Applicable Analysis and Discrete Mathematics
available online at http://pefmath.etf.rs

Appl. Anal. Discrete Math. **19** (2025), 150–175.
https://doi.org/10.2298/AADM231005024T

# SOME FIXED POINT THEOREMS FOR $\alpha$-ADMISSIBLE $\theta$-$\phi$-MULTIVALUED CONTRACTION MAPPINGS IN $b$-METRIC SPACES

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In this research paper, a new concept called $\theta$-$\phi$-multivalued contractions is introduced for an $\alpha$-admissible mapping in the context of $b$-metric spaces, which is an extension of the $\theta$-$\phi$-contractions originally proposed by Rossafi et al. in 2023. Several fixed point theorems are established, including the $\alpha$-admissible $\theta$-$\phi$-multivalued theorem of Kanan and the $\alpha$-admissible $\theta$-$\phi$-multivalued theorem of Riech, all in the framework of $b$-metric spaces. The applicability of these results is demonstrated using illustrative examples.

## 1. INTRODUCTION

The theory of multivalued mappings has diverse applications in control theory, convex optimization, differential equations, and economics, as is widely acknowledged. In his work, Nadler [**20**] extended the famous Banach contraction principle to multivalued mappings, which has served as a major source of inspiration for researchers in metric fixed point theory. Several attempts have been made by numerous authors to generalize this principle in various spaces, resulting in several notable generalizations that have been documented in [**11, 1, 7, 15, 26, 28**]. References [**8, 1, 24, 25, 27**] provide examples of various multi-valued mapping conditions formulated using the Hausdorff-Pompieu metric, as proposed by several authors.

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2020 Mathematics Subject Classification. 37C25 , 47H10, 47H09.
Keywords and Phrases. Admissible mapping, Fixed point, $\alpha$-Complete spaces, $\theta$-$\phi$-Multivalued contraction, $b$-Metric spaces.

Czerwik introduced the concept of b-metric space in [**6**], and subsequently, there have been numerous publications regarding the fixed point theory of different types of single-valued and multi-valued operators in $b$-metric spaces, as documented in [**3, 4**]. These spaces have been used to establish several fixed point results. If you want to learn more about $b$-metric spaces and related concepts, the reader can see [**9, 10, 12**]. The literature on fixed point problems involving $\alpha$-admissible mappings has experienced significant growth, see [**16, 17**]. The concept of $\alpha$-admissibility for multivalued mappings was introduced by Mohammadi [**19**]. Recently, Patel has expanded the class of $\alpha$-admissible multivalued mappings by defining the class of triangular $\alpha$-admissible multivalued mappings [**21**].

Based on the findings of [**21**] and [**19**], this article introduces various contraction-type mappings, such as $\alpha$-admissible $\theta$-multivalued contractive mappings and $\alpha$-admissible $\theta$-$\phi$-multivalued contractive mappings. Then we establish fixed point theorems for these mappings, and provide some practical examples and implications to demonstrate their applicability. In the following, we will present certain notations, definitions, and basic outcomes that will be required in the subsequent discussions.

**Definition 1.** [*6*] *A b-metric space is defined as a pair* $(\Upsilon, \mathrm{d}_b)$ *where* $\Upsilon$ *is a non-empty set and* $\mathrm{d}_b : \Upsilon \times \Upsilon \to [0, +\infty)$ *is a function satisfying the following properties for every* $\nu, \omega, \sigma \in \Upsilon$ *and a given real number* $b \geq 1$*:*

1. $\mathrm{d}_b(\nu, \omega) = 0$*, if and only if* $\nu = \omega$,
2. $\mathrm{d}_b(\nu, \omega) = \mathrm{d}_b(\omega, \nu)$,
3. $\mathrm{d}_b(\nu, \omega) \leq b[\mathrm{d}_b(\nu, \sigma) + \mathrm{d}_b(\sigma, \omega)]$.

Consider a $b$-metric space $(\Upsilon, \mathrm{d}_b)$ and a sequence $\nu_\mathrm{n}$ in $\Upsilon$. We can define the following concepts:

**1)** The sequence $\nu_\mathrm{n}$ is said to converge to a point $\nu \in \Upsilon$ if its limit exists, i.e., if $\lim_{\mathrm{n} \to +\infty} \mathrm{d}_b(\nu_\mathrm{n}, \nu) = 0$.

**2)** The sequence $\nu_\mathrm{n}$ is called a Cauchy sequence if, for any given $\epsilon > 0$, there is an integer $\mathrm{n}_\epsilon \in \mathbb{N}$ such that $\mathrm{d}_b(\nu_\mathrm{m}, \nu_\mathrm{n}) < \epsilon$, for every $\mathrm{n}, \mathrm{m} > \mathrm{n}_\epsilon$.

**3)** The $b$-metric space $(\Upsilon, \mathrm{d}_b)$ is said to be complete if every Cauchy sequence in $\Upsilon$ converges to a point in $\Upsilon$.

**Lemma 1.** [*2*]

1. *Suppose that two sequences* $\{\nu_\mathrm{n}\}$ *and* $\{\omega_\mathrm{n}\}$ *are convergent in a b-metric space* $(\Upsilon, \mathrm{d}_b)$*, such that*

$$\lim_{n \to +\infty} \mathrm{d}_b(\nu_\mathrm{n}, \nu) = 0,$$
$$\lim_{n \to +\infty} \mathrm{d}_b(\omega_\mathrm{n}, \omega) = 0.$$

*Then, we have*

$$\begin{aligned}\frac{1}{b^2}\mathrm{d}_b(\nu,\omega) &\leq \lim_{\mathrm{n}\to+\infty}\inf \mathrm{d}_b(\nu_\mathrm{n},\omega_\mathrm{n})\\ &\leq \lim_{\mathrm{n}\to+\infty}\sup \mathrm{d}_b(\nu_\mathrm{n},\omega_\mathrm{n})\\ &\leq b^2\mathrm{d}_b(\nu,\omega).\end{aligned}$$

**2.** *Furthermore, if* $\nu=\omega$*, then* $\lim_{\mathrm{n}\to+\infty}\mathrm{d}_b(\nu_\mathrm{n},\omega_\mathrm{n})=0$*. Additionally, for any* $\sigma\in\Upsilon$ *and* $\nu\in\Upsilon$*, we can get*

$$\begin{aligned}\frac{1}{b}\mathrm{d}_b(\nu,\sigma) &\leq \lim_{\mathrm{n}\to+\infty}\inf \mathrm{d}_b(\nu_\mathrm{n},\sigma)\\ &\leq \lim_{\mathrm{n}\to+\infty}\sup \mathrm{d}_b(\nu_\mathrm{n},\sigma)\\ &\leq b\mathrm{d}_b(\nu,\sigma),\end{aligned}$$

*for all* $\nu\in\Upsilon$*.*

**Lemma 2.** [*22*] *Let* $(\Upsilon,\mathrm{d}_b)$ *be a b-metric space and let* $\{\nu_\mathrm{n}\}$ *be a sequence in* $\Upsilon$ *satisfying the following condition*

$$\lim_{\mathrm{n}\to+\infty}\mathrm{d}_b(\nu_\mathrm{n},\nu_{\mathrm{n}+1})=0,$$

*If* $\{\nu_\mathrm{n}\}$ *is not a Cauchy sequence, then there is* $\epsilon>0$ *and two sequences of positive integers* $\{\mathrm{n(k)}\}$ *and* $\{\mathrm{m(k)}\}$ *such that*

***i)*** $\epsilon\leq\liminf_{\mathrm{k}\to+\infty}\mathrm{d}_b(\nu_{\mathrm{m}_{(\mathrm{k})}},\nu_{\mathrm{n}_{(\mathrm{k})}})\leq\limsup_{\mathrm{k}\to+\infty}\mathrm{d}_b(\nu_{\mathrm{m}_{(\mathrm{k})}},\nu_{\mathrm{n}_{(\mathrm{k})}})\leq b\epsilon,$

***ii)*** $\frac{\epsilon}{b}\leq\liminf_{\mathrm{k}\to+\infty}\mathrm{d}_b(\nu_{\mathrm{m}_{(\mathrm{k})}},\nu_{\mathrm{n}_{(\mathrm{k})+1}})\leq\limsup_{\mathrm{k}\to+\infty}\mathrm{d}_b(\nu_{\mathrm{m}_{(\mathrm{k})}},\nu_{\mathrm{n}_{(\mathrm{k})+1}})\leq b^2\epsilon,$

***iii)*** $\frac{\epsilon}{b}\leq\liminf_{\mathrm{k}\to+\infty}\mathrm{d}_b(\nu_{\mathrm{m}_{(\mathrm{k}+1)}},\nu_{\mathrm{n}_{(\mathrm{k})}})\leq\limsup_{\mathrm{k}\to+\infty}\mathrm{d}_b(\nu_{\mathrm{m}_{(\mathrm{k})+1}},\nu_{\mathrm{n}_{(\mathrm{k})}})\leq b^2\epsilon,$

***vi)*** $\frac{\epsilon}{b^2}\leq\liminf_{\mathrm{k}\to+\infty}\mathrm{d}_b(\nu_{\mathrm{m}_{(\mathrm{k})+1}},\nu_{\mathrm{n}_{(\mathrm{k})+1}})\leq\limsup_{\mathrm{k}\to+\infty}\mathrm{d}_b(\nu_{\mathrm{m}_{(\mathrm{k})+1}},\nu_{\mathrm{n}_{(\mathrm{k})+1}})\leq b^3\epsilon.$

In [14], Jleli et al. provided the following definition

**Definition 2.** [*14*] *Let* $\Theta$ *be the set of functions* $\theta:(0,+\infty)\to(1,+\infty)$ *that satisfy the following conditions*

*($\theta_1$)* $\theta$ *is nondecreasing,*

*($\theta_2$) for any sequence* $\{\nu_\mathrm{n}\}\subset(0,+\infty)$*, we can get*

$$\lim_{\mathrm{n}\to0}\nu_\mathrm{n}=0 \quad \textit{if and only if} \quad \lim_{\mathrm{n}\to+\infty}\theta(\nu_\mathrm{n})=1,$$

*($\theta_3$)* $\theta$ *is continuous on* $(0,+\infty)$,

*($\theta_4$)* *there exist* $r \in (0,1)$ *and* $l \in (0,+\infty)$ *such that* $\lim_{t\to 0^+} \frac{\theta(t)-1}{t^r} = l$.

**Definition 3.** [29] *Let* $\Phi$ *be the set of functions* $\phi : [1,+\infty) \to [1,+\infty)$ *that satisfy the following conditions*

*($\phi_1$)* $\phi$ *is nondecreasing;*

*($\phi_2$)* *for any* $\tau \in (1,+\infty)$, *we get*

$$\lim_{\mathrm{n}\to+\infty} \phi^{\mathrm{n}}(\tau) = 1;$$

*($\phi_3$)* $\phi$ *is continuous.*

**Lemma 3.** [29] *If* $\phi \in \Phi$, *then*

$$\phi(\tau) < \tau, \ \textit{for every } \tau \in (1,+\infty) \quad \textit{and} \quad \phi(1) = 1.$$

The notion of $\theta$-$\phi$-contraction on metric spaces was introduced in [**29**] by Zheng et al. who also demonstrated the following outcome.

**Definition 4.** [29] *Let* $(\Upsilon, \mathrm{d})$ *be a metric space, a mapping* $\mathcal{F} : \Upsilon \to \Upsilon$ *is called* $\theta$-$\phi$-*contraction if there is* $\phi \in \Phi$ *and* $\theta \in \Theta$ *such that*

$$\mathrm{d}\ (\mathcal{F}\nu, \mathcal{F}\omega) > 0 \ \textit{implies that } \theta[\mathrm{d}(\mathcal{F}\nu, \mathcal{F}\omega)] \le \phi(\theta[\mathrm{M}(\nu,\omega)]), \ \textit{for every } \nu,\omega \in \Upsilon,$$

*where*

$$\mathrm{M}(\nu,\omega) = \max\{\mathrm{d}(\nu,\omega),\ \mathrm{d}(\nu,\mathcal{F}\nu), \mathrm{d}(\omega,\mathcal{F}\omega)\}.$$

**Theorem 1.** [29] *Let* $(\Upsilon, \mathrm{d})$ *be a complete metric space and* $\mathcal{F} : \Upsilon \to \Upsilon$ *be a* $\theta$-$\phi$-*contraction mapping. Then* $\mathcal{F}$ *possesses a unique fixed point.*

In 2023, Rossafi et al. [**23**] presented some fixed point results for a $\theta$-$\phi$-contraction on $b$-metric spaces and showed the following result.

**Definition 5.** [23] *Let* $(\Upsilon, \mathrm{d}_b)$ *be a b-metric spaces with* $b > 1$, *a mapping* $\mathcal{F} : \Upsilon \to \Upsilon$ *is called* $\theta$-$\phi$-*contraction if there is* $\phi \in \Phi$ *and* $\theta \in \Theta$ *such that*

$$\mathrm{d}_b(\mathcal{F}\nu,\ \mathcal{F}\omega) > 0, \ \textit{implies that } \theta[b^3 \mathrm{d}_b(\mathcal{F}\nu,\ \mathcal{F}\omega)] \le \phi[\theta(\mathrm{M}(\nu,\omega))], \ \textit{for every } \nu,\omega \in \Upsilon,$$

*where*

$$\mathrm{M}(\nu,\omega) = \max\left\{\mathrm{d}_b(\nu,\omega), \mathrm{d}_b(\nu,\mathcal{F}\nu), \mathrm{d}_b(\omega,\mathcal{F}\omega), \frac{\mathrm{d}_b(\nu,\mathcal{F}\omega) + \mathrm{d}_b(\mathcal{F}\nu,\omega)}{2b^2}\right\}.$$

**Theorem 2.** [23] *Let* $(\Upsilon, \mathrm{d}_b)$ *be a complete b-metric space and* $\mathcal{F} : \Upsilon \to \Upsilon$ *be a* $\theta$-$\phi$-*contraction mapping. Then* $\mathcal{F}$ *possesses a unique fixed point.*

For a non-empty set $\Upsilon$, let $\mathcal{P}(\Upsilon)$ be the power set of $\Upsilon$ and $U, V \in \mathcal{P}(\Upsilon)$. If $(\Upsilon, \mathrm{d}_b)$ is a $b$-metric space, we define

$$\begin{aligned} \mathcal{N}(\Upsilon) &= \mathcal{P}(\Upsilon) \setminus \{\emptyset\}, \\ CB(\Upsilon) &= \{W \in \mathcal{N}(\Upsilon) : \ W \text{ is closed and bounded}\}, \\ \mathcal{K}(\Upsilon) &= \{W \in \mathcal{N}(\Upsilon) : \ W \text{ is compact}\}, \\ \mathrm{d}_b(\mu, W) &= \inf\{\mathrm{d}_b(\mu, \omega) : \ \omega \in W\}, \\ \rho(U, W) &= \sup\{\mathrm{d}_b(\mu, W) : \ \mu \in U\}, \\ \mathcal{H}(U, W) &= \max\{\rho(U, W), \rho(W, U)\}. \end{aligned}$$

**Lemma 4.** [5] *Given a b-metric space* $(\Upsilon, \mathrm{d}_b)$. *For all* $W, U, V \in CB(\Upsilon)$ *and* $\nu, \omega \in \Upsilon$, *we have*

1. $\mathrm{d}_b(\nu, U) \leq \mathrm{d}_b(\nu, \mu)$, *for all* $\mu \in U$,
2. $\rho(V, W) \leq \mathcal{H}(V, W)$,
3. $\mathrm{d}_b(\mathrm{v}, W) \leq \mathcal{H}(V, W)$, *for all* $\mathrm{v} \in V$,
4. $\mathcal{H}(V, V) = 0$,
5. $\mathcal{H}(V, W) = \mathcal{H}(W, V)$,
6. $\mathcal{H}(V, W) \leq b\big[\mathcal{H}(V, U) + \mathcal{H}(U, W)\big]$,
7. $\mathrm{d}_b(\nu, U) \leq b\big[\mathrm{d}_b(\nu, \omega) + \mathrm{d}_b(\omega, U)\big]$.

**Lemma 5.** [5] *Given a b-metric space* $(\Upsilon, \mathrm{d}_b)$. *If* $U \in CB(\Upsilon)$ *and* $\nu \in \Upsilon$, *then*

$$\mathrm{d}_b(\nu, U) = 0 \quad \text{is equivalent to} \quad \nu \in \overline{U} = U,$$

*where* $\overline{U}$ *is the closure of the set* $U$.

Mohammadi et al. [**19**] introduced the notion of $\alpha$-admissibility for multivalued mappings as follows:

**Definition 6.** [19] *Let* $\Upsilon$ *be a non-empty set and* $\mathcal{F} : \Upsilon \to \mathcal{N}(\Upsilon)$ *and* $\alpha : \Upsilon^2 \to [0, +\infty)$ *be two maps. Then* $\mathcal{F}$ *is called an* $\alpha$*-admissible whenever for each* $\nu \in \Upsilon$ *and* $\omega \in \mathcal{F}\nu$ *it holds*

$$\alpha(\nu, \omega) \geq 1 \text{ implies that } \alpha(\omega, \sigma) \geq 1 \text{ for all } \sigma \in T\omega.$$

**Definition 7.** [13] *Let* $(\Upsilon, \mathrm{d}_b)$ *be a b-metric space and* $\alpha : \Upsilon^2 \to [0, +\infty)$. *We say that the space* $(\Upsilon, \mathrm{d}_b)$ *is* $\alpha$*-complete if and only if every Cauchy sequence* $\{\nu_{\mathrm{n}}\}$ *where* $\alpha(\nu_{\mathrm{n}}, \nu_{\mathrm{n}+1}) \geq 1$ *for every* $\mathrm{n} \in \mathbb{N}$, *converges in* $\Upsilon$.

**Definition 8.** [18] *Let* $(\Upsilon, \mathrm{d}_b)$ *be a b-metric space and* $\alpha : \Upsilon^2 \to \mathbb{R}_+ \cup \{0\}$ *and* $\mathcal{F} : \Upsilon \to \mathcal{K}(\Upsilon)$ *be two maps. Then* $\mathcal{F}$ *is called* $\alpha$*-continuous multivalued mapping on* $(\mathcal{K}(\Upsilon), \mathcal{H})$ *if, for all sequences* $\{\nu_{\mathrm{n}}\}$ *with* $\alpha(\nu_{\mathrm{n}}, \nu_{\mathrm{n+1}}) \geq 1$ *for every* $\mathrm{n} \in \mathbb{N}$ *and* $\lim_{\mathrm{n}\to+\infty} \nu_{\mathrm{n}} = \nu \in \Upsilon$*, we have* $\lim_{\mathrm{n}\to+\infty} \mathcal{F}\nu_{\mathrm{n}} = \mathcal{F}\nu$ *so that*

$$\left.\begin{array}{l} \lim\limits_{\mathrm{n}\to+\infty} \mathrm{d}_b(\nu_{\mathrm{n}}, \nu) = 0 \\ \textit{and} \\ \alpha(\nu_{\mathrm{n}}, \nu_{\mathrm{n+1}}) \geq 1 \textit{ for every } n \in \mathbb{N} \end{array}\right\} \textit{ means that } \lim_{\mathrm{n}\to+\infty} \mathcal{H}(\mathcal{F}\nu_{\mathrm{n}}, \mathcal{F}\nu) = 0.$$

**Definition 9.** [21] *Let* $\Upsilon$ *be a non-empty set and* $\mathcal{F} : \Upsilon \to \mathcal{N}(\Upsilon)$ *and* $\alpha : \Upsilon^2 \to \mathbb{R}_+ \cup \{0\}$ *be two maps. Then* $\mathcal{F}$ *is called a triangular* $\alpha$*-admissible if* $\mathcal{F}$ *is* $\alpha$*-admissible which satisfies the following condition*

$$\left.\begin{array}{l} \alpha(\nu, \omega) \geq 1 \\ \textit{and} \\ \alpha(\omega, \sigma) \geq 1 \end{array}\right\} \textit{ implies that } \alpha(\nu, \sigma) \geq 1 \textit{ for all } \sigma \in \mathcal{F}\omega.$$

**Lemma 6.** [21] *Let* $\mathcal{F} : \Upsilon \to \mathcal{N}(\Upsilon)$ *be a triangular* $\alpha$*-admissible mapping. If there is* $\omega_0 \in \Upsilon$ *and* $\nu_1 \in \mathcal{F}\nu_0$ *such that* $\alpha(\nu_0, \nu_1) \geq 1$ *then for a sequence* $\{\nu_{\mathrm{n}}\}$*,* $\nu_{\mathrm{n+1}} \in \mathcal{F}\nu_n$*, we get* $\alpha(\nu_{\mathrm{n}}, \nu_{\mathrm{m}}) \geq 1$ *for every* $\mathrm{n}, \mathrm{m} \in \mathbb{N}$ *with* $\mathrm{n} < \mathrm{m}$.

## 2. MAIN RESULTS

By utilizing the concept proposed by Rossafi et al. [**23**], our paper introduces the notion of $\alpha$-admissible $\theta$-$\phi$-multivalued contraction in $b$-metric spaces and establishes several fixed point theorems that rely on this type of contraction. In the following definition, we present the concept of $\alpha$-admissible $\theta$-$\phi$-Multivalued contraction in $b$-metric spaces.

**Definition 10.** *Given a b-metric space* $(\Upsilon, \mathrm{d}_b)$ *with* $b > 1$*. Let* $\mathcal{F} : \Upsilon \to \mathcal{K}(\Upsilon)$ *be a mapping.*

***(1)*** $\mathcal{F}$ *is called an* $\alpha$*-admissible* $\theta$*-Multivalued contraction if there are* $\theta \in \Theta$, $L \geq 0$ *and* $\mathrm{r} \in (0, 1)$ *such that*
(1)
$\mathcal{H}(\mathcal{F}\nu,\ \mathcal{F}\omega) > 0$ *implies that* $\theta[\alpha(\nu, \omega) b^3 \mathcal{H}(\mathcal{F}\nu,\ \mathcal{F}\omega)] \leq \theta[\mathrm{M}(\nu, \omega)]^{\mathrm{r}} + L\mathrm{N}(\nu, \omega)$,

*where*

$$\mathrm{M}(\nu, \omega) = \max\left\{\mathrm{d}_b(\nu, \omega), \mathrm{d}_b(\nu, \mathcal{F}\nu), \mathrm{d}_b(\omega, \mathcal{F}\omega), \frac{\mathrm{d}_b(\nu, \mathcal{F}\omega) + \mathrm{d}_b(\mathcal{F}\nu, \omega)}{2b^2}\right\}$$
$$\mathrm{N}(\nu, \omega) = \min\left\{\mathrm{d}_b(\nu, \mathcal{F}\nu), \mathrm{d}_b(\mathcal{F}\nu, \omega), \mathrm{d}_b(\nu, \mathcal{F}\omega), \mathrm{d}_b(\mathcal{F}\nu, \omega)\right\},$$

**(2)** *$\mathcal{F}$ is called an $\alpha$-admissible $\theta$-$\phi$-Multivalued contraction if there are $\theta \in \Theta$, $L \geq 0$ and $\phi \in \Phi$ such that*

(2)

$$\mathcal{H}(\mathcal{F}\nu, \mathcal{F}\omega) > 0 \text{ implies that } \theta[\alpha(\nu,\omega) b^3 \mathcal{H}(\mathcal{F}\nu,\ \mathcal{F}\omega)] \leq \phi[\theta(\mathrm{M}(\nu,\omega))] + L\,\mathrm{N}(\nu,\omega),$$

*where*

$$\mathrm{M}(\nu,\omega) = \max\Big\{\mathrm{d}_b(\nu,\omega), \mathrm{d}_b(\nu,\mathcal{F}\nu), \mathrm{d}_b(\omega,\mathcal{F}\omega), \frac{\mathrm{d}_b(\nu,\mathcal{F}\omega) + \mathrm{d}_b(\mathcal{F}\nu,\omega)}{2b^2}\Big\}$$

$$\mathrm{N}(\nu,\omega) = \min\Big\{\mathrm{d}_b(\nu,\mathcal{F}\nu), \mathrm{d}_b(\{\omega,\omega), \mathrm{d}_b(\nu,\mathcal{F}\omega), \mathrm{d}_b(\mathcal{F}\nu,\omega)\Big\},$$

**(3)** *$\mathcal{F}$ is called an $\alpha$-admissible $\theta$-$\phi$-Multivalued Kannan-type contraction if there are $\theta \in \Theta$, $L \geq 0$ and $\phi \in \Phi$ such that for each $\nu, \omega \in \Upsilon$ for which $\mathcal{H}(\mathcal{F}\nu, \mathcal{F}\omega) > 0$ we get*

$$(3) \quad \theta[\alpha(\nu,\omega) b^3 \mathcal{H}(\mathcal{F}\nu, \mathcal{F}\omega)] \leq \phi\Big[\theta\Big(\frac{\mathrm{d}_b(\nu,\mathcal{F}\nu) + \mathrm{d}_b(\omega,\mathcal{F}\omega)}{2}\Big)\Big] + L\,\mathrm{N}(\nu,\omega),$$

*where*

$$\mathrm{N}(\nu,\omega) = \min\Big\{\mathrm{d}_b(\nu,\mathcal{F}\nu), \mathrm{d}_b(\{\omega,\omega), \mathrm{d}_b(\nu,\mathcal{F}\omega), \mathrm{d}_b(\mathcal{F}\nu,\omega)\Big\},$$

**(4)** *$\mathcal{F}$ is called an $\alpha$-Multivalued $\theta$-$\phi$-Multivalued Reich-type contraction if there are $\theta \in \Theta$, $L \geq 0$ and $\phi \in \Phi$ such that for each $\nu, \omega \in \Upsilon$ for which $\mathrm{d}_b(\mathcal{F}\nu,\ \mathcal{F}\omega) > 0$ we get*

(4)

$$\theta[\alpha(\nu,\omega) b^3 \mathcal{H}(\mathcal{F}\nu,\ \mathcal{F}\omega)] \leq \phi\Big[\theta\Big(\frac{\mathrm{d}_b(\nu,\omega) + \mathrm{d}_b(\nu,\mathcal{F}\nu) + \mathrm{d}_b(\omega,\mathcal{F}\omega)}{3}\Big)\Big] + L\,\mathrm{N}(\nu,\omega),$$

*where*

$$\mathrm{N}(\nu,\omega) = \min\Big\{\mathrm{d}_b(\nu,\mathcal{F}\nu), \mathrm{d}_b(\{\omega,\omega), \mathrm{d}_b(\nu,\mathcal{F}\omega), \mathrm{d}_b(\mathcal{F}\nu,\omega)\Big\},$$

*for every $\omega, \nu \in \Upsilon$.*

**Theorem 3.** *Let $(\Upsilon, \mathrm{d}_b)$ be a complete b-metric space and $\mathcal{F} : \Upsilon \to \mathcal{K}(\Upsilon)$ be an $\alpha$-admissible $\theta$-Multivalued contraction satisfying*

**(1)** *$(\Upsilon, \mathrm{d}_b)$ is an $\alpha$-complete metric space;*

**(2)** *there are $\nu_0 \in \Upsilon$ and $\nu_1 \in \mathcal{F}\nu_0$ so that $\alpha(\nu_0, \nu_1) \geq 1$;*

**(3)** *$\mathcal{F}$ is triangular $\alpha$-admissible;*

**(4)** *either*

  **(4a)** *$\mathcal{F}$ is an $\alpha$-continuous multivalued mapping*
  *or*

***(4b)*** *if* $\{\nu_n\} \subset \Upsilon$ *so that* $\alpha(\nu_n, \nu_{n+1}) \geq 1$ *for every* $n \in \mathbb{N}$ *and* $\lim_{n\to+\infty} \nu_n = \nu \in \Upsilon$*, then we get* $\alpha(\nu_n, \nu) \geq 1$ *for every* $n \in \mathbb{N}$.

*Then* $\mathcal{F}$ *possesses a fixed point.*

*Proof.* Let $\nu_0 \in \Upsilon$ and $\nu_1 \in \mathcal{F}\nu_0$ such that $\alpha(\nu_0, \nu_1) \geq 1$. We define a sequence $\{\nu_n\}$ by

$$\nu_{n+1} \in \mathcal{F}\nu_n$$

for all $n \in \mathbb{N}$. If there is $n_0 \in \mathbb{N}$ with $d_b(\nu_{n_0}, \nu_{n_0+1}) = 0$, then it follows that $\nu_{n_0}$ belongs to $\mathcal{F}\nu_{n_0}$, and we can conclude that $\nu_{n_0}$ is a fixed point of $\mathcal{F}$, thus completing the proof.

Now we assume that $d_b(\nu_n, \mathcal{F}\nu_n) > 0$ for all $n \in \mathbb{N}$. From the triangular $\alpha$-admissibility of $\mathcal{F}$ follows $\alpha(\nu_1, \nu_2) \geq 1$. By continuously performing this procedure, we can obtain

$$\alpha(\nu_n, \nu_{n+1}) \geq 1 \text{ for every } n \in \mathbb{N} \cup \{0\}. \tag{5}$$

If we substitute $\nu$ by $\nu_{n-1}$ and $\omega$ by $\nu_n$ in **(1)**, we can get

$$\begin{aligned} \theta\big[\mathcal{H}(\mathcal{F}\nu_{n-1}, \mathcal{F}\nu_n)\big] &\leq \theta\big[b^3\mathcal{H}(\mathcal{F}\nu_{n-1}, \mathcal{F}\nu_n)\big] \\ &\leq \theta\big[\alpha(\nu_{n-1}, \nu_n)b^3\mathcal{H}(\mathcal{F}\nu_{n-1}, \mathcal{F}\nu_n)\big] \\ &\leq \big[\theta(M(\nu_{n-1}, \nu_n))\big]^r + L\,N(\nu_{n-1}, \nu_n), \text{ for all } n \in \mathbb{N}, \end{aligned} \tag{6}$$

where

$$\begin{aligned} M(\nu_{n-1}, \nu_n) &= \max\Big\{d_b(\nu_{n-1}, \nu_n), d_b(\nu_{n-1}, \mathcal{F}\nu_{n-1}), d_b(\nu_n, \mathcal{F}\nu_n), \\ &\qquad \frac{d_b(\mathcal{F}\nu_{n-1}, \nu_n) + d_b(\nu_{n-1}, \mathcal{F}\nu_n)}{2b^2}\Big\} \\ &= \max\Big\{d_b(\nu_{n-1}, \nu_n), d_b(\nu_n, \mathcal{F}\nu_n), \frac{d_b(\nu_{n-1}, \mathcal{F}\nu_n)}{2b^2}\Big\} \end{aligned}$$

and

$$\begin{aligned} N(\nu_{n-1}, \nu_n) &= \min\Big\{d_b(\nu_{n-1}, \mathcal{F}\nu_{n-1}), d_b(\nu_n, \mathcal{F}\nu_n), d_b(\mathcal{F}\nu_{n-1}, \nu_n), d_b(\nu_{n-1}, \mathcal{F}\nu_n)\Big\} \\ &= \min\Big\{d_b(\nu_{n-1}, \mathcal{F}\nu_{n-1}), d_b(\nu_n, \mathcal{F}\nu_n), 0, d_b(\nu_{n-1}, \mathcal{F}\nu_n)\Big\} \\ &= 0. \end{aligned}$$

On the other hand, we have

$$\begin{aligned} \frac{1}{2b^2} d_b(\nu_{n-1}, \mathcal{F}\nu_n) &\leq \frac{1}{2b^2}\big[b\big(d_b(\nu_{n-1}, \nu_n) + d_b(\nu_n, \mathcal{F}\nu_n)\big)\big] \\ &= \frac{1}{2b}\big(d_b(\nu_{n-1}, \nu_n) + d_b(\nu_n, \mathcal{F}\nu_n)\big) \\ &\leq \frac{1}{2}\big(d_b(\nu_{n-1}, \nu_n) + d_b(\nu_n, \mathcal{F}\nu_n)\big) \\ &\leq \max\big\{d_b(\nu_{n-1}, \nu_n), d_b(\nu_n, \mathcal{F}\nu_n)\big\}, \end{aligned}$$

and we obtain

$$\mathrm{M}(\nu_{\mathrm{n}-1}, \nu_{\mathrm{n}}) = \max\big\{\mathrm{d}_b(\nu_{\mathrm{n}-1}, \nu_{\mathrm{n}}), \mathrm{d}_b(\nu_{\mathrm{n}}, \mathcal{F}\nu_{\mathrm{n}})\big\}.$$

If $\mathrm{M}(\nu_{\mathrm{n}-1}, \nu_{\mathrm{n}}) = \mathrm{d}_b(\nu_{\mathrm{n}}, \mathcal{F}\nu_{\mathrm{n}})$, since $\mathcal{F}$ is compact, it follows

$$d(\nu_{\mathrm{n}+1}, \nu_{\mathrm{n}}) \leq \mathcal{H}(\mathcal{F}\nu_{\mathrm{n}-1}, \mathcal{F}\nu_{\mathrm{n}}). \tag{7}$$

On the other hand, we have $\nu_{\mathrm{n}+1} \in \mathcal{F}\nu_{\mathrm{n}}$ which implies that $\mathrm{d}_b(\nu_{\mathrm{n}}, \mathcal{F}\nu_{\mathrm{n}}) \leq \mathrm{d}_b(\nu_{\mathrm{n}}, \nu_{\mathrm{n}+1})$. By using **(6)** and **(7)**, we have

$$\begin{aligned}
\theta(\mathrm{d}_b(\nu_{\mathrm{n}+1}, \nu_{\mathrm{n}})) &\leq \theta\big(\mathcal{H}(\mathcal{F}\nu_{\mathrm{n}-1}, \mathcal{F}\nu_{\mathrm{n}})\big) \\
&\leq \Big[\theta\big(\mathrm{M}(\nu_{\mathrm{n}-1}, \nu_{\mathrm{n}})\big)\Big]^{\mathrm{r}} + L\,\mathrm{N}(\nu_{\mathrm{n}-1}, \nu_{\mathrm{n}}) \\
&= \Big[\theta\big(\mathrm{M}(\nu_{\mathrm{n}-1}, \nu_{\mathrm{n}})\big)\Big]^{\mathrm{r}} \\
&< \theta\big(\mathrm{M}(\nu_{\mathrm{n}-1}, \nu_{\mathrm{n}})\big) \\
&= \theta\big(\mathrm{d}_b(\nu_{\mathrm{n}}, \mathcal{F}\nu_{\mathrm{n}})\big) \\
&\leq \theta\big(\mathrm{d}_b(\nu_{\mathrm{n}}, \nu_{\mathrm{n}+1})\big),
\end{aligned}$$

which leads to a contradiction, which implies that the assumption made earlier is incorrect. Consequently, $\mathrm{M}(\nu_{\mathrm{n}-1}, \nu_{\mathrm{n}}) = \mathrm{d}_b(\nu_{\mathrm{n}-1}, \nu_{\mathrm{n}})$. Therefore, using inequality **(6)**, we can derive the following

$$\begin{aligned}
\theta\big(\mathrm{d}_b(\nu_{\mathrm{n}+1}, \nu_{\mathrm{n}}) &\leq \theta\big(\mathcal{H}(\mathcal{F}\nu_{\mathrm{n}-1}, \mathcal{F}\nu_{\mathrm{n}})\big) \\
&\leq \Big[\theta\big(\mathrm{M}(\nu_{\mathrm{n}-1}, \nu_{\mathrm{n}})\big)\Big]^{\mathrm{r}} + L\,\mathrm{N}(\nu_{\mathrm{n}-1}, \nu_{\mathrm{n}}) \\
&= \Big[\theta\big(\mathrm{M}(\nu_{\mathrm{n}-1}, \nu_{\mathrm{n}})\big)\Big]^{\mathrm{r}} \\
&= \big[\theta\big(\mathrm{d}_b(\nu_{\mathrm{n}-1}, \nu_{\mathrm{n}})\big)\big]^{\mathrm{r}} \\
&< \theta\big(\mathrm{d}_b(\nu_{\mathrm{n}-1}, \nu_{\mathrm{n}})\big). \qquad (8)
\end{aligned}$$

From **(8)** and using **($\theta_1$)**, we get

$$\mathrm{d}_b(\nu_{\mathrm{n}}, \nu_{\mathrm{n}+1}) < \mathrm{d}_b(\nu_{\mathrm{n}-1}, \nu_{\mathrm{n}}).$$

As a result, the sequence of nonnegative real numbers $\{\mathrm{d}_b(\nu_{\mathrm{n}}, \nu_{\mathrm{n}+1})\}_{\mathrm{n}\in\mathbb{N}}$ is strictly decreasing. This implies the existence of a nonnegative numbers $\gamma$ that satisfy the following equality

$$\lim_{\mathrm{n}\to+\infty} \mathrm{d}_b(\nu_{\mathrm{n}+1}, \nu_{\mathrm{n}}) = \gamma.$$

Now, our assertion is that $\gamma = 0$. To demonstrate this, we will use a proof by contradiction and suppose that $\gamma > 0$. As the sequence $\{(\mathrm{d}_b(\nu_{\mathrm{n}}, \nu_{\mathrm{n}+1})\}_{\mathrm{n}\in\mathbb{N}}$ is both decreasing and positive, we can conclude that

$$\mathrm{d}_b(\nu_{\mathrm{n}}, \nu_{\mathrm{n}+1}) \geq \gamma, \quad \text{for all } \mathrm{n} \in \mathbb{N}.$$

Repeating **(8)**, we get

$$\begin{aligned}\theta(\mathrm{d}_b(\nu_{\mathrm{n}},\nu_{\mathrm{n}+1})) &\leq (\theta(\mathrm{d}_b(\nu_{\mathrm{n}-1},\nu_{\mathrm{n}})))^{\mathrm{r}}\\ &\leq (\theta(\mathrm{d}_b(\nu_{\mathrm{n}-2},\nu_{\mathrm{n}-1})))^{\mathrm{r}^2}\\ &\vdots\\ &\leq \theta(\mathrm{d}_b(\nu_0,\nu_1))^{\mathrm{r}^{\mathrm{n}}}. \end{aligned} \tag{9}$$

Using the property of $\theta$ and inequality **(9)**, we come to

$$1 < \theta(\gamma) \leq \theta(\mathrm{d}_b(\nu_0,\nu_1))^{\mathrm{r}^{\mathrm{n}}}. \tag{10}$$

As n tends to $+\infty$ in inequality **(10)**, we can conclude that

$$1 < \theta(\gamma) \leq 1.$$

This leads to a contradiction. Hence,

$$\lim_{\mathrm{n}\to+\infty} \mathrm{d}_b(\nu_{\mathrm{n}},\nu_{\mathrm{n}+1}) = 0. \tag{11}$$

Next, our goal is to establish the Cauchy sequence property of $\{\nu_{\mathrm{n}}\}_{\mathrm{n}\in\mathbb{N}}$. Suppose the opposite. According to Lemma **2**, there exists $\epsilon > 0$ such that for a given integer k we can find two sequences $\{\mathrm{m}_{(\mathrm{k})}\}$ and $\{\mathrm{n}_{(\mathrm{k})}\}$ for which it holds:

**i)** $\epsilon \leq \lim_{\mathrm{k}\to+\infty} \inf \mathrm{d}_b(\nu_{\mathrm{m}_{(\mathrm{k})}},\nu_{\mathrm{n}_{(\mathrm{k})}}) \leq \lim_{\mathrm{k}\to+\infty} \sup \mathrm{d}_b(\nu_{\mathrm{m}_{(\mathrm{k})}},\nu_{\mathrm{n}_{(\mathrm{k})}}) \leq b\epsilon,$

**ii)** $\dfrac{\epsilon}{b} \leq \lim_{\mathrm{k}\to+\infty} \inf \mathrm{d}_b(\nu_{\mathrm{n}_{(\mathrm{k})}},\nu_{\mathrm{m}_{(\mathrm{k})+1}}) \leq \lim_{\mathrm{k}\to+\infty} \sup \mathrm{d}_b(\nu_{\mathrm{n}_{(\mathrm{k})}},\nu_{\mathrm{m}_{(\mathrm{k})+1}}) \leq b^2\epsilon,$

**iii)** $\dfrac{\epsilon}{b} \leq \lim_{\mathrm{k}\to+\infty} \inf \mathrm{d}_b(\nu_{\mathrm{m}_{(\mathrm{k})}},\nu_{\mathrm{n}_{(\mathrm{k})+1}}) \leq \lim_{\mathrm{k}\to+\infty} \sup \mathrm{d}_b(\nu_{\mathrm{m}_{(\mathrm{k})}},\nu_{\mathrm{n}_{(\mathrm{k})+1}}) \leq b^2\epsilon,$

**vi)** $\dfrac{\epsilon}{b^2} \leq \lim_{\mathrm{k}\to+\infty} \inf \mathrm{d}_b(\nu_{\mathrm{m}_{(\mathrm{k})+1}},\nu_{\mathrm{n}_{(\mathrm{k})+1}}) \leq \lim_{\mathrm{k}\to+\infty} \sup \mathrm{d}_b(\nu_{\mathrm{m}_{(\mathrm{k})+1}},\nu_{\mathrm{n}_{(\mathrm{k})+1}}) \leq b^3\epsilon.$

As $\mathcal{F}$ is a triangular $\alpha$-admissible, then by using Lemma **6** we have

$$\alpha(\nu_{\mathrm{m}_{(\mathrm{k})}},\nu_{\mathrm{n}_{(\mathrm{k})}}) \geq 1. \tag{12}$$

We also have

$$\begin{aligned}\mathrm{M}(\nu_{\mathrm{m}_{(\mathrm{k})}},\nu_{\mathrm{n}_{(\mathrm{k})}}) = \max\Big\{&\mathrm{d}_b(\nu_{\mathrm{m}_{(\mathrm{k})}},\nu_{\mathrm{n}_{(\mathrm{k})}}),\mathrm{d}_b(\nu_{\mathrm{m}_{(\mathrm{k})}},\mathcal{F}\nu_{\mathrm{m}_{(\mathrm{k})}}),\mathrm{d}_b(\nu_{\mathrm{n}_{(\mathrm{k})}},\mathcal{F}\nu_{\mathrm{n}_{(\mathrm{k})}}),\\ &\frac{1}{2b^2}\big(\mathrm{d}_b(\nu_{\mathrm{n}_{(\mathrm{k})}},\mathcal{F}\nu_{\mathrm{m}_{(\mathrm{k})}})+\mathrm{d}_b(\nu_{\mathrm{m}_{(\mathrm{k})}},\mathcal{F}\nu_{\mathrm{n}_{(\mathrm{k})}})\big)\Big\}\\ \leq \max\Big\{&\mathrm{d}_b(\nu_{\mathrm{m}_{(\mathrm{k})}},\nu_{\mathrm{n}_{(\mathrm{k})}}),\mathrm{d}_b(\nu_{\mathrm{m}_{(\mathrm{k})}},\nu_{\mathrm{m}_{(\mathrm{k})+1}}),\mathrm{d}_b(\nu_{\mathrm{n}_{(\mathrm{k})}},\nu_{\mathrm{n}_{(\mathrm{k})+1}}),\\ &\frac{1}{2b^2}\big(\mathrm{d}_b(\nu_{\mathrm{n}_{(\mathrm{k})}},\nu_{\mathrm{m}_{(\mathrm{k})+1}})+\mathrm{d}_b(\nu_{\mathrm{m}_{(\mathrm{k})}},\nu_{\mathrm{n}_{(\mathrm{k})+1}})\big)\Big\}\end{aligned}$$

and

$$\begin{aligned}\mathrm{N}(\nu_{\mathrm{m}_{(\mathrm{k})}},\nu_{\mathrm{n}_{(\mathrm{k})}}) &= \min\Big\{\mathrm{d}_b(\nu_{\mathrm{m}_{(\mathrm{k})}},\mathcal{F}\nu_{\mathrm{m}_{(\mathrm{k})}}),\mathrm{d}_b(\nu_{\mathrm{n}_{(\mathrm{k})}},\mathcal{F}\nu_{\mathrm{n}_{(\mathrm{k})}}),\mathrm{d}_b(\nu_{\mathrm{n}_{(\mathrm{k})}},\mathcal{F}\nu_{\mathrm{m}_{(\mathrm{k})}}),\\ &\qquad \mathrm{d}_b(\nu_{\mathrm{m}_{(\mathrm{k})}},\mathcal{F}\nu_{\mathrm{n}_{(\mathrm{k})}})\Big\}\\ &\leq \min\Big\{\mathrm{d}_b(\nu_{\mathrm{m}_{(\mathrm{k})}},\nu_{\mathrm{m}_{(\mathrm{k})+1}}),\mathrm{d}_b(\nu_{\mathrm{n}_{(\mathrm{k})}},\nu_{\mathrm{n}_{(\mathrm{k})+1}}),\mathrm{d}_b(\nu_{\mathrm{n}_{(\mathrm{k})}},\nu_{\mathrm{m}_{(\mathrm{k})+1}}),\\ &\qquad \mathrm{d}_b(\nu_{\mathrm{m}_{(\mathrm{k})}},\nu_{\mathrm{n}_{(\mathrm{k})+1}})\Big\}.\end{aligned}$$

By applying Lemma **2** and taking $\lim_{\mathrm{k}\to+\infty}$ by using **(11)**, we obtain

$$\begin{aligned}\lim_{\mathrm{k}\to+\infty}\mathrm{M}(\nu_{\mathrm{m}_{(\mathrm{k})}},\nu_{\mathrm{n}_{(\mathrm{k})}}) &\leq \lim_{\mathrm{k}\to+\infty}\max\Bigg\{\mathrm{d}_b(\nu_{\mathrm{m}_{(\mathrm{k})}},\nu_{\mathrm{n}_{(\mathrm{k})}}),\mathrm{d}_b(\nu_{\mathrm{m}_{(\mathrm{k})}},\nu_{\mathrm{m}_{(\mathrm{k})+1}}),\\ &\qquad \mathrm{d}_b(\nu_{\mathrm{n}_{(\mathrm{k})}},\nu_{\mathrm{n}_{(\mathrm{k})+1}}),\frac{1}{2b^2}(\mathrm{d}_b(\nu_{\mathrm{n}_{(\mathrm{k})}},\nu_{\mathrm{m}_{(\mathrm{k})+1}})+\mathrm{d}_b(\nu_{\mathrm{m}_{(\mathrm{k})}},\nu_{\mathrm{n}_{(\mathrm{k})+11}})\Bigg\}\\ &\leq \max\Big\{b\epsilon,0,0,\frac{1}{2b^2}(b^2\epsilon+b^2\epsilon)\Big\}\\ &= b\epsilon\end{aligned}$$

and

$$\begin{aligned}\lim_{\mathrm{k}\to+\infty}\mathrm{N}(\nu_{\mathrm{m}_{(\mathrm{k})}},\nu_{\mathrm{n}_{(\mathrm{k})}}) &\leq \lim_{\mathrm{k}\to+\infty}\min\Big\{\mathrm{d}_b(\nu_{\mathrm{m}_{(\mathrm{k})}},\nu_{\mathrm{m}_{(\mathrm{k})+1}}),\mathrm{d}_b(\nu_{\mathrm{n}_{(\mathrm{k})}},\nu_{\mathrm{n}_{(\mathrm{k})+1}}),\\ &\qquad \mathrm{d}_b(\nu_{\mathrm{n}_{(\mathrm{k})}},\nu_{\mathrm{m}_{(\mathrm{k})+1}}),\mathrm{d}_b(\nu_{\mathrm{m}_{(\mathrm{k})}},\nu_{\mathrm{n}_{(\mathrm{k})+1}})\Big\}\\ &\leq \min\Big\{0,0,b^2\epsilon,b^2\epsilon\Big\}\\ &= 0.\end{aligned}$$

So, we have

$$\begin{aligned}&\lim_{\mathrm{k}\to+\infty}\mathrm{N}(\nu_{\mathrm{m}_{(\mathrm{k})}},\nu_{\mathrm{n}_{(\mathrm{k})}}) = 0,\\ &\lim_{\mathrm{k}\to+\infty}\mathrm{M}(\nu_{\mathrm{m}_{(\mathrm{k})}},\nu_{\mathrm{n}_{(\mathrm{k})}}) \leq b\epsilon.\end{aligned}\tag{13}$$

Now, from **(12)** and letting $\nu=\nu_{\mathrm{m}_{(\mathrm{k})}}$, $\omega=\nu_{\mathrm{n}_{(\mathrm{k})}}$ in **(1)**, we obtain

$$\begin{aligned}\theta[b^3\mathrm{d}_b(\nu_{\mathrm{m}_{(\mathrm{k})+1}},\nu_{\mathrm{n}_{(\mathrm{k})+1}})] &\leq \theta[b^3\mathcal{H}(\mathcal{F}\nu_{\mathrm{m}_{(\mathrm{k})}},\mathcal{F}\nu_{\mathrm{n}_{(\mathrm{k})}})]\\ &\leq \theta[\alpha(\nu_{\mathrm{m}_{(\mathrm{k})}},\nu_{\mathrm{n}_{(\mathrm{k})}})b^3\mathcal{H}(\mathcal{F}\nu_{\mathrm{m}_{(\mathrm{k})}},\mathcal{F}\nu_{\mathrm{n}_{(\mathrm{k})}})]\\ &\leq [\theta(\mathrm{M}(\nu_{\mathrm{m}_{(\mathrm{k})}},\nu_{\mathrm{n}_{(\mathrm{k})}}))]^{\mathrm{r}}+L\,\mathrm{N}(\nu_{\mathrm{m}_{(\mathrm{k})}},\nu_{\mathrm{n}_{(\mathrm{k})}}).\end{aligned}\tag{14}$$

By utilizing inequalities **(13)** and **(14)** and applying the continuity of $\theta$, we can obtain the inequality below by letting $k \longrightarrow +\infty$

$$\begin{aligned}
\theta(\epsilon b) &= \theta(\frac{\epsilon}{b^2}b^3) \\
&\leq \theta\Big(b^3 \lim_{\mathrm{k}\to+\infty} \mathrm{d}_b(\nu_{\mathrm{m}_{(\mathrm{k})+1}}, \nu_{\mathrm{n}_{(\mathrm{k})+1}})\Big) \\
&\leq \Big[\theta\Big(\lim_{\mathrm{k}\to+\infty} \mathrm{M}(\nu_{\mathrm{m}_{(\mathrm{k})}}, \nu_{\mathrm{n}_{(\mathrm{k})}})\Big)\Big]^{\mathrm{r}} + L \lim_{\mathrm{k}\to+\infty} \mathrm{N}(\nu_{\mathrm{m}_{(\mathrm{k})}}, \nu_{\mathrm{n}_{(\mathrm{k})}}) \\
&= \Big[\theta\Big(\lim_{\mathrm{k}\to+\infty} \mathrm{M}(\nu_{\mathrm{m}_{(\mathrm{k})}}, \nu_{\mathrm{n}_{(\mathrm{k})}})\Big)\Big]^{\mathrm{r}} \\
&\leq [\theta(b\epsilon)]^{\mathrm{r}} \\
&< \theta(b\epsilon).
\end{aligned}$$

Due to the fact that $\theta$ is a monotonically increasing function, we come to the contradiction

$$b\epsilon < b\epsilon.$$

Therefore, the sequence $\{\nu_{\mathrm{n}}\}_{\mathrm{n}\in\mathbb{N}}$ is a Cauchy sequence in $\Upsilon$. As a consequence of the completeness of $(\Upsilon, d_b)$, we can find $\sigma$ as element of $\Upsilon$ such that

$$\lim_{\mathrm{n}\to+\infty} \mathrm{d}_b(\nu_{\mathrm{n}}, \sigma) = 0.$$

**Case 1:** If $\mathcal{F}$ is $\alpha$-continuous multivalued mapping, according to **(4a)** we can conclude that

$$\lim_{n\to+\infty} \mathcal{H}(\mathcal{F}\nu_n, \mathcal{F}\sigma) \ = 0.$$

Thus we obtain

$$\mathrm{d}_b(\omega, \mathcal{F}\sigma) = \lim_{n\to+\infty} \mathrm{d}_b(\nu_{n+1}, \mathcal{F}\sigma) \leq \lim_{n\to+\infty} \mathcal{H}(\mathcal{F}\nu_n, \mathcal{F}\sigma) = 0.$$

Therefore, $\sigma \in \mathcal{F}\sigma$ and hence $\mathcal{F}$ possesses a fixed point.

**Case 2:** If $\mathcal{F}$ is not $\alpha$-continuous multivalued mapping, according to **(4b)** we can show that $\sigma \in \mathcal{F}\sigma$ using contradiction. Assume that

$$\sigma \notin \mathcal{F}\sigma.$$

We know that $0 \leq \mathrm{d}_b(\mathcal{F}\nu_{\mathrm{n}}, \sigma) \leq \mathrm{d}_b(\nu_{\mathrm{n}+1}, \sigma)$, hence

$$\lim_{\mathrm{n}\to+\infty} \mathrm{d}_b(\mathcal{F}\nu_{\mathrm{n}}, \sigma) = 0.$$

Since $\lim_{\mathrm{n}\to+\infty} \mathcal{F}\nu = \sigma$, based on Lemma **2.2**, we can conclude

$$\begin{aligned}
\frac{1}{b^2}\mathrm{d}_b(\sigma, \mathcal{F}\sigma) &\leq \lim_{\mathrm{n}\to+\infty} \inf \mathcal{H}(\mathcal{F}\nu_{\mathrm{n}}, \mathcal{F}\sigma) \\
&\leq \lim_{\mathrm{n}\to+\infty} \sup \mathcal{H}(\mathcal{F}\nu_{\mathrm{n}}, \mathcal{F}\sigma) \\
&\leq b^2\mathrm{d}_b(\sigma, \mathcal{F}\sigma). \qquad (15)
\end{aligned}$$

Now, letting $\nu = \nu_{\mathrm{n}}$ and $\omega = \sigma$ in **(1)**, we have

$$\begin{aligned}\theta\big(b^3\mathcal{H}(\mathcal{F}\nu_{\mathrm{n}},\mathcal{F}\sigma)\big) &\leq \theta\big(\alpha(\nu_{\mathrm{n}},\sigma)b^3\mathcal{H}(\mathcal{F}\nu_{\mathrm{n}},\mathcal{F}\sigma)\big)\\ &\leq [\theta(\mathrm{M}(\nu_{\mathrm{n}},\sigma))]^{\mathrm{r}} + L\,\mathrm{N}(\nu_{\mathrm{n}},\sigma), \text{ for all } \mathrm{n}\in\mathbb{N},\end{aligned}$$

where

$$\mathrm{M}(\nu_{\mathrm{n}},\sigma) = \max\Big\{\mathrm{d}_b(\nu_{\mathrm{n}},\sigma), \mathrm{d}_b(\nu_{\mathrm{n}},\mathcal{F}\nu_{\mathrm{n}}), \mathrm{d}_b(\sigma,\mathcal{F}\sigma), \frac{1}{2b^2}\big(\mathrm{d}_b(\sigma,\mathcal{F}\nu_{\mathrm{n}}) + \mathrm{d}_b(\nu_{\mathrm{n}},\mathcal{F}\sigma)\big)\Big\}$$

and

$$\mathrm{N}(\nu_{\mathrm{n}},\sigma) = \min\Big\{\mathrm{d}_b(\nu_{\mathrm{n}},\mathcal{F}\nu_{\mathrm{n}}), \mathrm{d}_b(\sigma,\mathcal{F}\sigma), \mathrm{d}_b(\sigma,\mathcal{F}\nu_{\mathrm{n}}), \mathrm{d}_b(\nu_{\mathrm{n}},\mathcal{F}\sigma)\Big\}.$$

Taking $\lim_{\mathrm{n}\longrightarrow+\infty}$, we obtain

$$\begin{aligned}\lim_{\mathrm{n}\to+\infty}\sup \mathrm{M}(\nu_{\mathrm{n}},\sigma) &= \lim_{\mathrm{n}\to+\infty}\sup\max\Big\{\mathrm{d}_b(\nu_{\mathrm{n}},\sigma), \mathrm{d}_b(\nu_{\mathrm{n}},\mathcal{F}\nu_{\mathrm{n}}), \mathrm{d}_b(\sigma,\mathcal{F}\sigma),\\ &\qquad \frac{1}{2b^2}\big(\mathrm{d}_b(\sigma,\mathcal{F}\nu_{\mathrm{n}}) + \mathrm{d}_b(\nu_{\mathrm{n}},\mathcal{F}\sigma)\big)\Big\}\\ &\leq \lim_{\mathrm{n}\to+\infty}\sup\max\Big\{\mathrm{d}_b(\nu_{\mathrm{n}},\sigma), \mathrm{d}_b(\nu_{\mathrm{n}},\nu_{\mathrm{n}+1}), \mathrm{d}_b(\sigma,\mathcal{F}\sigma),\\ &\qquad \frac{1}{2b^2}\big(\mathrm{d}_b(\sigma,\nu_{\mathrm{n}+1}) + \mathrm{d}_b(\nu_{\mathrm{n}},\mathcal{F}\sigma)\big)\Big\}\\ &\leq \mathrm{d}_b(\sigma,\mathcal{F}\sigma)\end{aligned}$$

and

$$\begin{aligned}\lim_{\mathrm{n}\to+\infty}\sup \mathrm{N}(\nu_{\mathrm{n}},\sigma) &= \lim_{\mathrm{n}\to+\infty}\sup\min\Big\{\mathrm{d}_b(\nu_{\mathrm{n}},\mathcal{F}\nu_{\mathrm{n}}), \mathrm{d}_b(\sigma,\mathcal{F}\sigma), \mathrm{d}_b(\sigma,\mathcal{F}\nu_{\mathrm{n}}), \mathrm{d}_b(\nu_{\mathrm{n}},\mathcal{F}\sigma)\Big\}\\ &\leq \lim_{\mathrm{n}\to+\infty}\sup\min\Big\{\mathrm{d}_b(\nu_{\mathrm{n}},\nu_{\mathrm{n}+1}), \mathrm{d}_b(\sigma,\mathcal{F}\sigma), \mathrm{d}_b(\sigma,\nu_{\mathrm{n}+1}), \mathrm{d}_b(\nu_{\mathrm{n}},\mathcal{F}\sigma)\Big\}\\ &= \min\Big\{0, \mathrm{d}_b(\sigma,\mathcal{F}\sigma), \mathrm{d}_b(\sigma,z), z,\mathcal{F}\sigma)\Big\}\\ &= 0.\end{aligned}$$

Therefore,

$$\begin{aligned}&\theta\big(b^3\mathcal{H}(\mathcal{F}\nu_{\mathrm{n}},\mathcal{F}\sigma)\big)\\ &\quad\leq \Big[\theta\Big(\max\Big\{\mathrm{d}_b(\nu_{\mathrm{n}},\sigma), \mathrm{d}_b(\nu_{\mathrm{n}},\mathcal{F}\nu_{\mathrm{n}}), \mathrm{d}_b(\sigma,\mathcal{F}\sigma), \frac{1}{2b^2}\big(\mathrm{d}_b(\sigma,\mathcal{F}\nu_{\mathrm{n}}) + \mathrm{d}_b(\nu_{\mathrm{n}},\mathcal{F}\sigma)\big)\Big\}\Big)\Big]^{\mathrm{r}}\\ &\quad\quad + L\,\min\Big\{\mathrm{d}_b(\nu_{\mathrm{n}},\mathcal{F}\nu_{\mathrm{n}}), \mathrm{d}_b(\sigma,\mathcal{F}\sigma), \mathrm{d}_b(\sigma,\mathcal{F}\nu_{\mathrm{n}}), \mathrm{d}_b(\nu_{\mathrm{n}},\mathcal{F}\sigma)\Big\}.\end{aligned} \tag{16}$$

Taking $\lim_{\mathrm{n}\longrightarrow+\infty}$ in **(15)**, **(16)** and using **(11)** and $(\theta_3)$, we obtain

$$\begin{aligned}
\theta[b\,\mathrm{d}_b(\sigma,\mathcal{F}\sigma)] &\leq \theta[b^3 \lim_{\mathrm{n}\to+\infty} \mathcal{H}(\mathcal{F}\nu_{\mathrm{n}},\mathcal{F}\sigma)] \\
&\leq \lim_{\mathrm{n}\to+\infty} \theta[b^3\mathcal{H}(\mathcal{F}\nu_{\mathrm{n}},\mathcal{F}\sigma)] \\
&\leq \lim_{\mathrm{n}\to+\infty} \theta[\alpha(\nu_{\mathrm{n}},z)b^3\mathcal{H}(\mathcal{F}\nu_{\mathrm{n}},\mathcal{F}\sigma)] \\
&\leq \lim_{\mathrm{n}\to+\infty} \Big(\theta\big[\mathrm{M}(\nu_{\mathrm{n}},z)\big]^r + L\,\mathrm{N}(\nu_{\mathrm{n}},z)\Big) \\
&\leq \theta\big[\lim_{\mathrm{n}\to+\infty} \mathrm{M}(\nu_{\mathrm{n}},z)\big]^r + L \lim_{\mathrm{n}\to+\infty} \mathrm{N}(\nu_{\mathrm{n}},z) \\
&\leq [\theta(\mathrm{d}_b(\sigma,\mathcal{F}\sigma))]^{\mathrm{r}} \\
&< \theta(\mathrm{d}_b(\sigma,\mathcal{F}\sigma)).
\end{aligned}$$

Using $(\theta_1)$, we obtain

$$b\mathrm{d}_b(\sigma,\mathcal{F}\sigma) < \mathrm{d}_b(\sigma,\mathcal{F}\sigma).$$

This implies that

$$\mathrm{d}_b(\sigma,\mathcal{F}\sigma)(b-1) < 0,$$

hence

$$b < 1.$$

This leads to a contradiction, so it follows that $\sigma$ belongs to $\mathcal{F}\sigma$. □

**Theorem 4.** *Let $(\Upsilon,\mathrm{d}_b)$ be a complete b-metric space and $\mathcal{F}:\Upsilon\to\mathcal{K}(\Upsilon)$ be an $\alpha$-admissible $\theta$-$\phi$-Multivalued contraction satisfying*

***(1)*** $(\Upsilon,\mathrm{d}_b)$ *is an $\alpha$-complete metric space;*

***(2)*** *there are $\nu_0\in\Upsilon$ and $\nu_1\in\mathcal{F}\nu_0$ so that $\alpha(\nu_0,\nu_1)\geq 1$;*

***(3)*** *$\mathcal{F}$ is a triangular $\alpha$-admissible;*

***(4)*** *either*

- ***(4a)*** *$\mathcal{F}$ is an $\alpha$-continuous multivalued mapping,*
  *or*
- ***(4b)*** *If $\{\nu_{\mathrm{n}}\}\subset\Upsilon$ so that $\alpha(\nu_{\mathrm{n}},\nu_{\mathrm{n}+1})\geq 1$ for every $\mathrm{n}\in\mathbb{N}$ and $\lim_{\mathrm{n}\to+\infty}\nu_{\mathrm{n}}=\nu\in\Upsilon$, then we get $\alpha(\nu_{\mathrm{n}},\nu)\geq 1$ for every $\mathrm{n}\in\mathbb{N}$.*

*Then $\mathcal{F}$ possesses a fixed point.*

*Proof.* Let $\nu_0 \in \Upsilon$ and $\nu_1 \in \mathcal{F}\nu_0$ such that $\alpha(\nu_0, \nu_1) \geq 1$. We define a sequence $\{\nu_{\mathrm{n}}\}$ by

$$\nu_{\mathrm{n}+1} \in \mathcal{F}\nu_{\mathrm{n}}$$

for all $\mathrm{n} \in \mathbb{N}$. If there is $\mathrm{n}_0 \in \mathbb{N}$ with $\mathrm{d}_b(\nu_{\mathrm{n}_0}, \nu_{\mathrm{n}_0+1}) = 0$, then it follows that $\nu_{\mathrm{n}_0}$ belongs to $\mathcal{F}\nu_{\mathrm{n}_0}$, and we can conclude that $\nu_{\mathrm{n}_0}$ is a fixed point of $\mathcal{F}$, thus completing the proof.

Now we assume that $\mathrm{d}_b(\nu_{\mathrm{n}}, \mathcal{F}\nu_{\mathrm{n}}) > 0$ for all $\mathrm{n} \in \mathbb{N}$. From the triangular $\alpha$-admissibility of $\mathcal{F}$ follows $\alpha(\nu_1, \nu_2) \geq 1$. By continuously performing this procedure, we can obtain

$$\alpha(\nu_{\mathrm{n}}, \nu_{\mathrm{n}+1}) \geq 1 \text{ for every } \mathrm{n} \in \mathbb{N} \cup \{0\}. \tag{17}$$

Since $\mathcal{F}$ is compact and if we substitute $\nu$ by $\nu_{\mathrm{n}-1}$ and $\omega$ by $\nu_{\mathrm{n}}$ in **(2)**, we can get

$$\begin{aligned}
\theta[\mathrm{d}_b(\nu_{\mathrm{n}-1}, \nu_{\mathrm{n}})] &\leq \theta[\mathcal{H}(\mathcal{F}\nu_{\mathrm{n}-1}, \mathcal{F}\nu_{\mathrm{n}})] \\
&\leq \theta[b^3\mathcal{H}(\mathcal{F}\nu_{\mathrm{n}-1}, \mathcal{F}\nu_{\mathrm{n}})] \\
&\leq \theta[\alpha(\nu_{\mathrm{n}-1}, \nu_{\mathrm{n}})b^3\mathcal{H}(\mathcal{F}\nu_{\mathrm{n}-1}, \mathcal{F}\nu_{\mathrm{n}})] \\
&\leq \phi[\theta\big(\mathrm{M}(\nu_{\mathrm{n}-1}, \nu_{\mathrm{n}})\big] + L\mathrm{N}(\nu_{\mathrm{n}-1}, \nu_{\mathrm{n}}), \text{ for all } \mathrm{n} \in \mathbb{N}, \qquad (18)
\end{aligned}$$

where

$$\begin{aligned}
\mathrm{M}(\nu_{\mathrm{n}-1}, \nu_{\mathrm{n}}) &= \max\Big\{\mathrm{d}_b(\nu_{\mathrm{n}-1}, \nu_{\mathrm{n}}), \mathrm{d}_b(\nu_{\mathrm{n}-1}, \mathcal{F}\nu_{\mathrm{n}-1}),\ \mathrm{d}_b(\nu_{\mathrm{n}}, \mathcal{F}\nu_{\mathrm{n}}), \\
&\qquad \frac{\mathrm{d}_b(\mathcal{F}\nu_{\mathrm{n}-1}, \nu_{\mathrm{n}}) + \mathrm{d}_b(\nu_{\mathrm{n}-1}, \mathcal{F}\nu_{\mathrm{n}})}{2b^2}\Big\} \\
&\leq \max\Big\{\mathrm{d}_b(\nu_{\mathrm{n}-1}, \nu_{\mathrm{n}}), \mathrm{d}_b(\nu_{\mathrm{n}}, \mathcal{F}\nu_{\mathrm{n}}), \frac{\mathrm{d}_b(\nu_{\mathrm{n}-1}, \mathcal{F}\nu_{\mathrm{n}})}{2b^2}\Big\}
\end{aligned}$$

and

$$\begin{aligned}
\mathrm{N}(\nu_{\mathrm{n}-1}, \nu_{\mathrm{n}}) &= \min\Big\{\mathrm{d}_b(\nu_{\mathrm{n}-1}, \mathcal{F}\nu_{\mathrm{n}-1}), \mathrm{d}_b(\nu_{\mathrm{n}}, \mathcal{F}\nu_{\mathrm{n}}), \mathrm{d}_b(\mathcal{F}\nu_{\mathrm{n}-1}, \nu_{\mathrm{n}}), \mathrm{d}_b(\nu_{\mathrm{n}-1}, \mathcal{F}\nu_{\mathrm{n}})\Big\} \\
&= \min\Big\{\mathrm{d}_b(\nu_{\mathrm{n}-1}, \mathcal{F}\nu_{\mathrm{n}-1}), \mathrm{d}_b(\nu_{\mathrm{n}}, \mathcal{F}\nu_{\mathrm{n}}), 0, \mathrm{d}_b(\nu_{\mathrm{n}-1}, \mathcal{F}\nu_{\mathrm{n}})\Big\} \\
&= 0.
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
\frac{1}{2b^2}\mathrm{d}_b(\nu_{\mathrm{n}-1}, \mathcal{F}\nu_{\mathrm{n}}) &\leq \frac{1}{2b^2}\big[b\big(\mathrm{d}_b(\nu_{\mathrm{n}-1}, \nu_{\mathrm{n}}) + \mathrm{d}_b(\nu_{\mathrm{n}}, \mathcal{F}\nu_{\mathrm{n}})\big)\big] \\
&= \frac{1}{2b}\big(\mathrm{d}_b(\nu_{\mathrm{n}-1}, \nu_{\mathrm{n}}) + \mathrm{d}_b(\nu_{\mathrm{n}}, \mathcal{F}\nu_{\mathrm{n}})\big) \\
&\leq \frac{1}{2}\big(\mathrm{d}_b(\nu_{\mathrm{n}-1}, \nu_{\mathrm{n}}) + \mathrm{d}_b(\nu_{\mathrm{n}}, \mathcal{F}\nu_{\mathrm{n}})\big) \\
&\leq \max\Big\{\mathrm{d}_b(\nu_{\mathrm{n}-1}, \nu_{\mathrm{n}}), \mathrm{d}_b(\nu_{\mathrm{n}}, \mathcal{F}\nu_{\mathrm{n}})\Big\},
\end{aligned}$$

and we obtain

$$\mathrm{M}(\nu_{n-1},\nu_n)=\max\left\{\mathrm{d}_b(\nu_{n-1},\nu_n),\mathrm{d}_b(\nu_n,\mathcal{F}\nu_n)\right\}.$$

If $\mathrm{M}(\nu_{n-1},\nu_n)=\mathrm{d}_b(\nu_n,\mathcal{F}\nu_n)$, since $\nu_{n+1}\in\mathcal{F}\nu_n$, it follows $\mathrm{d}_b(\nu_n,\mathcal{F}\nu_n)\le\mathrm{d}_b(\nu_n,\nu_{n+1})$. By using **(18)**, we have

$$\begin{aligned}\theta(\mathrm{d}_b(\nu_{n+1},\nu_n)) &\le \phi\Big(\theta\big(\mathrm{M}(\nu_{n-1},\nu_n)\big)\Big)+L\,\mathrm{N}(\nu_{n-1},\nu_n)\\ &=\phi\Big(\theta\big(\mathrm{M}(\nu_{n-1},\nu_n)\Big)\\ &=\phi\Big(\theta\big(\mathrm{d}_b(\nu_n,\mathcal{F}\nu_n)\big)\Big)\\ &\le\phi\Big(\theta\big(\mathrm{d}_b(\nu_n,\nu_{n+1})\big)\Big)\\ &<\theta\big(\mathrm{d}_b(\nu_n,\nu_{n+1})\big),\end{aligned}$$

which leads to a contradiction, which implies that the assumption made earlier is incorrect. Consequently, $\mathrm{M}(\nu_{n-1},\nu_n)=\mathrm{d}_b(\nu_{n-1},\nu_n)$. Thus

$$\begin{aligned}\theta(\mathrm{d}_b(\nu_n,\nu_{n+1})) &\le\phi\Big(\theta\big(\mathrm{d}_b(\nu_{n-1},\nu_n)\big)\Big)\\ &<\theta\big(\mathrm{d}_b(\nu_{n-1},\nu_n)\big).\end{aligned}\tag{19}$$

From **(19)** and using Lemma **3** and $(\theta_1)$, we get

$$\mathrm{d}_b(\nu_n,\nu_{n+1})<\mathrm{d}_b(\nu_{n-1},\nu_n).$$

As a result, the sequence of nonnegative real numbers $\{\mathrm{d}_b(\nu_n,\nu_{n+1})\}_{n\in\mathbb{N}}$ is strictly decreasing. This implies the existence of a nonnegative numbers $\gamma$ that satisfy the following equality

$$\lim_{n\to+\infty}\mathrm{d}_b(\nu_{n+1},\nu_n)=\gamma.$$

Now, our assertion is that $\gamma=0$. To demonstrate this, we will use a proof by contradiction, assuming that $\gamma>0$. Since $\{\mathrm{d}_b(\nu_n,\nu_{n+1})\}_{n\in\mathbb{N}}$ is decreasing sequence of nonegative numbers, it follows that

$$\mathrm{d}_b(\nu_n,\nu_{n+1})\ge\gamma,\quad\text{for all } n\in\mathbb{N}.$$

Repeating **(19)**, we get

$$\begin{aligned}\theta(\mathrm{d}_b(\nu_n,\nu_{n+1})) &\le\phi\Big(\theta\big(\mathrm{d}_b(\nu_{n-1},\nu_n)\big)\Big)\\ &\le\phi^2\Big(\theta\big(\mathrm{d}_b(\nu_{n-2},\nu_{n-1})\big)\Big)\\ &\ \ \vdots\\ &\le\phi^n\Big(\theta\big(\mathrm{d}_b(\nu_0,\nu_1)\big)\Big).\end{aligned}$$

This implies that

$$\begin{aligned} 1 &< \theta(\alpha) \\ &\leq \theta\big(\mathrm{d}_b(\nu_{\mathrm{n}}, \nu_{\mathrm{n}+1})\big) \\ &\leq \phi^{\mathrm{n}}\Big(\theta\big(\mathrm{d}_b(\nu_0, \nu_1)\big)\Big). \end{aligned}$$

Taking $\lim_{\mathrm{n}\longrightarrow+\infty}$ and applying the property of $\phi$ and $\theta$, we can conclude that

$$\begin{aligned} 1 &< \theta(\alpha) \\ &\leq \lim_{\mathrm{n}\to+\infty} \phi^{\mathrm{n}}\Big(\theta\big(\mathrm{d}_b(\nu_0, \nu_1)\big)\Big) \\ &= 1, \end{aligned}$$

which contradicts our assumptions. Therefore, we can conclude that $\gamma = 0$, and consequently, we obtain the following result

$$\lim_{\mathrm{n}\to+\infty} \mathrm{d}_b(\nu_{\mathrm{n}}, \nu_{\mathrm{n}+1}) = 0.$$

Now we will demonstrate that $\{\nu_{\mathrm{n}}\}_{\mathrm{n}\in\mathbb{N}}$ is a Cauchy sequence. Let us assume the opposite. According to Lemma **2**, there exists $\epsilon > 0$ such that for some integer $K$ there are two sequences $\{\mathrm{m}_{(\mathrm{k})}\}$ and $\{\mathrm{n}_{(\mathrm{k})}\}$ for which it holds:

**i)** $\epsilon \leq \lim_{\mathrm{k}\to+\infty} \inf \mathrm{d}_b(\nu_{\mathrm{m}_{(\mathrm{k})}}, \nu_{\mathrm{n}_{(\mathrm{k})}}) \leq \lim_{\mathrm{k}\to+\infty} \sup \mathrm{d}_b(\nu_{\mathrm{m}_{(\mathrm{k})}}, \nu_{\mathrm{n}_{(\mathrm{k})}}) \leq b\epsilon,$

**ii)** $\dfrac{\epsilon}{b} \leq \lim_{\mathrm{k}\to+\infty} \inf \mathrm{d}_b(\nu_{\mathrm{n}_{(\mathrm{k})}}, \nu_{\mathrm{m}_{(\mathrm{k})+1}}) \leq \lim_{\mathrm{k}\to+\infty} \sup \mathrm{d}_b(\nu_{\mathrm{n}_{(\mathrm{k})}}, \nu_{\mathrm{m}_{(\mathrm{k})+1}}) \leq b^2\epsilon,$

**iii)** $\dfrac{\epsilon}{b} \leq \lim_{\mathrm{k}\to+\infty} \inf \mathrm{d}_b(\nu_{\mathrm{m}_{(\mathrm{k})}}, \nu_{\mathrm{n}_{(\mathrm{k})+1}}) \leq \lim_{\mathrm{k}\to+\infty} \sup \mathrm{d}_b(\nu_{\mathrm{m}_{(\mathrm{k})}}, \nu_{\mathrm{n}_{(\mathrm{k})+1}}) \leq b^2\epsilon,$

**vi)** $\dfrac{\epsilon}{b^2} \leq \lim_{\mathrm{k}\to+\infty} \inf \mathrm{d}_b(\nu_{\mathrm{m}_{(\mathrm{k})+1}}, \nu_{\mathrm{n}_{(\mathrm{k})+1}}) \leq \lim_{\mathrm{k}\to+\infty} \sup \mathrm{d}_b(\nu_{\mathrm{m}_{(\mathrm{k})+1}}, \nu_{\mathrm{n}_{(\mathrm{k})+1}}) \leq b^3\epsilon.$

As $\mathcal{F}$ satisfies the properties of a triangular $\alpha$-admissible mapping, we can apply Lemma **6** to conclude that

$$\alpha(\nu_{\mathrm{m}_{(\mathrm{k})}}, \nu_{\mathrm{n}_{(\mathrm{k})}}) \geq 1. \tag{20}$$

We also have

$$\begin{aligned} &\lim_{\mathrm{k}\to+\infty} \mathrm{M}(\nu_{\mathrm{m}_{(\mathrm{k})}}, \nu_{\mathrm{n}_{(\mathrm{k})}}) \leq b\epsilon \\ &\lim_{\mathrm{k}\to+\infty} \mathrm{N}(\nu_{\mathrm{m}_{(\mathrm{k})}}, \nu_{\mathrm{n}_{(\mathrm{k})}}) = 0. \end{aligned} \tag{21}$$

Now, from **(20)** and letting $\nu = \nu_{\mathrm{m}_{(\mathrm{k})}}$, $\omega = \nu_{\mathrm{n}_{(\mathrm{k})}}$ in **(2)**, we obtain

$$\begin{aligned} \theta\big[b^3\mathrm{d}_b(\nu_{\mathrm{m}_{(\mathrm{k})+1}}, \nu_{\mathrm{n}_{(\mathrm{k})+1}})\big] &\leq \theta\big[b^3\mathcal{H}(\mathcal{F}\nu_{\mathrm{m}_{(\mathrm{k})}}, \mathcal{F}\nu_{\mathrm{n}_{(\mathrm{k})}})\big] \\ &\leq \theta\big[\alpha\big(\nu_{\mathrm{m}_{(\mathrm{k})}}, \nu_{\mathrm{n}_{(\mathrm{k})}}\big)b^3\mathcal{H}(\mathcal{F}\nu_{\mathrm{m}_{(\mathrm{k})}}, \mathcal{F}\nu_{\mathrm{n}_{(\mathrm{k})}}\big)\big] \\ &\leq \phi\big[\theta\big(\mathrm{M}(\nu_{\mathrm{m}_{(\mathrm{k})}}, \nu_{\mathrm{n}_{(\mathrm{k})}})\big)\big] + L\,\mathrm{N}(\nu_{\mathrm{m}_{(\mathrm{k})}}, \nu_{\mathrm{n}_{(\mathrm{k})}}). \end{aligned}$$

Taking $\lim_{\mathrm{k}\longrightarrow+\infty}$ in the aforementioned inequality, and utilizing the continuity of both $\phi$ and $\theta$, along with the formula **(21)**, we can derive the following

$$\begin{aligned}\theta(\frac{\epsilon}{b^2}b^3) &= \theta(\epsilon b)\\ &\leq \theta\Big(b^3 \lim_{\mathrm{k}\to+\infty} \mathrm{d}_b(\nu_{\mathrm{m}_{(\mathrm{k})+1}}, \nu_{\mathrm{n}_{(\mathrm{k})+1}})\Big)\\ &\leq \phi\Big[\theta\Big(\lim_{\mathrm{k}\to+\infty} \mathrm{M}(\nu_{\mathrm{m}_{(\mathrm{k})}}, \nu_{\mathrm{n}_{(\mathrm{k})}})\Big)\Big] + L \lim_{\mathrm{k}\to+\infty} \mathrm{N}(\nu_{\mathrm{m}_{(\mathrm{k})}}, \nu_{\mathrm{n}_{(\mathrm{k})}})\\ &= \phi\Big[\theta\Big(\lim_{\mathrm{k}\to+\infty} \mathrm{M}(\nu_{\mathrm{m}_{(\mathrm{k})}}, \nu_{\mathrm{n}_{(\mathrm{k})}})\Big)\Big]\\ &\leq \phi\big[\theta(b\epsilon)\big].\end{aligned}$$

Applying Lemma **3**, we obtain the following inequality

$$\theta(b\epsilon) \leq \phi[\theta(b\epsilon)] < \theta(b\epsilon).$$

Since $\theta$ is increasing function, it follows that

$$b\epsilon < b\epsilon,$$

which contradicts our initial assumption. Therefore, we can conclude that $\{\nu_{\mathrm{n}}\}$ is a Cauchy sequence in the space $\Upsilon$. Furthermore, due to the completeness of $(\Upsilon, \mathrm{d}_b)$, we can deduce that there is $\sigma$ belonging to $\Upsilon$ such that

$$\lim_{\mathrm{n}\to+\infty} \mathrm{d}_b(\nu_{\mathrm{n}}, \sigma) = 0.$$

**Case 1:** If $\mathcal{F}$ is $\alpha$-continuous multivalued mapping, from **(4a)** we conclude that

$$\lim_{n\to+\infty} \mathcal{H}(\mathcal{F}\nu_n, \mathcal{F}\sigma) = 0.$$

Thus, we can obtain

$$\mathrm{d}_b(\sigma, \mathcal{F}\sigma) = \lim_{n\to+\infty} \mathrm{d}_b(\nu_{n+1}, \mathcal{F}\sigma) \leq \lim_{n\to+\infty} \mathcal{H}(\mathcal{F}\nu_n, \mathcal{F}\sigma) = 0.$$

Therefore, $\sigma \in \mathcal{F}\sigma$ and hence $\mathcal{F}$ possesses a fixed point.

**Case 2:** If $\mathcal{F}$ is not $\alpha$-continuous multivalued mapping, from **(4b)** we show that $\sigma \in \mathcal{F}\sigma$ by contradiction. Suppose that

$$\mathrm{d}_b(\mathcal{F}\sigma, \sigma) > 0.$$

Since $\lim_{\mathrm{n}\to+\infty} \nu_{\mathrm{n}} = \sigma$, by using Lemma **2.2** we can conclude that

$$\begin{aligned}\frac{1}{b^2}\mathrm{d}_b(\sigma, \mathcal{F}\sigma) &\leq \lim_{\mathrm{n}\to+\infty} \inf \mathcal{H}(\mathcal{F}\nu_{\mathrm{n}}, \mathcal{F}\sigma)\\ &\leq \lim_{\mathrm{n}\to+\infty} \sup \mathcal{H}(\mathcal{F}\nu_{\mathrm{n}}, \mathcal{F}\sigma)\\ &\leq b^2\mathrm{d}_b(\sigma, \mathcal{F}\sigma). \qquad (22)\end{aligned}$$

Now, letting $\nu = \nu_{\mathrm{n}}$ and $\omega = \sigma$ in **(2)**, we have

$$\begin{aligned} \theta\big[b^3\mathcal{H}(\mathcal{F}\nu_{\mathrm{n}}, \mathcal{F}\sigma)\big] &\leq \theta\big[\alpha(\nu_{\mathrm{n}}, \sigma)b^3\mathcal{H}(\mathcal{F}\nu_{\mathrm{n}}, \mathcal{F}\sigma)\big] \\ &\leq \phi\big[\theta\big(\mathrm{M}(\nu_{\mathrm{n}}, \sigma)\big)\big] + L\,\mathrm{N}(\nu_{\mathrm{n}}, \sigma), \text{ for all } \mathrm{n} \in \mathbb{N}, \end{aligned} \tag{23}$$

where

$$\mathrm{M}(\nu_{\mathrm{n}}, \sigma) = \max\Big\{\mathrm{d}_b(\nu_{\mathrm{n}}, \sigma), \mathrm{d}_b(\nu_{\mathrm{n}}, \mathcal{F}\nu_{\mathrm{n}}), \mathrm{d}_b(\sigma, \mathcal{F}\sigma), \frac{1}{2b^2}\big(\mathrm{d}_b(\sigma, \mathcal{F}\nu_{\mathrm{n}}) + \mathrm{d}_b(\nu_{\mathrm{n}}, \mathcal{F}\sigma)\big)\Big\}$$

and

$$\mathrm{N}(\nu_{\mathrm{n}}, \sigma) = \max\Big\{\mathrm{d}_b(\nu_{\mathrm{n}}, \mathcal{F}\nu_{\mathrm{n}}), \mathrm{d}_b(\sigma, \mathcal{F}\sigma), \mathrm{d}_b(\sigma, \mathcal{F}\nu_{\mathrm{n}}, \mathrm{d}_b(\nu_{\mathrm{n}}, \mathcal{F}\sigma)\Big\}.$$

Similar to the proof of Theorem **3**, we obtain

$$\begin{aligned} &\lim_{\mathrm{n}\to+\infty} \sup \mathrm{M}(\nu_{\mathrm{n}}, \sigma) \leq \mathrm{d}_b(\sigma, \mathcal{F}\sigma), \\ &\lim_{\mathrm{n}\to+\infty} \sup \mathrm{N}(\nu_{\mathrm{n}}, \sigma) = 0. \end{aligned}$$

Consequently, taking $\mathrm{n} \to +\infty$ in formula **(23)** and utilizing the properties of both $\theta$ and $\phi$, we obtain the following result

$$\begin{aligned} \theta\big(b^3\frac{1}{b}\mathrm{d}_b(\sigma, \mathcal{F}\sigma\big) &= \theta\big(b\mathrm{d}_b(\sigma, \mathcal{F}\sigma)\big) \\ &\leq \theta\big(b^3 \lim_{\mathrm{n}\to+\infty} \mathcal{H}(\mathcal{F}\nu_{\mathrm{n}}, \mathcal{F}\sigma)\big) \\ &\leq \lim_{\mathrm{n}\to+\infty} \theta\big(b^3\mathcal{H}(\mathcal{F}\nu_{\mathrm{n}}, \mathcal{F}\sigma)\big) \\ &\leq \lim_{\mathrm{n}\to+\infty} \theta\big(\alpha(\nu_{\mathrm{n}}, z)b^3\mathcal{H}(\mathcal{F}\nu_{\mathrm{n}}, \mathcal{F}\sigma)\big) \\ &\leq \lim_{\mathrm{n}\to+\infty} \Big(\phi\big[\theta\big(\mathrm{M}(\nu_{\mathrm{n}}, z)\big)\big] + L\,\mathrm{N}(\nu_{\mathrm{n}}, z)\Big) \\ &\leq \phi\big[\theta\big(\lim_{\mathrm{n}\to+\infty} \mathrm{M}(\nu_{\mathrm{n}}, z)\big)\big] \\ &\leq \phi\big(\theta(\mathrm{d}_b(\sigma, \mathcal{F}\sigma\big) \\ &< \theta(\mathrm{d}_b(\sigma, \mathcal{F}\sigma)). \end{aligned}$$

Using $(\theta_1)$, we obtain

$$b\mathrm{d}_b(\sigma, \mathcal{F}\sigma) < \mathrm{d}_b(\sigma, \mathcal{F}\sigma).$$

This implies that

$$\mathrm{d}_b(\sigma, \mathcal{F}\sigma)(b-1) < 0 \Rightarrow b < 1,$$

which leads to a contradiction. Hence, $\sigma$ belongs to $\mathcal{F}\sigma$.

□

The following fixed point theorems for $\alpha$-admissible $\theta$-$\phi$-multivalued Reich-type contraction and $\alpha$-admissible $\theta$-$\phi$-multivalued Kannan-type contraction can be obtained from Theorem **4**. These results enhance and expand upon the existing findings related to Reich-type contraction and Kannan-type contraction on $b$-metric space that were previously established.

**Theorem 5.** *Let $(\Upsilon, \mathrm{d}_b)$ be a complete b-metric space and $\mathcal{F} : \Upsilon \to \mathcal{K}(\Upsilon)$ be a given mapping. Assume that there are $\theta \in \Theta$ and $\phi \in \Phi$ such that for any $\nu, \omega \in \Upsilon$, we have*

$$\mathcal{H}(\mathcal{F}\nu,\ \mathcal{F}\omega) > 0 \text{ implies that } \theta[b^3\mathcal{H}(\mathcal{F}\nu,\ \mathcal{F}\omega)] \leq \phi[\theta(\mathcal{D}(\nu, \omega))],$$

*where*

$$\mathcal{D}(\nu, \omega) = \max\{\mathrm{d}_b(\nu, \omega),\ \mathrm{d}_b(\nu, \mathcal{F}\nu), \mathrm{d}_b(\omega, \mathcal{F}\omega)\}.$$

*Then $\mathcal{F}$ possesses a fixed point.*

*Proof.* We have

$$\begin{aligned}\mathcal{D}(\nu, \omega) &= \max\{\mathrm{d}_b(\nu, \omega),\ \mathrm{d}_b(\nu, \mathcal{F}\nu), \mathrm{d}_b(\omega, \mathcal{F}\omega)\} \\ &\leq \max\Big\{\mathrm{d}_b(\nu_\mathrm{n}, \sigma), \mathrm{d}_b(\nu_\mathrm{n}, \mathcal{F}\nu_\mathrm{n}), \mathrm{d}_b(\sigma, \mathcal{F}\sigma), \frac{1}{2b^2}\big(\mathrm{d}_b(\sigma, \mathcal{F}\nu_\mathrm{n}) + \mathrm{d}_b(\nu_\mathrm{n}, \mathcal{F}\sigma)\big)\Big\} \\ &= \mathrm{M}(\nu_\mathrm{n}, \sigma).\end{aligned}$$

Therefore, $\mathcal{F}$ is an $\alpha$-admissible $\theta$-$\phi$-multivalued type contraction, where $\alpha(\nu, \omega) = 1$, for every $\nu, \omega \in \Upsilon$. According to Theorem **4**, we can deduce that $\mathcal{F}$ possesses a fixed point. □

**Theorem 6.** *Let $(\Upsilon, \mathrm{d}_b)$ be a complete b-metric space and $\mathcal{F} : \Upsilon \to \mathcal{K}(\Upsilon)$ be an $\alpha$-admissible $\theta$-$\phi$-multivalued Kannan-type contraction satisfying*

***(1)*** *$(\Upsilon, \mathrm{d}_b)$ is an $\alpha$-complete metric space;*

***(2)*** *there are $\nu_0 \in \Upsilon$ and $\nu_1 \in \mathcal{F}\nu_0$ so that $\alpha(\nu_0, \nu_1) \geq 1$;*

***(3)*** *$\mathcal{F}$ is a triangular $\alpha$-admissible;*

***(4)*** *either*

- ***(4a)*** *$\mathcal{F}$ is an $\alpha$-continuous multivalued mapping,*
  *or*
- ***(4b)*** *If $\{\nu_\mathrm{n}\} \subset \Upsilon$ so that $\alpha(\nu_\mathrm{n}, \nu_{\mathrm{n}+1}) \geq 1$ for every $\mathrm{n} \in \mathbb{N}$ and $\lim_{\mathrm{n}\to+\infty} \nu_\mathrm{n} = \nu \in \Upsilon$, then we get $\alpha(\nu_\mathrm{n}, \nu) \geq 1$ for every $\mathrm{n} \in \mathbb{N}$.*

*Then $\mathcal{F}$ possesses a fixed point.*

*Proof.* Since $\mathcal{F}$ is an $\alpha$-admissible $\theta$-$\phi$-multivalued Kannan-type contraction, from **(3)** there are $\theta \in \Theta$ and $\phi \in \Phi$ so that

$$\begin{aligned}\theta\big[\alpha(\nu,\omega)b^3\mathcal{H}(\mathcal{F}\nu,\mathcal{F}\omega)\big] &\le \phi\big[\theta(\frac{\mathrm{d}_b(\mathcal{F}\nu,\nu)+\mathrm{d}_b(\{\omega,\omega)}{2})\big] \\ &\le \phi\Big[\theta\big(\max\big\{\mathrm{d}_b(\nu,\mathcal{F}\nu),\mathrm{d}_b(\omega,\mathcal{F}\omega)\big\}\big)\Big] \\ &\le \phi\Big[\theta\big(\mathrm{M}(\nu,\omega)\big)\Big].\end{aligned}$$

Therefore, $\mathcal{F}$ is an $\alpha$-admissible $\theta$-$\phi$-multivalued type contraction. According to Theorem **4**, we conclude that $\mathcal{F}$ possesses a fixed point. □

**Theorem 7.** *Let $(\Upsilon,\mathrm{d}_b)$ be a complete b-metric space and $\mathcal{F}:\Upsilon\to\mathcal{K}(\Upsilon)$ be an $\alpha$-admissible $\theta$-$\phi$-multivalued Reich-type contraction satisfying*

***(1)*** *$(\Upsilon,\mathrm{d}_b)$ is an $\alpha$-complete metric space;*

***(2)*** *there are $\nu_0\in\Upsilon$ and $\nu_1\in\mathcal{F}\nu_0$ so that $\alpha(\nu_0,\nu_1)\ge 1$;*

***(3)*** *$\mathcal{F}$ is triangular $\alpha$-admissible;*

***(4)*** *either*

- ***(4a)*** *$\mathcal{F}$ is an $\alpha$-continuous multivalued mapping,*
  *or*
- ***(4b)*** *If $\{\nu_\mathrm{n}\}\subset\Upsilon$ so that $\alpha(\nu_\mathrm{n},\nu_{\mathrm{n}+1})\ge 1$ for every $\mathrm{n}\in\mathbb{N}$ and $\lim_{n\to+\infty}\nu_\mathrm{n}=\nu\in\Upsilon$, then we get $\alpha(\nu_\mathrm{n},\nu)\ge 1$ for every $\mathrm{n}\in\mathbb{N}$.*

*Then $\mathcal{F}$ possesses a fixed point.*

*Proof.* Since $\mathcal{F}$ is an $\alpha$-admissible $\theta$-$\phi$-multivalued Reich-type contraction, from **(4)** there are $\phi\in\Phi$ and $\theta\in\Theta$ so that

$$\begin{aligned}\theta\big[\alpha(\nu,\omega)b^3\mathcal{H}(\mathcal{F}\nu,\mathcal{F}\omega)\big] &\le \phi\big[\theta\big(\frac{\mathrm{d}_b(\nu,\omega)+\mathrm{d}_b(\mathcal{F}\nu,\nu)+\mathrm{d}_b(\{\omega,\omega)}{3})\big] \\ &\le \phi\big[\theta\big(\max\{\mathrm{d}_b(\nu,\omega),\mathrm{d}_b(\mathcal{F}\nu,\nu),\mathrm{d}_b(\{\omega,\omega)\}\big)\big] \\ &\le \phi\big[\theta\big(\mathrm{M}(\nu,\omega)\big)\big].\end{aligned}$$

Then, $\mathcal{F}$ is an $\alpha$-admissible $\theta$-$\phi$-multivalued type contraction. Moreover, according to Theorem **4**, we prove that $\mathcal{F}$ possesses a fixed point. □

## 3. CONSEQUENCES

Our main results enable the deduction of several classical fixed point outcomes with ease, as demonstrated in this section.

**Corollary 1.** *Let $(\Upsilon, \mathrm{d}_b)$ be a complete b-metric space and $\mathcal{F} : \Upsilon \to \mathcal{K}(\Upsilon)$ be a mapping. Assume that there are $\mathrm{k} \in (0,1)$ and $\theta \in \Theta$ such that for every $\nu, \omega \in \Upsilon$, we obtain*

$$\mathcal{H}(\mathcal{F}\nu, \mathcal{F}\omega) > 0 \text{ implies that } \theta[b^3\mathcal{H}(\mathcal{F}\nu, \mathcal{F}\omega)] \leq [\theta(\mathrm{d}_b(\nu, \omega))]^{\mathrm{k}}.$$

*Then $\mathcal{F}$ possesses a fixed point.*

**Corollary 2.** *Let $(\Upsilon, \mathrm{d}_b)$ be a complete b-metric space and $\mathcal{F} : \Upsilon \to \mathcal{K}(\Upsilon)$ be a given mapping. Assume that there are $\phi \in \Phi$ and $\theta \in \Theta$ such that for every $\nu, y \in \Upsilon$, we obtain*

$$\mathcal{H}(\mathcal{F}\nu,\ \mathcal{F}\omega) > 0 \text{ implies that } \theta[b^3\mathcal{H}(\mathcal{F}\nu,\ \mathcal{F}\omega)] \leq \phi[\theta(\mathrm{d}_b(\nu, \omega))].$$

*Then $\mathcal{F}$ possesses a fixed point.*

**Corollary 3.** *Let $(\Upsilon, \mathrm{d}_b)$ be a complete b-metric space and $\mathcal{F} : \Upsilon \to \mathcal{K}(\Upsilon)$ be a given mapping. Assume that there are $\phi \in \Phi$ and $\theta \in \Theta$ such that for every $\nu, y \in \Upsilon$, we obtain*

$$\theta[b^3\mathcal{H}(\mathcal{F}\nu, \mathcal{F}\omega)] \ \leq \phi\Big[\theta\Big(\frac{\mathrm{d}_b(\nu, \mathcal{F}\nu) + \mathrm{d}_b(\omega, \mathcal{F}\omega)}{2}\Big)\Big].$$

*Then $\mathcal{F}$ possesses a fixed point.*

**Corollary 4.** *Let $(\Upsilon, \mathrm{d}_b)$ be a complete b-metric space and $\mathcal{F} : \Upsilon \to \mathcal{K}(\Upsilon)$ be a given mapping. Assume that there are $\phi \in \Phi$ and $\theta \in \Theta$ such that for every $\nu, \omega \in \Upsilon$, we obtain*

$$\theta[\alpha(\nu, \omega)b^3\mathcal{H}(\mathcal{F}\nu,\ \mathcal{F}\omega)] \ \leq \phi\Big[\theta\Big(\frac{\mathrm{d}_b(\nu, \omega) + \mathrm{d}_b(\nu, \mathcal{F}\nu) + \mathrm{d}_b(\omega, \mathcal{F}\omega)}{3}\Big)\Big].$$

*Then $\mathcal{F}$ possesses a fixed point.*

**Example 1.** *Consider the b-metric on the set $\Upsilon = [0, +\infty)$ defined by $\mathrm{d}_b(\nu, \omega) = (\nu - \omega)^2$ for every $\nu, \omega \in \Upsilon$. Let us define $\mathcal{F} : \Upsilon \to \mathcal{K}(\Upsilon)$ by*

$$\mathcal{F}\nu = \begin{cases} \left[0, \frac{\nu}{5}\right] & \text{if } \omega \in [0,5] \\ \left[\nu^2, \nu^3\right] & \text{otherwise,} \end{cases}$$

*$\alpha : \Upsilon \to [0, +\infty)$ by*

$$\alpha(\nu, \omega) = \begin{cases} 1 & \text{if } \nu, \omega \in [0,5] \\ 0 & \text{otherwise} \end{cases}$$

*and the functions $\theta : [0, +\infty) \to [1, +\infty)$ by $\phi(t) = 1 + \sqrt{t}$.*

*We will now demonstrate that $\mathcal{F}$ satisfies the conditions of being a triangular $\alpha$-admissible mapping. Let $\nu \in \Upsilon$ and $\omega \in \mathcal{F}\nu$ such that $\alpha(\nu,\omega) \geq 1$. Then, $\nu, \omega \in [0,5]$. Let $\mu \in \mathcal{F}\omega$, then*

$$\mu \in \left[0, \frac{\omega}{5}\right] \subset \left[0, \frac{\nu}{5^2}\right] \subset [0,1].$$

*So, $\alpha(\omega,\mu) \geq 1$. Hence, $\mathcal{F}$ is an $\alpha$-admissible. Moreover, let $\nu, \omega \in \Upsilon$ and $\mu \in \mathcal{F}\omega$ such that*

$$\alpha(\nu,\omega) \geq 1 \quad \text{and} \quad \alpha(\omega,\mu) \geq 1.$$

*Then, $\nu, \omega \in [0,5]$ and $\mu \in \left[0, \frac{\omega}{5}\right] \subset [0,1]$ which implies that $\nu, \mu \in [0,5]$, thus $\alpha(\nu,\mu) \geq 1$. So, $\mathcal{F}$ is triangular $\alpha$-admissible.*

*For $\nu_0 = \frac{1}{5} \in \Upsilon$ and $\nu_1 = \frac{1}{30} \in \mathcal{F}\nu_0$ we have $\alpha(\nu_0,\nu_1) \geq 1$.*

*In addition, for any sequence $\{\nu_{\mathrm{n}}\} \subset \Upsilon$, where $\nu_0 \in \Upsilon$, $\lim_{\mathrm{n}\to+\infty} \nu_{\mathrm{n}} = \nu$ and $\alpha(\nu_{\mathrm{n}},\nu_{\mathrm{n}+1}) \geq 1$ for all $\mathrm{n} \in \mathbb{N}$, it holds that $\nu_{\mathrm{n}} \in [0,5]$ for any $\mathrm{n} \in \mathbb{N}$ and thus $\nu_{\mathrm{n}}, \nu \in [0,5]$, so $\alpha(\nu_{\mathrm{n}},\nu) \geq 1$ for all $\mathrm{n} \in \mathbb{N}$. Hence, the condition **(4b)** holds.*

*Subsequently, we will prove that the assumptions of Theorem **3** are satisfied for every $\nu, \omega \in \Upsilon$ and $b = 2$.*

***Case 1:*** *If $\nu, \omega \in [0,5]$ we have $\alpha(\nu,\omega) \geq 1$ and thus*

$$\begin{aligned}\theta\big(\alpha(\nu,\omega)b^3\mathcal{H}(\{\nu,\mathcal{F}\omega)\big) &= \sqrt{\alpha(\nu,\omega)b^3\mathcal{H}(\nu,\omega)} + 1\\ &= \sqrt{\alpha(\nu,\omega)b^3\mathcal{H}(\{\nu,\mathcal{F}\omega)} + 1\\ &\leq \sqrt{8\mathcal{H}(\mathcal{P}\nu,\mathcal{F}\omega)} + 1.\end{aligned}$$

*Conversely, we can say that*

$$\begin{aligned}\mathcal{H}(\mathcal{F}\nu,\mathcal{F}\omega) &= \max\Big\{\rho(\mathcal{F}\nu,\{\omega),\rho(\mathcal{F}\omega,\mathcal{F}\nu)\Big\}\\ &= \max\Big\{\sup_{\mu\in\mathcal{F}\nu} \mathrm{d}_b(\mu,\mathcal{F}\omega), \sup_{\sigma\in\mathcal{F}\omega} \mathrm{d}_b(\sigma,\mathcal{F}\nu)\Big\}\\ &= \frac{(\nu-\omega)^2}{25}.\end{aligned}$$

*Thus,*

$$\begin{aligned}\theta\big(\alpha(\nu,\omega)b^3\mathcal{H}(\mathcal{F}\nu,\mathcal{F}\omega)\big) &\leq \sqrt{8\frac{(\nu-\omega)^2}{25}} + 1\\ &\leq \sqrt{\mathrm{d}_b(\nu,\omega)} + 1\\ &\leq \sqrt{\mathrm{M}(\nu,\omega)} + 1\\ &\leq \theta\big(\mathrm{M}(\nu,\omega)\big) + L\,\mathrm{N}(\nu,\omega).\end{aligned}$$

***Case 2:*** *If* $\nu, \omega \in (5, +\infty)$, *we have*

$$\alpha(\nu, \omega) = 0$$

*and thus*

$$\begin{aligned}\theta\big(\alpha(\nu,\omega)b^3\mathcal{H}(\{\nu,\mathcal{F}\omega)\big) &= \theta(0)\\ &\le \theta\big((\nu-\omega)^2\big)\\ &\le \theta\big(\mathrm{M}(\nu,\omega)\big) + L\,\mathrm{N}(\nu,\omega).\end{aligned}$$

*Then,* $\mathcal{F}$ *satisfies all assumptions of Theorem* ***3*** *and it possesses a fixed point, where* 0 *is such a fixed point.*

**Acknowledgements.** The research of the second author is partially supported by the Serbian Ministry of Education, Science and Technological Development, under project 451-03-47/2023-01/200103. The authors are thankful to the referees for their valuable suggestions towards the improvement of the paper.

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(Received 05. 10. 2023.)
(Revised 18. 12. 2024.)

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