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SOME PROPERTIES OF THE CATALAN NUMBERS

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This paper is dedicated to my father for his 83rd birthday.

In the paper, the author presents two expansions, complete monotonicity, minimality, some determinantal inequalities, and product inequalities of the Catalan numbers and other sequences involving the Catalan numbers in combinatorial number theory and verifies the equivalence of two integral representations for the Catalan numbers.

1. INTRODUCTION AND MAIN RESULTS

It is known [**3, 15**] in combinatorial analysis that the Catalan numbers C_n for $n \ge 0$ form a sequence of natural numbers that occur in tree enumeration problems such as "In how many ways can a regular n-gon be divided into $n-2$ triangles if different orientations are counted separately?" whose solution is the Catalan number C_{n-2}. The Catalan numbers C_n can be generated by

$$\frac{2}{1+\sqrt{1-4x}} = \sum_{n=0}^{\infty} C_n x^n = 1 + x + 2x^2 + 5x^3 + 14x^4 + 42x^5 + 132x^6 + \cdots .$$

Explicit formulas of C_n for $n \ge 0$ include, for examples,

$$C_n = \frac{1}{n+1}\binom{2n}{n} = \frac{2^n(2n-1)!!}{(n+1)!} = (-1)^n 2^{2n+1}\binom{1/2}{n+1} = \frac{4^n\Gamma(n+1/2)}{\sqrt{\pi}\,\Gamma(n+2)},$$

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where

$$\Gamma(z)=\int_0^\infty t^{z-1}\,\mathrm{e}^{-t}\,\mathrm{d}\,t,\quad \Re(z)>0$$

is the classical Euler gamma function. In [**2, 3, 15, 16**], the asymptotic expansion

$$C_x\triangleq\frac{4^x\Gamma\left(x+\frac{1}{2}\right)}{\sqrt{\pi}\,\Gamma(x+2)}\sim\frac{4^x}{\sqrt{\pi}}\left(\frac{1}{x^{3/2}}-\frac{9}{8}\frac{1}{x^{5/2}}+\frac{145}{128}\frac{1}{x^{7/2}}+\cdots\right)\tag{1}$$

was given for the Catalan function C_x.

In [**6**, Theorem 2], [**11**, Theorem 1.4], [**12**, Theorem 1.3], and [**14**, Theorem 1], three integral representations

$$C_x=\frac{\mathrm{e}^{3/2}\,4^x(x+1/2)^x}{\sqrt{\pi}\,(x+2)^{x+3/2}}\exp\left[\int_0^\infty\left(\frac{1}{\mathrm{e}^t-1}-\frac{1}{t}+\frac{1}{2}\right)\frac{\mathrm{e}^{-t/2}-\mathrm{e}^{-2t}}{t}e^{-xt}\,\mathrm{d}\,t\right],\tag{2}$$

$$C_n=\frac{1}{\pi}\int_0^\infty\frac{\sqrt{t}}{(t+1/4)^{n+2}}\,\mathrm{d}\,t,\tag{3}$$

and

$$C_n=\frac{1}{2\pi}\int_0^4\sqrt{\frac{4-t}{t}}\,t^n\,\mathrm{d}\,t\tag{4}$$

for the Catalan function C_x and C_n are respectively given, where $n\ge0$ and $x\ge0$.

Recall from monographs [**8**, pp. 372–373] and [**17**, p. 108, Definition 4] that a sequence $\{\mu_n\}_{0\le n\le\infty}$ is said to be completely monotonic if its elements are non-negative and its successive differences are alternatively non-negative, equivalently speaking, $(-1)^k\Delta^k\mu_n\ge0$ for $n,k\ge0$, where

$$\Delta^k\mu_n=\sum_{m=0}^k(-1)^m\binom{k}{m}\mu_{n+k-m}.$$

Recall from [**17**, p. 163, Definition 14a] that a completely monotonic sequence $\{a_n\}_{n\ge0}$ is minimal if it ceases to be completely monotonic when a_0 is decreased.

In the paper [**11**], by virtue of the integral representation (3) and other techniques, some determinantal and product inequalities and logarithmic convexity for the sequences $\{C_n\}_{n\ge0}$, $\{n!C_n\}_{n\ge0}$, and $\left\{\frac{C_n}{4^n}\right\}_{n\ge0}$ were presented.

In this paper, by virtue of the integral representations (2) and (4), we establish two expansions, complete monotonicity, minimality, and some determinantal and product inequalities of the Catalan numbers C_n and C_x and other sequences involving the Catalan numbers C_n. On the other hand, we also verify the equivalence of the integral representations (3) and (4).

Our main results can be formulated in details as the following theorems.

Theorem 1. *For $n\ge0$, the Catalan numbers C_n have the expansion*

$$C_n=\frac{4^{n+1}}{\pi}\left[\frac{1}{2n+1}-\sum_{k=1}^\infty\frac{(2k-3)!!}{2^kk!}\frac{1}{2n+2k+1}\right],\tag{5}$$

where the double factorial of negative odd integers $-2n-1$ is defined by

$$(-2n-1)!! = \frac{(-1)^n}{(2n-1)!!} = (-1)^n \frac{2^n n!}{(2n)!}, \quad n \ge 0.$$

Consequently,

1. *the sequences* $\left\{\frac{C_n}{4^n}\right\}_{n\ge0}$ *and*

$$\left\{\frac{4}{\pi}\left[\frac{1}{2n+1} - \sum_{k=1}^{N} \frac{(2k-3)!!}{2^k k!} \frac{1}{2n+2k+1}\right] - \frac{C_n}{4^n}\right\}_{n\ge0} \tag{6}$$

   for $N \in \{0\} \cup \mathbb{N}$ are completely monotonic and minimal, where an empty sum is understood to be 0;

2. *for $m \ge 1$ and any non-negative integers $a_0, a_1, \dots, a_m$, we have*

$$\begin{aligned} &\left(\frac{4}{\pi}\frac{1}{2a_0+1} - \frac{C_{a_0}}{4^{a_0}}\right)^{m-1} \left(\frac{4}{\pi}\frac{1}{1+2\sum_{k=0}^{m} a_k} - \frac{C_{\sum_{k=0}^{m} a_k}}{4^{\sum_{k=0}^{m} a_k}}\right) \\ &\qquad \ge \prod_{k=1}^{m} \left[\frac{4}{\pi}\frac{1}{2(a_0+a_k)+1} - \frac{C_{a_0+a_k}}{4^{a_0+a_k}}\right] \end{aligned} \tag{7}$$

   and

$$\left|\frac{4}{\pi}\frac{1}{2(a_i+a_j)+1} - \frac{C_{a_i+a_j}}{4^{a_i+a_j}}\right|_m \ge 0, \tag{8}$$

   where $|e_{kj}|_m$ denotes a determinant of order m with elements e_{kj}.

Theorem 2. *The Catalan function C_x have the exponential expansion*

$$C_x = \frac{e^{3/2}}{\sqrt{\pi}} \frac{4^x (x+1/2)^x}{(x+2)^{x+3/2}} \exp\left\{\sum_{j=1}^{\infty} \frac{B_{2j}}{2j(2j-1)}\left[\frac{1}{(x+1/2)^{2j-1}} - \frac{1}{(x+2)^{2j-1}}\right]\right\} \tag{9}$$

and satisfy the double inequality

$$\begin{aligned} &\exp\left\{\sum_{j=1}^{2m} \frac{B_{2j}}{2j(2j-1)}\left[\frac{1}{(x+1/2)^{2j-1}} - \frac{1}{(x+2)^{2j-1}}\right]\right\} \\ &\qquad < \frac{\sqrt{\pi}}{e^{3/2}} \frac{(x+2)^{x+3/2}}{4^x (x+1/2)^x} C_x \\ &\qquad < \exp\left\{\sum_{j=1}^{2m-1} \frac{B_{2j}}{2j(2j-1)}\left[\frac{1}{(x+1/2)^{2j-1}} - \frac{1}{(x+2)^{2j-1}}\right]\right\}, \end{aligned} \tag{10}$$

where $m\in\mathbb{N}$ and B_{2j} are the Bernoulli numbers which are defined by

$$\frac{x}{\mathrm{e}^x-1}=\sum_{i=0}^{\infty}B_i\frac{x^i}{i!}=1-\frac{x}{2}+\sum_{j=1}^{\infty}B_{2j}\frac{x^{2j}}{(2j)!},\quad |x|<2\pi. \tag{11}$$

The sequences

$$\left\{\ln\left[\frac{\sqrt{\pi}}{\mathrm{e}^{3/2}}\frac{(n+2)^{n+3/2}}{4^n(n+1/2)^n}C_n\right]\right\}_{n\ge0}\quad \text{and}\quad \left\{\frac{(n+2)^{n+3/2}}{4^n(n+1/2)^n}C_n\right\}_{n\ge0} \tag{12}$$

are completely monotonic and minimal and satisfy

$$\begin{aligned}\ln^{m-1}&\left[\frac{\sqrt{\pi}}{\mathrm{e}^{3/2}}\frac{(a_0+2)^{a_0+3/2}}{4^{a_0}(a_0+1/2)^n}C_{a_0}\right]\\ &\times\ln\left[\frac{\sqrt{\pi}}{\mathrm{e}^{3/2}}\frac{\left(2+\sum_{k=0}^m a_k\right)^{3/2+\sum_{k=0}^m a_k}}{4^{\sum_{k=0}^m a_k}\left(\frac12+\sum_{k=0}^m a_k\right)^{\sum_{k=0}^m a_k}}C_{\sum_{k=0}^m a_k}\right]\\ &\qquad\ge\prod_{k=1}^m\ln\left[\frac{\sqrt{\pi}}{\mathrm{e}^{3/2}}\frac{(a_0+a_k+2)^{a_0+a_k+3/2}}{4^{a_0+a_k}(a_0+a_k+1/2)^n}C_{a_0+a_k}\right],\end{aligned} \tag{13}$$

$$\begin{aligned}\left[\frac{(a_0+2)^{a_0+3/2}}{4^{a_0}(a_0+1/2)^n}C_{a_0}\right]^{m-1}&\left[\frac{\left(2+\sum_{k=0}^m a_k\right)^{3/2+\sum_{k=0}^m a_k}}{4^{\sum_{k=0}^m a_k}\left(\frac12+\sum_{k=0}^m a_k\right)^{\sum_{k=0}^m a_k}}C_{\sum_{k=0}^m a_k}\right]\\ &\ge\prod_{k=1}^m\left[\frac{(a_0+a_k+2)^{a_0+a_k+3/2}}{4^{a_0+a_k}(a_0+a_k+1/2)^n}C_{a_0+a_k}\right],\end{aligned}$$

$$\left|\ln\left[\frac{\sqrt{\pi}}{\mathrm{e}^{3/2}}\frac{(a_i+a_j+2)^{a_i+a_j+3/2}}{4^{a_i+a_j}(a_i+a_j+1/2)^{a_i+a_j}}C_{a_i+a_j}\right]\right|_m\ge0,$$

and

$$\left|\frac{(a_i+a_j+2)^{a_i+a_j+3/2}}{4^{a_i+a_j}(a_i+a_j+1/2)^{a_i+a_j}}C_{a_i+a_j}\right|_m\ge0, \tag{14}$$

where $a_0,a_1,\dots,a_m$ for $m\ge1$ are non-negative integers and $|e_{kj}|_m$ denotes a determinant of order m with elements e_{kj}.

Corollary 1. *The sequences $\left\{\frac{C_n}{4^n}\right\}_{n\ge0}$ and (6) are convex and the second sequence in (12) is logarithmically convex.*

Theorem 3. *The integral representations (3) and (4) are equivalent to each other.*

2. PROOFS OF THEOREMS 1 TO 3

Now we start out to prove Theorems 1 to 3.

Proof of Theorem 1. From the integral representation (4), it follows that

$$\begin{aligned}
C_n &= \frac{4^n}{2\pi}\int_0^4 \sqrt{1-\frac{t}{4}}\left(\frac{t}{4}\right)^{n-1/2} \,\mathrm{d}\, t \\
&= \frac{4^n}{2\pi}\int_0^4 \left(\frac{t}{4}\right)^{n-1/2} \sum_{k=0}^{\infty}(-1)^k \frac{\langle 1/2\rangle_k}{k!}\left(\frac{t}{4}\right)^{k} \,\mathrm{d}\, t \\
&= \frac{4^n}{2\pi}\sum_{k=0}^{\infty}(-1)^k \frac{\langle 1/2\rangle_k}{k!}\int_0^4 \left(\frac{t}{4}\right)^{n+k-1/2} \,\mathrm{d}\, t \\
&= \frac{4^n}{2\pi}\sum_{k=0}^{\infty}(-1)^k \frac{\langle 1/2\rangle_k}{k!}\frac{8}{2k+2n+1} \\
&= \frac{4^{n+1}}{\pi}\sum_{k=0}^{\infty}(-1)^k \frac{\langle 1/2\rangle_k}{k!}\frac{1}{2k+2n+1} \\
&= \frac{4^{n+1}}{\pi}\sum_{k=0}^{\infty}(-1)^k \frac{1}{k!}\frac{(-1)^{k-1}(2k-3)!!}{2^k}\frac{1}{2k+2n+1} \\
&= \frac{4^{n+1}}{\pi}\left[\frac{1}{2n+1}-\sum_{k=1}^{\infty}\frac{(2k-3)!!}{2^k k!}\frac{1}{2n+2k+1}\right],
\end{aligned}$$

where

$$\langle x\rangle_n = \prod_{k=0}^{n-1}(x-k) = \begin{cases} x(x-1)\cdots(x-n+1), & n\ge 1 \\ 1, & n=0 \end{cases}$$

is called the falling factorial. The asymptotic expansion (5) is thus proved.

Recall from [**8**, Chapter XIII], [**13**, Chapter 1], and [**17**, Chapter IV] that an infinitely differentiable function f is said to be completely monotonic on an interval I if it satisfies $0 \le (-1)^k f^{(k)}(x) < \infty$ on I for all $k \ge 0$. Theorem 14b in [**17**, p. 164] reads that a necessary and sufficient condition that there should exist a completely monotonic function $f(x)$ in $0 \le x < \infty$ such that $f(n) = a_n$ for $n \ge 0$ is that $\{a_n\}_0^\infty$ should be a minimal completely monotonic sequence. From the integral representation (4), it follows that

$$\frac{C_n}{4^n} = \frac{1}{2\pi}\int_0^4 \sqrt{\frac{4-t}{t}}\left(\frac{t}{4}\right)^n \,\mathrm{d}\, t.$$

Then the fact that the function a^t for $a \in (0,1)$ is completely monotonic on $(0,\infty)$ results in the complete monotonicity and minimality of the sequence $\left\{\frac{C_n}{4^n}\right\}_{n\ge 0}$.

From the expansion (5), it follows that

$$\frac{1}{2n+1}-\sum_{k=1}^{N}\frac{(2k-3)!!}{2^kk!}\frac{1}{2n+2k+1}-\frac{\pi C_n}{4^{n+1}}=\sum_{k=N+1}^{\infty}\frac{(2k-3)!!}{2^kk!}\frac{1}{2n+2k+1}.$$

Then combining the complete monotonicity of $\frac{1}{2x+2k+1}$ for $k\in\mathbb{N}$ on $(0,\infty)$ with the above mentioned [**17**, p. 164, Theorem 14b] arrives at the complete monotonicity and minimality of the sequence (6).

In [**7**] and [**8**, pp. 369 and 374], it was obtained that if f is completely monotonic on $[0,\infty)$ and $m\ge 1$, then

$$[f(x_0)]^{m-1}f\left(\sum_{k=0}^{m}x_k\right)\ge\prod_{k=1}^{m}f(x_0+x_k) \tag{15}$$

and

$$|f(x_i+x_j)|_m\ge 0 \tag{16}$$

for non-negative numbers $x_0,x_1,\dots,x_m$. Now we consider the function

$$\mathfrak{h}(x)=\frac{4}{\pi}\sum_{k=1}^{\infty}\frac{(2k-3)!!}{2^kk!}\frac{1}{2x+2k+1},\quad x\ge 0.$$

It is easy to see that the function $\mathfrak{h}(x)$ is completely monotonic on $[0,\infty)$. Replacing the function f and non-negative numbers $x_0,x_1,\dots,x_m$ in (15) and (16) by the function $\mathfrak{h}(x)$ and non-negative integers $a_0,a_1,\dots,a_m$ respectively yields

$$[\mathfrak{h}(a_0)]^{m-1}\mathfrak{h}\left(\sum_{k=0}^{m}a_k\right)\ge\prod_{k=1}^{m}\mathfrak{h}(a_0+a_k) \tag{17}$$

and

$$|\mathfrak{h}(a_i+a_j)|_m\ge 0. \tag{18}$$

Since $\mathfrak{h}(a_k)=\frac{4}{\pi}\frac{1}{2a_k+1}-\frac{C_{a_k}}{4^{a_k}}$, the inequalities (17) and (18) can be reformulated as (7) and (8). The proof of Theorem 1 is complete. □

Proof of Theorem 2. By virtue of (2) and (11), it follows that

$$\begin{aligned}\frac{\sqrt{\pi}}{e^{3/2}}\frac{(x+2)^{x+3/2}}{4^x(x+1/2)^x}C_x&=\exp\left[\int_0^\infty\left(\sum_{j=1}^{\infty}B_{2j}\frac{t^{2j-1}}{(2j)!}\right)\frac{e^{-t/2}-e^{-2t}}{t}e^{-xt}\,\mathrm{d}\,t\right]\\&=\exp\left[\sum_{j=1}^{\infty}\frac{B_{2j}}{(2j)!}\int_0^\infty t^{2j-2}\left(e^{-t/2}-e^{-2t}\right)e^{-xt}\,\mathrm{d}\,t\right]\end{aligned}$$

$$= \exp\left\{\sum_{j=1}^{\infty}\frac{B_{2j}}{2j(2j-1)}\left[\frac{1}{(x+1/2)^{2j-1}}-\frac{1}{(x+2)^{2j-1}}\right]\right\}.$$

In [**4**, Theorem 3], it was obtained that

$$1-\frac{x}{2}+\sum_{j=1}^{2m}\frac{B_{2j}}{(2j)!}x^{2j}<\frac{x}{e^x-1}<1-\frac{x}{2}+\sum_{j=1}^{2m-1}\frac{B_{2j}}{(2j)!}x^{2j}$$

for $m\in\mathbb{N}$ and $x>0$. Substituting this double inequality into (2) leads to (10).

From the integral representation (2), it is easy to see that the functions

$$\ln\left[\frac{\sqrt{\pi}}{e^{3/2}}\frac{(x+2)^{x+3/2}}{4^x(x+1/2)^x}C_x\right] \quad \text{and} \quad \frac{(x+2)^{x+3/2}}{4^x(x+1/2)^x}C_x \tag{19}$$

are completely monotonic on $[0,\infty)$. See also [**6, 12, 14**]. Hence, by virtue of the above-mentioned [**17**, p. 164, Theorem 14b], the complete monotonicity and minimality of the sequences in (12) follow immediately.

Applying f and non-negative numbers $x_0,x_1,\dots,x_m$ in (15) and (16) to the functions in (19) and to non-negative integers $a_0,a_1,\dots,a_m$ respectively derives the inequalities (13) to (14). The proof of Theorem 2 is complete. □

Proof of Corollary 1. These properties follow readily from complete monotonicity of the sequence (6) and the first one in (12). □

Proof of Theorem 3. Letting $\frac{4-t}{t}=s$ in the integral (4) gives

$$\begin{aligned}
\frac{1}{2\pi}\int_0^4\sqrt{\frac{4-t}{t}}\,t^n\,\mathrm{d}\,t &= \frac{1}{2\pi}\int_\infty^0\sqrt{s}\left(\frac{4}{1+s}\right)^n\left[-\frac{4}{(1+s)^2}\right]\mathrm{d}\,s\\
&= \frac{1}{2\pi}4^{n+1}\int_0^\infty\frac{\sqrt{s}}{(1+s)^{n+2}}\,\mathrm{d}\,s\\
&= \frac{1}{2\pi}4^{n+1}\int_0^\infty\frac{\sqrt{4t}}{(1+4t)^{n+2}}\,\mathrm{d}(4t)\\
&= \frac{1}{2\pi}4^{n+2}\int_0^\infty\frac{2\sqrt{t}}{(1+4t)^{n+2}}\,\mathrm{d}\,t\\
&= \frac{1}{\pi}\int_0^\infty\frac{\sqrt{t}}{(t+1/4)^{n+2}}\,\mathrm{d}\,t.
\end{aligned}$$

The proof of Theorem 3 is complete. □

3. REMARKS

Finally we list several remarks to explain more about our main results.

Remark 1. The complete monotonicity and minimality for $\left\{\frac{C_n}{4^n}\right\}_{n\ge0}$ in Theorem 1 recover one of conclusions obtained in [**11**, Theorem 1.4].

Remark 2. From the complete monotonicity of the sequence (6), we can deduce the inequality

$$C_n < \frac{4^{n+1}}{\pi}\left[\frac{1}{2n+1} - \sum_{k=1}^{N}\frac{(2k-3)!!}{2^k k!}\frac{1}{2n+2k+1}\right]$$

for all $n, N \ge 0$.

Remark 3. It is obvious that the asymptotic expansion (1) is generalized by the asymptotic expansion (9) in Theorem 2.

Remark 4. This paper is a revised version of the preprint [**9**]. For new developments in recent years, please refer to the literature in the papers [**1, 5, 10**].

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