1/3}} > 0.$$

Elementary calculations yield

$$\frac{3(\cosh x)^{4/3}}{\sinh x} g'(x) = \frac{3(\cosh x)^{1/3} - 3\cosh x}{\sinh x} + x =: h(x),$$

$$h'(x) = \frac{\left((\cosh x)^{2/3}+1\right)\left((\cosh x)^{2/3}-1\right)^3}{\sinh^2 x(\cosh x)^{2/3}} > 0.$$

Hence, $h(x)$ is strictly increasing for $x > 0$, and we have, for $x > 0$,

$$h(x) > \lim_{t\to 0} h(t) = 0 \Longrightarrow g'(x) > 0 \Longrightarrow g(x) > \lim_{t\to 0} g(t) = 0 \Longrightarrow f'(x) > 0.$$

Therefore, the function $f(x)$ is strictly increasing for $x > 0$. The proof of Theorem 4 is complete. □

In relation to the previous theorem, we can consider the family of functions $\{\varphi_p(x)\}_{p\in P}$, $P = [0, \infty)$, for $x \in (0, \infty)$, defined by

$$\varphi_p(x) = \frac{\mathrm{gd}(x)}{x} - \frac{1}{\sqrt{1 + p\,x^2}}. \tag{15}$$

The family is increasingly stratified in the sense of [19], see also [7, 20, 21]. Graph of the family members with highlighted parameter values $p = \frac{1}{3}$ and $p = \frac{4}{\pi^2}$ (from the previous theorem) is given by Figure 1.

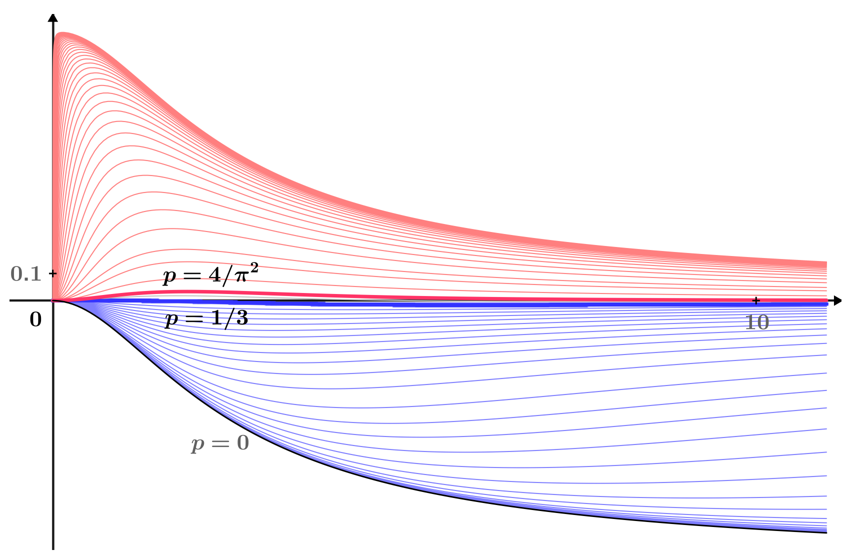

Figure 1: Stratified family of functions defined by (15)

We introduce the approximations family

$$\mathrm{gd}(x) \approx \frac{ax}{b + (1 + cx^2)^d}, \qquad x \to 0,$$

and $a, b, c, d \in \mathbb{R}$ are parameters. By using the computer program MAPLE 11, we determine the values

$$a = \frac{59}{35}, \quad b = \frac{24}{35}, \quad c = \frac{47}{105}, \quad d = \frac{59}{94}, \tag{16}$$

which provide the best approximation near the origin:

$$\operatorname{gd}(x) \approx \frac{\frac{59}{35}x}{\frac{24}{35} + (1 + \frac{47}{105}x^2)^{\frac{59}{94}}}. \tag{17}$$

We write $\operatorname{gd}(x) - \frac{ax}{b+(1+cx^2)^d}$ as power series

$$\begin{aligned} \operatorname{gd}(x) - \frac{ax}{b+(1+cx^2)^d} &= \frac{b-a+1}{b+1}x + \frac{6acd - b^2 - 2b - 1}{6(b+1)^2}x^3 \\ &+ \frac{12abc^2d^2 - 12ac^2d^2 - 12abc^2d - 12ac^2d + b^3 + 3b^2 + 3b + 1}{24(b+1)^3}x^5 \\ &+ \frac{1}{5040(b+1)^4}\Big(840ab^2c^3d^3 - 3360abc^3d^3 + 840ac^3d^3 - 2520ab^2c^3d^2 \\ &+ 2520ac^3d^2 + 1680ab^2c^3d + 3360abc^3d + 1680ac^3d - 61b^4 - 244b^3 \\ &- 366b^2 - 244b - 61\Big)x^7 + O(x^9). \end{aligned} \tag{18}$$

This produces the best approximation from (18):

$$\begin{cases} b - a + 1 = 0 \\ 6acd - b^2 - 2b - 1 = 0 \\ 12abc^2d^2 - 12ac^2d^2 - 12abc^2d - 12ac^2d + b^3 + 3b^2 + 3b + 1 = 0 \\ 840ab^2c^3d^3 - 3360abc^3d^3 + 840ac^3d^3 - 2520ab^2c^3d^2 + 2520ac^3d^2 \\ \quad +1680ab^2c^3d + 3360abc^3d + 1680ac^3d - 61b^4 - 244b^3 - 366b^2 - 244b - 61, \end{cases}$$

that is, by (16). We thus obtain the best approximation (17). The formula (17) led us to pose the following conjecture:

Conjecture 5. *Let $x > 0$. Then*

$$\operatorname{gd}(x) < \frac{\frac{59}{35}x}{\frac{24}{35} + (1 + \frac{47}{105}x^2)^{\frac{59}{94}}}.$$

We introduce the approximations family

$$\frac{\operatorname{gd}^{-1}(x)}{x} \approx \frac{p_1}{q_1 + \sqrt{1 + r_1x^2}}, \qquad x \to 0,$$

and $p_1, q_1, r_1 \in \mathbb{R}$ are parameters. By using the computer program MAPLE 11, we determine the values

$$p_1 = 1, \quad q_1 = 0, \quad r_1 = -\frac{1}{3},$$

which provide the best approximation near the origin:

$$\frac{\operatorname{gd}^{-1}(x)}{x} \approx \frac{1}{\sqrt{1 - \frac{1}{3}x^2}}. \tag{19}$$

In view of (19) it is natural to ask: What is the largest number α and what is the smallest number β such that the inequality

$$\frac{1}{\sqrt{1-\alpha x^2}} < \frac{\mathrm{gd}^{-1}(x)}{x} < \frac{1}{\sqrt{1-\beta x^2}},$$

holds for $0 < x < \pi/2$? Theorem 6 answers this question.

Theorem 6. *For $0 < x < \pi/2$,*

$$\frac{1}{\sqrt{1-\frac{1}{3}x^2}} < \frac{\mathrm{gd}^{-1}(x)}{x} < \frac{1}{\sqrt{1-\left(\frac{2}{\pi}\right)^2 x^2}}, \tag{20}$$

where the constants $\frac{1}{3}$ and $\left(\frac{2}{\pi}\right)^2$ are the best possible.

Proof. For $0 < x < \pi/2$, let

$$F(x) = \frac{\mathrm{gd}^{-1}(x)}{\frac{x}{\sqrt{1-px^2}}} = \frac{\ln(\sec x + \tan x)}{\frac{x}{\sqrt{1-px^2}}} = \frac{F_1(x)}{F_2(x)},$$

where $p = \frac{1}{3}$ or $p = (2/\pi)^2$, and

$$F_1(x) = \ln(\sec x + \tan x) \quad \text{and} \quad F_2(x) = \frac{x}{\sqrt{1-px^2}}.$$

Elementary calculations yield

$$\frac{F_1'(x)}{F_2'(x)} = \frac{(1-px^2)^{3/2}}{\cos x} =: F_3(x),$$

and

$$\frac{\cos^2 x}{\sqrt{1-px^2}\,x(3\cos x + x\sin x)} F_3'(x) = G(x) - p,$$

where

$$G(x) = \frac{\sin x}{x(3\cos x + x\sin x)} = \frac{\tan x}{x(3 + x\tan x)}.$$

Elementary calculations yield

$$\begin{aligned}[x\cos x(3 + x\tan x)]^2 G'(x) &= 2x\cos^2 x - 3\sin x\cos x + x \\ &= x\cos(2x) - \frac{3}{2}\sin(2x) + 2x =: H(x),\end{aligned}$$

$$\frac{1}{2}H'(x) = -\cos(2x) - x\sin(2x) + 1 =: I(x),$$

$$I'(x) = \sin(2x) - 2x\cos(2x) =: J(x),$$

$$J'(x) = 4x\sin(2x) > 0 \quad \text{for} \quad 0 < x < \frac{\pi}{2}.$$

Therefore, the function $J(x)$ is strictly increasing on $(0, \pi/2)$, and we have, for $0 < x < \pi/2$,

$$J(x) > J(0) = 0 \Longrightarrow I'(x) > 0 \Longrightarrow I(x) > I(0) = 0 \Longrightarrow H'(x) > 0 \\ \Longrightarrow H(x) > H(0) = 0 \Longrightarrow G'(x) > 0.$$

Hence, the function $G(x)$ is strictly increasing on $(0, \pi/2)$, and we have, for $0 < x < \pi/2$,

$$\frac{1}{3} = \lim_{x\to 0} G(x) < G(x) = \frac{\tan x}{x(3 + x\tan x)} < \lim_{x\to\frac{\pi}{2}} G(x) = \left(\frac{2}{\pi}\right)^2.$$

First of all, we show that the left-hand side of (20) with $p = \frac{1}{3}$ is valid for $0 < x < \pi/2$. When $p = \frac{1}{3}$, we have $G(x) > p$ and $F_3'(x) > 0$ for $0 < x < \pi/2$. Hence, $F_3(x)$ and $\frac{F_1'(x)}{F_2'(x)}$ are both strictly increasing on $(0, \pi/2)$. By Lemma 1, the function

$$F(x) = \frac{F_1(x)}{F_2(x)} = \frac{F_1(x) - F_1(0)}{F_2(x) - F_2(0)}$$

is strictly increasing for $0 < x < \pi/2$, and we have

$$F(x) = \frac{\mathrm{gd}^{-1}(x)}{\frac{x}{\sqrt{1-\frac{1}{3}x^2}}} = \frac{\ln(\sec x + \tan x)}{\frac{x}{\sqrt{1-\frac{1}{3}x^2}}} > \lim_{x\to 0} F(x) = 1. \tag{21}$$

This means that the left-hand side of (20) holds for $0 < x < \pi/2$.

We now show that the right-hand side of (20) with $p = (2/\pi)^2$ is valid for $0 < x < \pi/2$. When $p = (2/\pi)^2$, we have $G(x) < p$ and $F_3'(x) < 0$ for $0 < x < \pi/2$. Hence, $F_3(x)$ and $\frac{F_1'(x)}{F_2'(x)}$ are both strictly decreasing on $(0, \pi/2)$. By Lemma 1, the function

$$F(x) = \frac{F_1(x)}{F_2(x)} = \frac{F_1(x) - F_1(0)}{F_2(x) - F_2(0)}$$

is strictly decreasing for $0 < x < \pi/2$, and we have

$$F(x) = \frac{\mathrm{gd}^{-1}(x)}{\frac{x}{\sqrt{1-(2/\pi)^2x^2}}} = \frac{\ln(\sec x + \tan x)}{\frac{x}{\sqrt{1-(2/\pi)^2x^2}}} < \lim_{x\to 0} F(x) = 1.$$

This means that the right-hand side of (20) holds for $0 < x < \pi/2$.

If we write (20) as

$$\frac{1}{3} < \frac{1 - \left(\frac{x}{\mathrm{gd}^{-1}(x)}\right)^2}{x^2} < \left(\frac{2}{\pi}\right)^2,$$

we find that

$$\lim_{x\to 0}\frac{1-\left(\frac{x}{\mathrm{gd}^{-1}(x)}\right)^2}{x^2}=\frac{1}{3}\quad\text{and}\quad \lim_{x\to \frac{\pi}{2}}\frac{1-\left(\frac{x}{\mathrm{gd}^{-1}(x)}\right)^2}{x^2}=\left(\frac{2}{\pi}\right)^2.$$

Hence, the inequality (20) holds for $0<x<\pi/2$, and the constants $\frac{1}{3}$ and $\left(\frac{2}{\pi}\right)^2$ are the best possible. The proof of Theorem 6 is complete. □

In relation to the previous theorem, we can consider the family of functions $\{\varphi_p(x)\}_{p\in P}$, $P=\left(-\infty,\frac{\pi^2}{4}\right]$, for $x\in\left(0,\frac{\pi}{2}\right)$, defined by

$$\varphi_p(x)=\frac{\mathrm{gd}^{-1}(x)}{x}-\frac{1}{\sqrt{1-p\,x^2}}. \tag{22}$$

The family is decreasingly stratified in the sense of [19], see also [7, 20, 21]. Graph of the family members with highlighted parameter values $p=\frac{1}{3}$ and $p=\frac{4}{\pi^2}$ (from the previous theorem) is given by Figure 2.

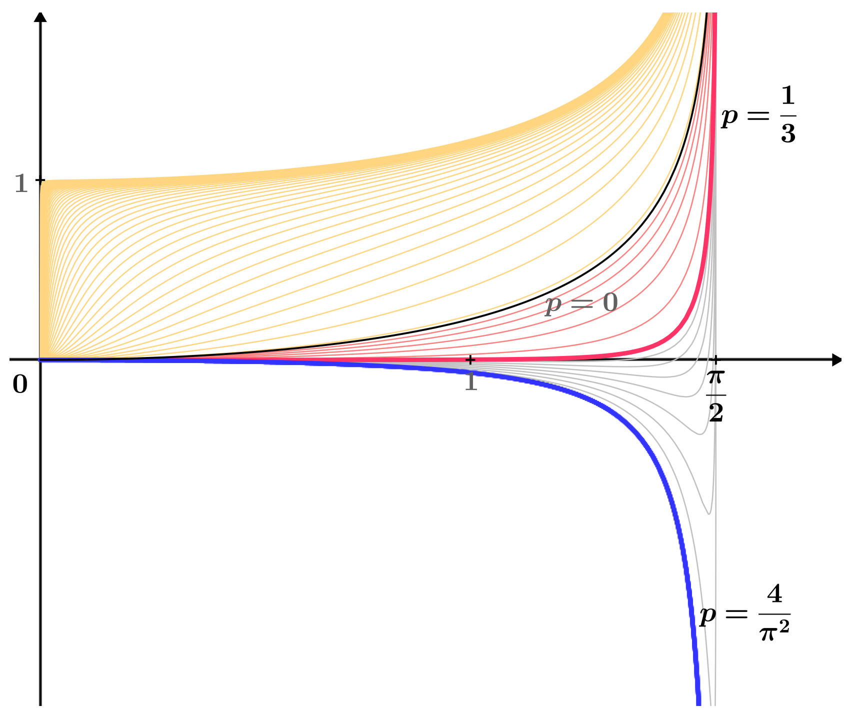

Figure 2: Stratified family of functions defined by (22)

We introduce the approximations family

$$\mathrm{gd}^{-1}(x) \approx \frac{a_1 x}{b_1 + (1 - c_1 x^2)^{d_1}}, \qquad x \to 0,$$

and $a_1, b_1, c_1, d_1 \in \mathbb{R}$ are parameters. By using the computer program MAPLE 11, we determine the values

$$a_1 = \frac{59}{35}, \quad b_1 = \frac{24}{35}, \quad c_1 = \frac{47}{105}, \quad d_1 = \frac{59}{94},$$

which provide the best approximation near the origin:

$$\mathrm{gd}^{-1}(x) \approx \frac{\frac{59}{35}x}{\frac{24}{35} + (1 - \frac{47}{105}x^2)^{\frac{59}{94}}}. \tag{23}$$

The formula (23) led us to pose the following conjecture:

Conjecture 7. *Let* $0 < x < \sqrt{105/47}$. *Then*

$$\mathrm{gd}^{-1}(x) < \frac{\frac{59}{35}x}{\frac{24}{35} + (1 - \frac{47}{105}x^2)^{\frac{59}{94}}}.$$

4. LAZAREVIĆ AND CUSA TYPE INEQUALITIES

It is known in the literature that

$$(\cos x)^{1/3} < \frac{\sin x}{x} < \frac{2 + \cos x}{3} \tag{24}$$

for $0 < |x| < \frac{\pi}{2}$, and

$$(\cosh x)^{1/3} < \frac{\sinh x}{x} < \frac{2 + \cosh x}{3} \tag{25}$$

for $x \neq 0$. The left-hand side inequality (24) first appeared in [23, p. 238], while the right-hand side inequality (24) was first mentioned by the German philosopher and theologian Nicolaus de Cusa (1401-1464), by a geometrical method. A rigorous proof of the right-hand side inequality (24) was given by Huygens [12], who used the right-hand side of (24) to estimate the number π. The right-hand side inequality (24) is now known as Cusa's inequality [5, 9, 24, 27, 34, 45–47]. Further interesting historical facts about Cusa's inequality can be found in [34]. The first inequality in (25) was established by Lazarević [14] (see, e.g., [23, p. 238]), while the second inequality in (25) appeared in [27]. The left-hand side of inequality (25) is now known as Lazarević-type inequality [23, p. 238]. For more information on inequalities involving trigonometric and hyperbolic functions, please refer to [7, 19–21] and references therein.

The inequalities (24) and (25) can be written in the following form:

$$\big(f(x)\big)^{1/3} < \frac{F(x)}{x} < \frac{2+f(x)}{3}, \quad \text{where} \quad F'(x) = f(x).$$

In analogy with (24) and (25), we establish Lazarević- and Cusa-type inequalities for the Gudermannian function given by Theorem 8.

Theorem 8. *For $x > 0$,*

$$(\operatorname{sech} x)^p < \frac{\operatorname{gd}(x)}{x} < (1-q) + q \operatorname{sech} x, \tag{26}$$

where the constants $p = \frac{1}{3}$ and $q = \frac{1}{3}$ are the best possible, in the sense that $p = \frac{1}{3}$ can not be replaced by a smaller number, and $q = \frac{1}{3}$ can not be replaced by a greater number.

Proof. First of all, we show that the left-hand side of (26) with $p = 1/3$ is valid for $x > 0$. For $x > 0$, let

$$L(x) = \operatorname{gd}(x) - x(\operatorname{sech} x)^{1/3}.$$

Elementary calculations yield

$$\frac{3(\cosh x)^{4/3}}{\sinh x} L'(x) = \frac{3(\cosh x)^{1/3} - 3\cosh x}{\sinh x} + x =: L_1(x),$$

$$L_1'(x) = \frac{\big((\cosh x)^{2/3} + 1\big)\big((\cosh x)^{2/3} - 1\big)^3}{\sinh^2 x (\cosh x)^{2/3}} > 0.$$

Hence, $L_1(x)$ is strictly increasing for $x > 0$, and we have, for $x > 0$,

$$L_1(x) > \lim_{t\to 0} L_1(t) = 0 \Longrightarrow L'(x) > 0 \Longrightarrow L(x) > \lim_{t\to 0} L(t) = 0.$$

This means that the left-hand side of (26) with $p = 1/3$ is valid for $x > 0$.

We now show that the right-hand side of (26) with $q = 1/3$ is valid for $x > 0$. For $x > 0$, let

$$U(x) = x\left(\frac{2 + \operatorname{sech} x}{3}\right) - \operatorname{gd}(x).$$

Elementary calculations yield

$$\frac{3(\cosh x)^2}{\sinh x} U'(x) = \frac{2\cosh^2 x - 2\cosh x}{\sinh x} - x = U_1(x),$$

$$U_1'(x) = \frac{(2\cosh x + 3)(\cosh x - 1)}{\cosh x + 1} > 0.$$

Hence, $U_1(x)$ is strictly increasing for $x > 0$, and we have, for $x > 0$,

$$U_1(x) > \lim_{t\to 0} U_1(t) = 0 \Longrightarrow U'(x) > 0 \Longrightarrow U(x) > \lim_{t\to 0} U(t) = 0.$$

This means that the right-hand side of (26) with $q = 1/3$ is valid for $x > 0$.

If we write (26) as

$$\frac{\ln \frac{\mathrm{gd}(x)}{x}}{\ln \operatorname{sech} x} < p \quad \text{and} \quad q < \frac{1 - \frac{\mathrm{gd}(x)}{x}}{1 - \operatorname{sech} x},$$

we find that

$$\lim_{x\to 0} \frac{\ln \frac{\mathrm{gd}(x)}{x}}{\ln \operatorname{sech} x} = \frac{1}{3} \quad \text{and} \quad \lim_{x\to 0} \frac{1 - \frac{\mathrm{gd}(x)}{x}}{1 - \operatorname{sech} x} = \frac{1}{3}.$$

Hence, the double inequality (26) holds for $x > 0$, and the constants $p = \frac{1}{3}$ and $q = \frac{1}{3}$ are the best possible. The proof of Theorem 8 is complete. □

Remark 9. *The choice* $(p, q) = (\frac{1}{3}, \frac{1}{3})$ *in* (26) *yields*

$$(\operatorname{sech} x)^{1/3} < \frac{\mathrm{gd}(x)}{x} < \frac{2 + \operatorname{sech} x}{3}, \qquad x > 0, \tag{27}$$

which is an analogue of the inequality (25).

Some computer experiments led us to propose the following conjecture.

Conjecture 10. *For* $x > 0$,

$$\frac{\mathrm{gd}(x)}{x} < \frac{4 + 5 \operatorname{sech}\left(\sqrt{\frac{3}{5}}\, x\right)}{9}. \tag{28}$$

Remark 11. *The upper bound in* (28) *is sharper than the upper bound in* (27), *that is to say,*

$$\frac{4 + 5 \operatorname{sech}\left(\sqrt{\frac{3}{5}}\, x\right)}{9} < \frac{2 + \operatorname{sech} x}{3}, \qquad x > 0. \tag{29}$$

We now prove the inequality (29). *It suffices to show that*

$$p(x) = \frac{1}{3} \operatorname{sech} x - \frac{5}{9} \operatorname{sech}\left(\sqrt{\frac{3}{5}}\, x\right) + \frac{2}{9} > 0, \qquad x > 0.$$

Differentiating $p(x)$ *with respect to* x *yields*

$$\frac{3}{x} p'(x) = \frac{\operatorname{sech}\left(\sqrt{\frac{3}{5}}\, x\right) \tanh\left(\sqrt{\frac{3}{5}}\, x\right)}{\sqrt{\frac{3}{5}}\, x} - \frac{\operatorname{sech} x \tanh x}{x}. \tag{30}$$

For $x > 0$, let

$$q(x) = \frac{\operatorname{sech} x \tanh x}{x}.$$

Elementary calculations yield

$$-x^2 \cosh^3 x q'(x) = x \cosh^2 x + \sinh x \cosh x - 2x =: r(x),$$

$$r'(x) = \sinh x(2x \cosh x + 3 \sinh x) > 0, \qquad x > 0.$$

Hence, $r(x)$ is strictly increasing for $x > 0$, and we have, for $x > 0$,

$$r(x) > \lim_{t\to 0} r(t) = 0 \Longrightarrow q'(x) < 0.$$

Hence, $q(x)$ is strictly decreasing on $(0, \infty)$. We see from (30) that $p'(x) > 0$ for $x > 0$. And hence, $p(x)$ is strictly increasing on $(0, \infty)$, and we have $p(x) > p(0) = 0$ for $x > 0$. The proof of (29) is complete.

We here give a partial solution to Conjecture 10. More precisely, we prove that the inequality (28) is valid for $0 < x < 6/5$.

Theorem 12. *The inequality (28) holds for $0 < x < 6/5$.*

Proof. Using (6) and (7), we have

$$\begin{aligned}
\frac{4 + 5 \operatorname{sech}\left(\sqrt{\frac{3}{5}}\, x\right)}{9} - \frac{\operatorname{gd}(x)}{x} &= \frac{4}{9} + \frac{5}{9}\sum_{n=0}^{\infty} \frac{(-1)^n |E_{2n}|}{(2n)!}\left(\frac{3}{5}\right)^n x^{2n} - \sum_{n=0}^{\infty}\frac{(-1)^n|E_{2n}|}{(2n+1)!}x^{2n} \\
&= \sum_{n=3}^{\infty}(-1)^{n-1}\left(\frac{1}{2n+1} - \frac{1}{3}\left(\frac{3}{5}\right)^{n-1}\right)\frac{|E_{2n}|}{(2n)!}x^{2n} \\
&= \frac{61}{31500}x^6 - \frac{3047}{2268000}x^8 + \sum_{n=5}^{\infty}(-1)^{n-1}u_n(x), \qquad (31)
\end{aligned}$$

where

$$u_n(x) = \left(\frac{1}{2n+1} - \frac{1}{3}\left(\frac{3}{5}\right)^{n-1}\right)\frac{|E_{2n}|}{(2n)!}x^{2n}.$$

Using (10), we find that for $0 < x < \frac{6}{5}$ and $n \geq 5$,

$$\begin{aligned}
\frac{u_{n+1}(x)}{u_n(x)} &= \frac{\left(\frac{1}{2n+3} - \frac{1}{3}\left(\frac{3}{5}\right)^n\right)\frac{|E_{2n+2}|}{(2n+2)!}}{\left(\frac{1}{2n+1} - \frac{1}{3}\left(\frac{3}{5}\right)^{n-1}\right)\frac{|E_{2n}|}{(2n)!}} x^2 \\
&< \frac{\left(\frac{1}{2n+3} - \frac{1}{3}\left(\frac{3}{5}\right)^n\right)\frac{4^{n+2}}{\pi^{2n+3}}}{\left(\frac{1}{2n+1} - \frac{1}{3}\left(\frac{3}{5}\right)^{n-1}\right)\frac{4^{n+1}}{\pi^{2n+1}}\left(\frac{1}{1+3^{-1-2n}}\right)} \left(\frac{6}{5}\right)^2 \\
&= \frac{\left(\frac{1}{2n+3} - \frac{1}{3}\left(\frac{3}{5}\right)^n\right)\left(\frac{2}{\pi}\right)^2}{\left(\frac{1}{2n+1} - \frac{1}{3}\left(\frac{3}{5}\right)^{n-1}\right)\left(\frac{1}{1+3^{-1-2n}}\right)} \left(\frac{6}{5}\right)^2.
\end{aligned}$$

Noting that the sequence

$$\frac{1}{1+3^{-1-2n}}$$

is strictly increasing for $n \geq 5$, we have

$$\frac{1}{1+3^{-1-2n}} \geq \left[\frac{1}{1+3^{-1-2n}}\right]_{n=5} = \frac{177147}{177148}, \qquad n \geq 5.$$

Clearly, $(\frac{2}{\pi})^2 < \frac{1}{2}$. We then obtain that for $0 < x < \frac{6}{5}$ and $n \geq 5$,

$$\begin{aligned}
\frac{u_{n+1}(x)}{u_n(x)} &< \frac{\left(\frac{1}{2n+3} - \frac{1}{3}\left(\frac{3}{5}\right)^n\right)\frac{1}{2}}{\left(\frac{1}{2n+1} - \frac{1}{3}\left(\frac{3}{5}\right)^{n-1}\right)\frac{177147}{177148}} \left(\frac{6}{5}\right)^2 \\
&= \frac{\left(\frac{1}{2n+3} - \frac{1}{3}\left(\frac{3}{5}\right)^n\right)}{\left(\frac{1}{2n+1} - \frac{1}{3}\left(\frac{3}{5}\right)^{n-1}\right)} \frac{54296}{492075} < 1.
\end{aligned}$$

Therefore, for every $x \in (0, 6/5)$, the sequence $n \longmapsto u_n(x)$ is strictly decreasing for $n \geq 5$. We then obtain from (31) that

$$\frac{4 + 5\,\mathrm{sech}\left(\sqrt{\frac{3}{5}}\,x\right)}{9} - \frac{\mathrm{gd}(x)}{x} > x^6\left(\frac{61}{31500} - \frac{3047}{2268000}x^2\right) > 0$$

for $0 < x < 6/5$. The proof of Theorem 12 is complete. □

Theorem 13. *For* $0 < x < \pi/2$,

$$(1-\xi)+\xi\sec\left(\sqrt{\frac{3}{5}}\,x\right) < \frac{\operatorname{gd}^{-1}(x)}{x} < (\sec x)^{\eta}, \tag{32}$$

where the constants $\xi = \frac{5}{9}$ *and* $\eta = \frac{1}{3}$ *is the best possible.*

Proof. First of all, we show that the left-hand side of (32) with $\xi = \frac{5}{9}$ is valid for $0 < x < \pi/2$. Using (8) and (9), we obtain that

$$\begin{aligned}\frac{\operatorname{gd}^{-1}(x)}{x} - \frac{4+5\sec\left(\sqrt{\frac{3}{5}}\,x\right)}{9} &= \sum_{n=0}^{\infty}\frac{|E_{2n}|}{(2n+1)!}x^{2n} - \frac{4}{9} - \frac{5}{9}\sum_{n=0}^{\infty}\frac{|E_{2n}|}{(2n)!}\left(\frac{3}{5}\right)^{n}x^{2n}\\ &= \sum_{n=3}^{\infty}\left(\frac{1}{2n+1} - \frac{1}{3}\left(\frac{3}{5}\right)^{n-1}\right)\frac{|E_{2n}|}{(2n)!}x^{2n} > 0.\end{aligned}$$

We now show that the right-hand side of (32) with $\eta = 1/3$ is valid for $0 < x < \pi/2$. For $0 < x < \pi/2$, let

$$V(x) = x(\operatorname{sech} x)^{1/3} - \operatorname{gd}^{-1}(x).$$

Elementary calculations yield

$$\frac{3(\cos x)^{4/3}}{\sin x}V'(x) = \frac{3\cos x - 3(\cos x)^{1/3}}{\sin x} + x =: V_1(x),$$

$$V_1'(x) = \frac{\left(1+(\cos x)^{2/3}\right)\left(1-(\cos x)^{2/3}\right)^3}{\sin^2 x(\cos x)^{2/3}} > 0.$$

Hence, $V_1(x)$ is strictly increasing on $(0, \pi/2)$, and we have, for $0 < x < \pi/2$,

$$V_1(x) > \lim_{t\to 0} V_1(t) = 0 \Longrightarrow V'(x) > 0 \Longrightarrow V(x) > \lim_{t\to 0} V(t) = 0.$$

This means that the right-hand side of (32) with $\eta = 1/3$ is valid for $0 < x < \pi/2$.

If we write (32) as

$$\xi < \frac{\frac{\operatorname{gd}^{-1}(x)}{x} - 1}{\sec\left(\sqrt{\frac{3}{5}}\,x\right) - 1} \quad \text{and} \quad \frac{\ln\frac{\operatorname{gd}^{-1}(x)}{x}}{\ln\sec x} < \eta,$$

we find that

$$\lim_{x\to 0}\frac{\frac{\operatorname{gd}^{-1}(x)}{x} - 1}{\sec\left(\sqrt{\frac{3}{5}}\,x\right) - 1} = \frac{5}{9} \quad \text{and} \quad \lim_{x\to 0}\frac{\ln\frac{\operatorname{gd}^{-1}(x)}{x}}{\ln\sec x} = \frac{1}{3}.$$

Hence, the double inequality (32) holds for $0 < x < \pi/2$, and the constants $\xi = \frac{5}{9}$ and $\eta = \frac{1}{3}$ are the best possible. The proof of Theorem 13 is complete. □

Remark 14. *Noting that*

$$(\sec x)^{1/3} < \frac{2+\sec x}{3}$$

holds for $0 < x < \pi/2$*, we then obtain the following inequality chain:*

$$\frac{4+5\sec\left(\sqrt{\frac{3}{5}}\,x\right)}{9} < \frac{\mathrm{gd}^{-1}(x)}{x} < (\sec x)^{1/3} < \frac{2+\sec x}{3}, \qquad 0 < x < \frac{\pi}{2}.$$

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