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FORMULAE FOR ANTI-TRIANGULAR BLOCK MATRICES WHICH INCLUDE THE DRAZIN INVERSE

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The expressions for the Drazin inverse of two kinds of anti-triangular block matrices are developed under new and weaker assumptions relative to those already used recently in this subject. Applying our results concerning the Drazin inverse and anti-triangular block matrices, we propose some characterizations and representations of the Drazin inverse of a 2×2 block matrix. In this way, we expand some notable achievements in characterizing and representing generalized inverses of partitioned matrices.

1. INTRODUCTION AND MOTIVATION

Let the symbol $\mathbb{C}^{m\times n}$ denote the set of $m \times n$ complex matrices. The *Drazin inverse* of $F \in \mathbb{C}^{n\times n}$ is the unique matrix F^d which satisfies

$$FF^d = F^dF, \quad F^dFF^d = F^d, \quad F^k = F^{k+1}F^d,$$

for the index k of F (denoted by $\mathrm{ind}(F)$), which presents the minimal non-negative integer satisfying $\mathrm{rank}(F^k) = \mathrm{rank}(F^{k+1})$. For $F^e = FF^d$ and the identity matrix I of suitable dimensions, $F^\pi = I - F^e$ denotes the spectral idempotent of F associated with $\{0\}$. In particular, if $\mathrm{ind}(F) = 1$, $F^d = F^\#$ is the group inverse, as the specific phenomenon of the Drazin inverse.

The Drazin inverse of block matrices is applicable in different fields such as finite Markov chains, cryptography, iterative methods, systems of differential or difference equations, and some other applications [**7, 8, 21, 22, 25, 26, 27, 29**].

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To solve second-order singular differential equations, the problem of finding the Drazin inverse of anti-triangular block (or shortly ATB) matrices $M = \begin{bmatrix} F & E \\ G & 0 \end{bmatrix}$ was initiated in [**6**]. A representation for M^d without any restrictions on blocks involved in M is still not available, although many researchers have considered this problem [**2, 4, 9, 13, 17, 20, 23, 31**]. Interestingly, M^d was expressed in [**14**] under restriction $EGF = 0$.

Many studies have appeared in the literature under various conditions [**16, 18, 30, 32, 33**] with the intention to present the Drazin inverse of a partitioned matrix $N = \begin{bmatrix} F & E \\ G & H \end{bmatrix}$. Some expressions for N^d were given under the following assumptions:

1. In [**5**], $F = 0$ and $H = 0$;
2. In [**15**], $EG = 0, EH = 0$ and $HG = 0$;
3. In [**16**], $EG = 0, EHG = 0$ and $EH^2 = 0$;
4. In [**18**], $EG = 0, HG = 0$ (or $EH = 0$) and H is nilpotent;
5. In [**24**], $EGF = 0, FEH = 0$ and $GEH = 0$;
6. In [**10**], $EGF = 0, HG = 0$ and H is nilpotent;
7. In [**10**], $EGF = 0, EH = 0$ and $HG = 0$ (or EG is nilpotent);
8. In [**30**], $EGF = 0, EGE = 0, HGF = 0$ and $HGE = 0$;
9. In [**1**], $EGF = 0, GEGE = 0, F^\pi EGE = 0, EHG = 0$ and $EH^2 = 0$;
10. In [**1**], $FEH = 0, GEH = 0, EGF = 0, HGF = 0, EGEG = 0$ and $H^\pi GEG = 0$;
11. In [**1**], $HGF = 0, EGF = 0, GEH = 0, FEH = 0, GEGE = 0$ and $F^\pi EGE = 0$.

Motivated by the applications and theoretical significance of the Drazin inverse of various block matrices as well as growing research interest in this subject, the central topic of this study is the development of representations of the Drazin inverse on two categories of ATB matrices and the general block matrix under new and weaker conditions than considered in the research available right now. Thus, we extend some well-known results. The main topics of this research are briefly described now.

(1) Firstly, we consider the Drazin inverse of the ATB matrix

$$\bar{M} = \begin{bmatrix} F & E \\ I & 0 \end{bmatrix} \tag{1}$$

under new assumptions $EF^2 = 0$, $(EF)^2 = 0$ and $E^2F = 0$.

(2) Applying derived representations for $\bar{M}^d$, we present certain specific representations for M^d, such that

$$M = \begin{bmatrix} F & E \\ G & 0 \end{bmatrix}, \tag{2}$$

wherein $EGF^2 = 0$, $(EGF)^2 = 0$ and $(EG)^2F = 0$.

(3) Applying the Drazin inverse of ATB matrices of the form (2), the expressions for the Drazin inverse of the more general block complex matrix

$$N = \begin{bmatrix} F & E \\ G & H \end{bmatrix} \tag{3}$$

are established in terms of its blocks.

This research is organized as follows. Section 2 contains preliminary results, which are essential in subsequent sections. In Section 3, new explicit formulae for the Drazin inverse of ATB matrices $\bar{M}$ and M are derived. Section 4 involves representations of N^d obtained using the results of Section 3. Some concluding remarks are placed in Section 5.

2. PRELIMINARY RESULTS

This section contains a few preliminary results that will be used to prove the main results. Cline's formula is stated in the beginning. Assuming $F \in \mathbb{C}^{n\times n}$ and $k \in \mathbb{N}$, we adopt the mark $(F^d)^k = F^{kd} = F^{dk}$ for the equality $(F^d)^k = (F^k)^d$. The definition of an integral function stands for the truncated integer of x as $[x]$.

Lemma 1. [**11**](Cline's Formula) *The identity* $(EF)^d = E(FE)^{2d}F$ *holds for arbitrary* $F \in \mathbb{C}^{m\times n}$ *and* $E \in \mathbb{C}^{n\times m}$.

Lemma 2 gives the representations of the Drazin inverse of block triangular matrices. According to the common convention, a sum with a lower limit greater than its upper equals 0.

Lemma 2. [**19, 28**] *Let* $N = \begin{bmatrix} F & E \\ 0 & H \end{bmatrix}$ *and* $M = \begin{bmatrix} H & 0 \\ E & F \end{bmatrix} \in \mathbb{C}^{n\times n}$ *are defined using square blocks* F *and* H *satisfying* $\mathrm{ind}(F) = r$, $\mathrm{ind}(H) = s$, *and appropriate block* E. *Then*

$$N^d = \begin{bmatrix} F^d & Z \\ 0 & H^d \end{bmatrix} \quad \text{and} \quad M^d = \begin{bmatrix} H^d & 0 \\ Z & F^d \end{bmatrix},$$

where

$$Z = \sum_{i=0}^{s-1} (F^d)^{i+2} E H^i H^\pi + F^\pi \sum_{i=0}^{r-1} F^i E (H^d)^{i+2} - F^d E H^d.$$

Some useful assertions which are related to the Drazin inverse of the sum of two matrices are developed in Lemma 3.

Lemma 3. *Let* $A, B \in \mathbb{C}^{n\times n}$ *satisfy* $\mathrm{ind}(A) = t$ *and* $\mathrm{ind}(B) = s$ *as well as* $B^2A = 0$, $BA^2 = 0$ *and* $(BA)^2 = 0$. *Then*

$$\begin{aligned}(A+B)^d = {} & \sum_{k=0}^{s-1}(A^d)^{k+1}B^kB^\pi + \sum_{k=2}^{s+1}(A^d)^{k+1}BAB^{k-2}B^\pi + \sum_{k=0}^{t-1}A^\pi A^k(B^d)^{k+1} \\ & + \sum_{k=2}^{t+1}A^\pi A^{k-2}BA(B^d)^{k+1} + X,\end{aligned}$$

where

$$\begin{aligned} X &= UB^\pi + A^\pi V + AUV + UVB - A^d - 2A^dBAB^{2d} \\ & \quad -2A^{2d}BAB^d - A^{3d}BA - BAB^{3d} - B^d, \\ U &= A^{3d}BA + A^d, \\ V &= B^d + BAB^{3d}. \end{aligned}$$

Proof. For arbitrary $k \ge 0$, we denote $k' = [(k-1)/2]$ as well as $\alpha = 0$ if k is even, and $\alpha = 1$ otherwise. Note that $\mathrm{ind}(BA) \le 2$, because $(BA)^2(BA)^\pi = 0$. By [**10**, Theorem 2.3], we obtain

$$(A+B)^d = UB^\pi + A^\pi V + AUV + UVB + \sum_{k=0}^{s+2}(A^d)^{k+1}\Gamma_{k+2}B + \sum_{k=0}^{t+2}A\mathbb{E}_{k+2}(B^d)^{k+1},$$

where

$$\begin{aligned} U &= A^{3d}BA + A^d, \\ V &= B^d + BAB^{3d}, \\ \mathbb{E}_{k+2} &= -A^{\alpha d}U(BA)^{k'+1} + Z_k \quad (k \ge 0), \\ \Gamma_{k+2} &= -(BA)^{k'+1}VB^{\alpha d} + T_k \quad (k \ge 0), \end{aligned}$$

such that

$$Z_0 = 0, \quad Z_1 = A^\pi, \quad Z_2 = A^\pi A, \quad Z_j = A^\pi A^{j-1} + A^\pi A^{j-3}BA \quad (j \ge 3),$$

$$T_0 = 0, \quad T_1 = B^\pi, \quad T_2 = BB^\pi, \quad T_j = B^{j-1}B^\pi + BAB^{j-3}B^\pi \quad (j \ge 3).$$

Routine calculations show that

$$\begin{aligned}
&\sum_{k=0}^{s+2}(A^d)^{k+1}\Gamma_{k+2}B\\
&= A^d(-V)B + A^{2d}(-BAVB^d + T_1)B + A^{3d}(-BAV + T_2)B + \sum_{k=3}^{s+2}(A^d)^{k+1}T_kB\\
&= -A^dVB + A^{2d}(-BAVB^d + B^\pi)B + A^{3d}(-BAV + BB^\pi)B\\
&\quad + \sum_{k=3}^{s+2}(A^d)^{k+1}(B^{k-1}B^\pi + BAB^{k-3}B^\pi)B\\
&= -A^dVB - A^{2d}BAVB^e - A^{3d}BAVB + \sum_{k=1}^{s+2}(A^d)^{k+1}B^kB^\pi\\
&\quad + \sum_{k=3}^{s+2}(A^d)^{k+1}BAB^{k-2}B^\pi\\
&= -A^d - A^dBAB^{2d} - A^{2d}BAB^d - A^{3d}BA + \sum_{k=0}^{s+2}(A^d)^{k+1}B^kB^\pi\\
&\quad + \sum_{k=2}^{s+2}(A^d)^{k+1}BAB^{k-2}B^\pi,
\end{aligned}$$

and

$$\begin{aligned}
&\sum_{k=0}^{t+2}A\mathbb{E}_{k+2}(B^d)^{k+1}\\
&= -AUB^d + A(-A^dUBA + Z_1)B^{2d} + A(-UBA + Z_2)B^{3d} + \sum_{k=3}^{t+2}AZ_k(B^d)^{k+1}\\
&= -AUB^d - A^eUBAB^{2d} + AA^\pi B^{2d} - AUBAB^{3d} + A^2A^\pi B^{3d}\\
&\quad + \sum_{k=3}^{t+2}A(A^\pi A^{k-1} + A^\pi A^{k-3}BA)(B^d)^{k+1}\\
&= -(A^{2d}BA + A^e)B^d - A^dBAB^{2d} + AA^\pi B^{2d} - A^eBAB^{3d} + A^2A^\pi B^{3d}\\
&\quad + \sum_{k=3}^{t+2}A(A^\pi A^{k-1} + A^\pi A^{k-3}BA)(B^d)^{k+1}\\
&= -A^{2d}BAB^d - A^dBAB^{2d} - BAB^{3d} - B^d + \sum_{k=0}^{t+2}A^\pi A^k(B^d)^{k+1}\\
&\quad + \sum_{k=2}^{t+2}A^\pi A^{k-2}BA(B^d)^{k+1}
\end{aligned}$$

as desired. □

The following additive properties of the Drazin inverse were proved in [**30**].

Lemma 4. [**30**, Theorem 2.1] *Let the matrices* $A, B \in \mathbb{C}^{n\times n}$ *satisfy* $\mathrm{ind}(A) = r$ *and* $\mathrm{ind}(B) = s$ *in common with* $ABA = 0$ *and* $AB^2 = 0$. *In this case,*

$$(A+B)^d = B^\pi \sum_{i=0}^{s-1} B^i A^{(i+1)d} + \sum_{i=0}^{r-1} B^{(i+1)d} A^i A^\pi + B^\pi \sum_{i=0}^{s-1} B^i A^{(i+2)d} B + \sum_{i=0}^{r-2} B^{(i+3)d} A^{i+1} A^\pi B - B^d A^d B - B^{2d} A A^d B.$$

Lemma 5. [**30**, Theorem 2.2] *Let* $A, B \in \mathbb{C}^{n\times n}$ *satisfy* $\mathrm{ind}(A) = r$, $\mathrm{ind}(B) = s$, $BAB = 0$ *and* $A^2B = 0$. *Under these conditions, it follows*

$$(A+B)^d = B^\pi \sum_{i=0}^{s-1} B^i (A^d)^{i+1} + \sum_{i=0}^{r-1} (B^d)^{i+1} A^i A^\pi + A \sum_{i=0}^{r-1} B^{(i+2)d} A^i A^\pi + A \sum_{i=0}^{s-2} B^\pi B^{i+1} A^{(i+3)d} - AB^d A^d - ABB^d (A^d)^2.$$

3. DRAZIN INVERSE OF ATB MATRICES

This section gives some novel expressions for the Drazin inverse of ATB matrices $\bar{M}$ and M, represented by (1) and (2), respectively. These expressions are proved under new conditions that generalize some already existing results.

Theorem 6 presents a formula for the Drazin inverse of $\bar{M}$ under specific constraints. The notation $[y]$ will designate the truncated integer of y.

Theorem 6. *For square matrices* F *and* E *of identical dimensions and included in* $\bar{M}$ *given by* (1), *let* $\mathrm{ind}(F) = r$, $\mathrm{ind}(E) = s$. *If the restrictions*

$$EF^2 = 0, \quad (EF)^2 = 0 \quad \text{and} \quad E^2F = 0$$

are satisfied, then $\bar{M}^d$ *possesses the block representation*

$$\bar{M}^d = \begin{bmatrix} M_1 & M_2 \\ M_3 & M_4 \end{bmatrix},$$

where

$$M_1 = \sum_{k=0}^{s-1} F^{(2k+1)d}E^kE^\pi + \sum_{k=0}^{[\frac{r}{2}]} F^\pi F^{2k+1}E^{(k+1)d} + \sum_{k=2}^{s+1} F^{(2k)d}EFE^{k-2}E^\pi + \sum_{k=1}^{[\frac{r}{2}]+1} F^\pi F^{2k-2}EFE^{(k+1)d} - F^{2d}EFE^d,$$

$$M_2 = \sum_{k=1}^{s} F^{(2k+1)d}EFE^{k-1}E^\pi + \sum_{k=1}^{s-1} F^{2kd}E^kE^\pi + \sum_{k=0}^{[\frac{r}{2}]} F^\pi F^{2k}E^{kd} + \sum_{k=0}^{[\frac{r}{2}]} F^\pi F^{2k+1}EFE^{(k+2)d} - F^\pi E^\pi - F^dEFE^d - F^{3d}EF,$$

$$M_3 = \sum_{k=0}^{s-1} F^{(2k+2)d}E^kE^\pi + \sum_{k=0}^{[\frac{r}{2}]} F^\pi F^{2k}E^{(k+1)d} + \sum_{k=2}^{s+1} F^{(2k+1)d}EFE^{k-2}E^\pi + \sum_{k=2}^{[\frac{r}{2}]+2} F^\pi F^{2k-3}EFE^{(k+1)d} - F^{3d}EFE^d - F^dEFE^{2d},$$

$$M_4 = \sum_{k=1}^{s} F^{(2k+2)d}EFE^{k-1}E^\pi + \sum_{k=0}^{[\frac{r}{2}]} F^\pi F^{2k+1}E^{(k+1)d} + \sum_{k=0}^{s-1} F^{(2k+1)d}E^kE^\pi + \sum_{k=0}^{[\frac{r}{2}]} F^\pi F^{2k}EFE^{(2k+2)d} - F^d - F^{2d}EFE^d - F^{4d}EF.$$

Proof. We can represent $\bar{M}^2$ as

$$\bar{M}^2 = \begin{bmatrix} F^2+E & FE \\ F & E \end{bmatrix} = \begin{bmatrix} F^2 & 0 \\ F & F^eE \end{bmatrix} + \begin{bmatrix} E & FE \\ 0 & F^\pi E \end{bmatrix} := A + B. \tag{4}$$

Notice that the next equalities hold:

$$(F^eE)^d = 0, \quad (F^\pi E)^n = F^\pi E^n, n \geq 1.$$

By Lemma 1, it follows

$$(F^\pi E)^d = F^\pi[(EF^\pi)^d]^2E = F^\pi[(EF^\pi)^2]^dE = F^\pi(E^2)^dE = F^\pi E^d.$$

According to Lemma 2, we derive

$$A^d = \begin{bmatrix} F^{2d} & 0 \\ \sum_{i=0}^{r-1}(F^eE)^iF((F^2)^d)^{i+2} & 0 \end{bmatrix} = \begin{bmatrix} F^{2d} & 0 \\ F^{3d} & 0 \end{bmatrix},$$

$$B^d = \begin{bmatrix} E^d & \sum_{i=0}^{r-1} E^iFE((F^\pi E)^d)^{i+2} \\ 0 & F^\pi E^d \end{bmatrix} = \begin{bmatrix} E^d & EFE^{2d} + FE^d \\ 0 & F^\pi E^d \end{bmatrix},$$

and

$$A^\pi = \begin{bmatrix} F^\pi & 0 \\ -F^d & I \end{bmatrix}, \quad B^\pi = \begin{bmatrix} E^\pi & -EFE^d - FE^e \\ 0 & I - F^\pi E^e \end{bmatrix}.$$

It can be checked that

$$A^n = \begin{bmatrix} F^{2n} & 0 \\ F^{2n-1} & 0 \end{bmatrix}, \quad n \geq 3;$$

$$B^n = \begin{bmatrix} E^n & EFE^{n-1} + FE^n \\ 0 & F^\pi E^n \end{bmatrix}, \quad n \geq 2;$$

and

$$A^{nd} = \begin{bmatrix} F^{2nd} & 0 \\ F^{(2n+1)d} & 0 \end{bmatrix}, \quad B^{nd} = \begin{bmatrix} E^{nd} & EFE^{(n+1)d} + FE^{nd} \\ 0 & F^\pi E^{nd} \end{bmatrix}, \quad n \geq 1.$$

We observe that $\mathrm{ind}(F) = r$ and $\mathrm{ind}(E) = s$, respectively, give $F^r F^\pi = 0$ and $E^s E^\pi = 0$. It is clear that

$$t - 2 \leq 2\left[\frac{t}{2}\right] - 1 \leq t - 1, \quad t - 1 \leq 2\left[\frac{t}{2}\right] \leq t, \ t \leq 2\left[\frac{t}{2}\right] + 1 \leq t + 1$$

for arbitrary nonnegative integer t. Because

$$A^i A^\pi = \begin{bmatrix} F^{2i} & 0 \\ F^{2i-1} & 0 \end{bmatrix} \begin{bmatrix} F^\pi & 0 \\ -F^d & I \end{bmatrix} = \begin{bmatrix} F^{2i} F^\pi & 0 \\ F^{2i-1} F^\pi & 0 \end{bmatrix}, \quad i \geq 3,$$

and

$$\begin{aligned} B^i B^\pi &= \begin{bmatrix} E^i & EFE^{i-1} + FE^i \\ 0 & F^\pi E^i \end{bmatrix} \begin{bmatrix} E^\pi & -EFE^d - FE^e \\ 0 & I - F^\pi E^e \end{bmatrix} \\ &= \begin{bmatrix} E^i E^\pi & EFE^{i-1} E^\pi + FE^i E^\pi \\ 0 & F^\pi E^i E^\pi \end{bmatrix}, \quad i \geq 2, \end{aligned}$$

we conclude that $\mathrm{ind}(A) = [\frac{r}{2}] + 1$ and $\mathrm{ind}(B) = s + 1$. In particular,

$$AA^\pi = \begin{bmatrix} F^2 F^\pi & 0 \\ FF^\pi & F^e E \end{bmatrix}, A^2 A^\pi = \begin{bmatrix} F^4 F^\pi & 0 \\ F^3 F^\pi + F^e EF & 0 \end{bmatrix}$$

and

$$BB^\pi = \begin{bmatrix} EE^\pi & -EFE^e + FEE^\pi \\ 0 & F^\pi EE^\pi \end{bmatrix}.$$

Direct calculations show that the assumptions $B^2 A = 0, BA^2 = 0$ and $(BA)^2 = 0$ from Lemma 3 are satisfied. Making substitutions into Lemma 3 according to the following expressions

$$\begin{aligned} UB^\pi &= \begin{bmatrix} F^{5d} EF + F^{2d} & 0 \\ F^{6d} EF + F^{3d} & 0 \end{bmatrix} \begin{bmatrix} E^\pi & -EFE^d - FE^e \\ 0 & I - F^\pi E^e \end{bmatrix} \\ &= \begin{bmatrix} F^{5d} EFE^\pi + F^{2d} E^\pi & -F^{2d} EFE^d - F^d E^e \\ F^{6d} EFE^\pi + F^{3d} E^\pi & -F^{3d} EFE^d - F^{2d} E^e \end{bmatrix}, \end{aligned}$$

$$A^\pi V = \begin{bmatrix} F^\pi FEFE^{3d} + F^\pi E^d & F^\pi EFE^{2d} + F^\pi FE^d \\ -F^e EFE^{3d} - F^d E^d + F^\pi EFE^{3d} & -F^d EFE^{2d} - F^e E^d + F^\pi E^d \end{bmatrix},$$

$$AUV = \begin{bmatrix} F^{3d}EFE^d + F^e E^d + F^e FEFE^{3d} & F^e EFE^{2d} + F^e FE^d \\ F^{4d}EFE^d + F^e EFE^{3d} + F^d E^d & F^d EFE^{2d} + F^e E^d \end{bmatrix},$$

$$UVB = \begin{bmatrix} F^{5d}EFE^e + F^d EFE^{2d} + F^{2d}E^e & F^{2d}EFE^d + F^d E^e \\ F^{6d}EFE^e + F^{2d}EFE^{2d} + F^{3d}E^e & F^{3d}EFE^d + F^{2d}E^e \end{bmatrix},$$

$$-A^d - 2A^d BAB^{2d} - 2A^{2d}BAB^d = \begin{bmatrix} -F^{2d} - 2F^d EFE^{2d} - 2F^{3d}EFE^d & 0 \\ -F^{3d} - 2F^{2d}EFE^{2d} - 2F^{4d}EFE^d & 0 \end{bmatrix},$$

$$-B^d - BAB^{3d} - A^{3d}BA = \begin{bmatrix} -E^d - FEFE^{3d} - F^{5d}EF & -EFE^{2d} - FE^d \\ -F^\pi EFE^{3d} - F^{6d}EF & -F^\pi E^d \end{bmatrix},$$

$$\sum_{k=0}^{[\frac{r}{2}]+3} A^\pi A^k (B^d)^{k+1} =$$

$$= \left[\begin{array}{c|c} \sum_{k=0}^{[\frac{r}{2}]+3} F^\pi F^{2k} E^{(k+1)d} & \sum_{k=0}^{[\frac{r}{2}]+3} F^\pi F^{2k} EFE^{(k+2)d} + \sum_{k=0}^{[\frac{r}{2}]+3} F^\pi F^{2k+1} E^{(k+1)d} \\ \hline \sum_{k=1}^{[\frac{r}{2}]+3} F^\pi F^{2k-1} E^{(k+1)d} & \sum_{k=1}^{[\frac{r}{2}]+3} F^\pi F^{2k-1} EFE^{(k+2)d} + \sum_{k=0}^{[\frac{r}{2}]+3} F^\pi F^{2k} E^{(k+1)d} \\ +F^e EFE^{3d} - F^d E^d & -F^d EFE^{2d} \end{array}\right],$$

$$\sum_{k=2}^{[\frac{r}{2}]+3} A^\pi A^{k-2} BA (B^d)^{k+1} = \begin{bmatrix} \sum_{k=2}^{[\frac{r}{2}]+3} F^\pi F^{2k-3} EFE^{(k+1)d} & 0 \\ \sum_{k=2}^{[\frac{r}{2}]+3} F^\pi F^{2k-4} EFE^{(k+1)d} - F^e EFE^{3d} & 0 \end{bmatrix},$$

$$\sum_{k=2}^{s+3} A^{(k+1)d} BAB^{k-2} B^\pi = \begin{bmatrix} \sum_{k=2}^{s+3} F^{(2k+1)d} EFE^{k-2} E^\pi & 0 \\ \sum_{k=2}^{s+3} F^{(2k+2)d} EFE^{k-2} E^\pi & 0 \end{bmatrix},$$

$$\sum_{k=0}^{s+3} A^{(k+1)d} B^k B^\pi =$$

$$= \left[\begin{array}{c|c} \sum_{k=0}^{s+3} F^{(2k+2)d} E^k E^\pi & \sum_{k=1}^{s+3} F^{(2k+1)d} E^k E^\pi + \sum_{k=2}^{s+3} F^{(2k+2)d} EFE^{k-1} E^\pi \\ & -F^d E^e - F^{2d} EFE^d - F^{4d} EFE^e \\ \hline \sum_{k=0}^{s+3} F^{(2k+3)d} E^k E^\pi & \sum_{k=1}^{s+3} F^{(2k+2)d} E^k E^\pi + \sum_{k=2}^{s+3} F^{(2k+3)d} EFE^{k-1} E^\pi \\ & -F^{2d} E^e - F^{3d} EFE^d - F^{5d} EFE^e \end{array}\right],$$

(4) leads to

$$(\bar{M}^d)^2 = (A+B)^d = \begin{bmatrix} m_1 & m_2 \\ m_3 & m_4 \end{bmatrix},$$

where

$$\begin{aligned} m_1 = & \sum_{k=0}^{s+3} F^{(2k+2)d} E^k E^\pi + \sum_{k=0}^{[\frac{r}{2}]+3} F^\pi F^{2k} E^{(k+1)d} + \sum_{k=2}^{s+3} F^{(2k+1)d} EFE^{k-2} E^\pi \\ & + \sum_{k=2}^{[\frac{r}{2}]+3} F^\pi F^{2k-3} EFE^{(k+1)d} - F^{3d} EFE^d - F^d EFE^{2d}, \end{aligned}$$

$$\begin{aligned} m_2 = & \sum_{k=2}^{s+3} F^{(2k+2)d} EFE^{k-1} E^\pi + \sum_{k=1}^{s+3} F^{(2k+1)d} E^k E^\pi + \sum_{k=0}^{[\frac{r}{2}]+3} F^\pi F^{2k+1} E^{(k+1)d} \\ & + \sum_{k=0}^{[\frac{r}{2}]+3} F^\pi F^{2k} EFE^{(2k+2)d} - F^d E^e - F^{2d} EFE^d - F^{4d} EFE^e, \end{aligned}$$

$$\begin{aligned} m_3 = & \sum_{k=0}^{s+3} F^{(2k+3)d} E^k E^\pi + \sum_{k=1}^{[\frac{r}{2}]+3} F^\pi F^{2k-1} E^{(k+1)d} + \sum_{k=2}^{[\frac{r}{2}]+3} F^\pi F^{2k-4} EFE^{(k+1)d} \\ & + \sum_{k=2}^{s+3} F^{(2k+2)d} EFE^{k-2} E^\pi - F^{4d} EFE^d - F^{2d} EFE^{2d} - F^d E^d, \end{aligned}$$

$$\begin{aligned} m_4 = & \sum_{k=1}^{[\frac{r}{2}]+3} F^\pi F^{2k-1} EFE^{(k+2)d} + \sum_{k=2}^{s+3} F^{(2k+3)d} EFE^{k-1} E^\pi + \sum_{k=1}^{s+3} F^{(2k+2)d} E^k E^\pi \\ & + \sum_{k=0}^{[\frac{r}{2}]+3} F^\pi F^{2k} E^{(k+1)d} - F^{2d} E^e - F^{3d} EFE^d - F^{5d} EFE^e - F^d EFE^{2d}. \end{aligned}$$

Applying the equality $\bar{M}^d = \bar{M}\bar{M}^{2d}$, we find

$$\bar{M}^d = \begin{bmatrix} M_1 & M_2 \\ M_3 & M_4 \end{bmatrix},$$

where

$$M_1 = Fm_1 + Em_3 = \sum_{k=0}^{s+3} F^{(2k+1)d} E^k E^\pi + \sum_{k=0}^{[\frac{r}{2}]+3} F^\pi F^{2k+1} E^{(k+1)d}$$
$$+ \sum_{k=2}^{s+3} F^{(2k)d} EFE^{k-2} E^\pi + \sum_{k=1}^{[\frac{r}{2}]+3} F^\pi F^{2k-2} EFE^{(k+1)d} - F^{2d} EFE^d,$$
$$M_2 = Fm_2 + Em_4 = \sum_{k=2}^{s+3} F^{(2k+1)d} EFE^{k-1} E^\pi$$
$$+ \sum_{k=1}^{s+3} F^{2kd} E^k E^\pi + \sum_{k=0}^{[\frac{r}{2}]+3} F^\pi F^{2k+2} E^{(k+1)d}$$
$$+ \sum_{k=0}^{[\frac{r}{2}]+3} F^\pi F^{2k+1} EFE^{(k+2)d} + F^\pi E^e - F^d EFE^d - F^{3d} EFE^e,$$
$$M_3 = m_1 = \sum_{k=0}^{s+3} F^{(2k+2)d} E^k E^\pi + \sum_{k=0}^{[\frac{r}{2}]+3} F^\pi F^{2k} E^{(k+1)d} + \sum_{k=2}^{s+3} F^{(2k+1)d} EFE^{k-2} E^\pi$$
$$+ \sum_{k=2}^{[\frac{r}{2}]+3} F^\pi F^{2k-3} EFE^{(k+1)d} - F^{3d} EFE^d - F^d EFE^{2d},$$
$$M_4 = m_2 = \sum_{k=2}^{s+3} F^{(2k+2)d} EFE^{k-1} E^\pi + \sum_{k=0}^{[\frac{r}{2}]+3} F^\pi F^{2k+1} E^{(k+1)d} + \sum_{k=1}^{s+3} F^{(2k+1)d} E^k E^\pi$$
$$+ \sum_{k=0}^{[\frac{r}{2}]+3} F^\pi F^{2k} EFE^{(2k+2)d} - F^d E^e - F^{2d} EFE^d - F^{4d} EFE^e.$$

The proof can be finished by setting the upper and lower bounds of the involved sums. □

An example is given for finding the Drazin inverse of adequate ATB matrix utilizing Theorem 6.

Example 7. *Consider* 2×2 *complex block matrices*

$$F' = \begin{bmatrix} 0 & 0 \\ 0 & H \end{bmatrix}, \quad E' = \begin{bmatrix} 0 & 0 \\ HQ & 0 \end{bmatrix}$$

and 4×4 *block matrix*

$$\bar{M} = \begin{bmatrix} F' & E' \\ I & 0 \end{bmatrix} = \left[\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ 0 & H & HQ & 0 \\ \hline I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \end{array}\right],$$

where H and Q are matrices of appropriate dimensions. Simple calculation gives $(E')^2 = 0$ and $E'F' = 0$. So, conditions of Theorem 6 are satisfied. Further, E' and F' satisfy $s = \operatorname{ind}(E') = 2$ and $r = \operatorname{ind}(F') = \operatorname{ind}(H)$. The Drazin inverses of blocks E' and F' are equal to $(E')^d = 0$ and $(F')^d = \begin{bmatrix} 0 & 0 \\ 0 & H^d \end{bmatrix}$, which imply

$$\bar{M}^d = \begin{bmatrix} F' & E' \\ I & 0 \end{bmatrix}^d = \left[\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ 0 & H & HQ & 0 \\ \hline I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \end{array}\right]^d = \begin{bmatrix} M_1 & M_2 \\ M_3 & M_4 \end{bmatrix},$$

in which blocks $M_i, i = 1,2,3,4$ are defined as in Theorem 6. Necessary calculation gives

$$\begin{aligned}
M_1 &= \sum_{k=0}^{1} (F')^{(2k+1)d}(E')^k(E')^\pi = (F')^d + (F')^{3d}E' \\
&= \begin{bmatrix} 0 & 0 \\ 0 & H^d \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ H^{3d}HQ & 0 \end{bmatrix}, \\
M_2 &= (F')^{2d}E' = \begin{bmatrix} 0 & 0 \\ 0 & H^{2d} \end{bmatrix} \begin{bmatrix} 0 & 0 \\ HQ & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ H^{2d}HQ & 0 \end{bmatrix}, \\
M_3 &= \sum_{k=0}^{1} (F')^{(2k+2)d}(E')^k(E')^\pi = (F')^{2d} + (F')^{4d}E' \\
&= \begin{bmatrix} 0 & 0 \\ 0 & H^{2d} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ H^{4d}HQ & 0 \end{bmatrix}, \\
M_4 &= \sum_{k=0}^{1} (F')^{(2k+1)d}(E')^k(E')^\pi - (F')^d = (F')^{3d}E' = \begin{bmatrix} 0 & 0 \\ H^{3d}HQ & 0 \end{bmatrix}.
\end{aligned}$$

Following the results of Theorem 6, it can be obtained

$$\begin{bmatrix} F' & E' \\ I & 0 \end{bmatrix}^d = \left[\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ H^{2d}Q & H^d & H^dQ & 0 \\ \hline 0 & 0 & 0 & 0 \\ H^{3d}Q & H^{2d} & H^{2d}Q & 0 \end{array}\right].$$

Now, the goal of our research is the block matrix M defined in (2). The representation of M^d is developed in Theorem 8 using the representation of $\bar{M}^d$ in Theorem 6.

Theorem 8. *Let M be represented as in (2) such that $\operatorname{ind}(F) = r$, $\operatorname{ind}(EG) = s$, and let square matrices F and EG be of the same size. If the constraints*

$$EGF^2 = 0, \quad (EGF)^2 = 0 \quad \text{and} \quad (EG)^2F = 0$$

are satisfied, then

$$M^d = \begin{bmatrix} N_1 & N_2 \\ N_3 & N_4 \end{bmatrix}, \tag{5}$$

where

$$\begin{aligned}
N_1 = {} & \sum_{k=1}^{s} F^{(2k+2)d} EGF(EG)^{k-1}(EG)^{\pi} + \sum_{k=0}^{s-1} F^{(2k+1)d}(EG)^k(EG)^{\pi} \\
& + \sum_{k=0}^{[\frac{r}{2}]} F^{\pi} F^{2k+1}(EG)^{(k+1)d} + \sum_{k=1}^{[\frac{r}{2}]+1} F^{\pi} F^{2k-2} EGF(EG)^{(k+1)d} \\
& - F^{2d} EGF(EG)^d, \\
N_2 = {} & \sum_{k=2}^{s+1} F^{(2k+1)d} EGF(EG)^{k-2}(EG)^{\pi} E + \sum_{k=0}^{[\frac{r}{2}]} F^{\pi} F^{2k+1} EGF(EG)^{(k+3)d} E \\
& + \sum_{k=0}^{[\frac{r}{2}]} F^{\pi} F^{2k}(EG)^{(k+1)d} E + \sum_{k=0}^{s-1} F^{(2k+2)d}(EG)^k(EG)^{\pi} E \\
& - F^d EGF(EG)^{2d} E - F^{3d} EGF(EG)^d E, \\
N_3 = {} & \sum_{k=0}^{[\frac{r}{2}]} GF^{\pi} F^{2k}(EG)^{(k+1)d} + \sum_{k=1}^{s} GF^{(2k+3)d} EGF(EG)^{k-1}(EG)^{\pi} \\
& + \sum_{k=0}^{s-1} GF^{(2k+2)d}(EG)^k(EG)^{\pi} + \sum_{k=2}^{[\frac{r}{2}]+2} GF^{\pi} F^{2k-3} EGF(EG)^{(k+1)d} \\
& - GF^{3d} EGF(EG)^d - GF^d EGF(EG)^{2d}, \\
N_4 = {} & \sum_{k=2}^{s+1} GF^{(2k+2)d} EGF(EG)^{k-2}(EG)^{\pi} E + \sum_{k=0}^{s-1} GF^{(2k+3)d}(EG)^k(EG)^{\pi} E \\
& + \sum_{k=0}^{[\frac{r}{2}]} GF^{\pi} F^{2k+1}(EG)^{(k+2)d} E + \sum_{k=0}^{[\frac{r}{2}]} GF^{\pi} F^{2k} EGF(EG)^{(2k+3)d} E \\
& - GF^d(EG)^d E - GF^{2d} EGF(EG)^{2d} E - GF^{4d} EGF(EG)^d E.
\end{aligned}$$

Proof. Let $M = AB$, where

$$A = \begin{bmatrix} I & 0 \\ 0 & G \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} F & E \\ I & 0 \end{bmatrix}.$$

Because

$$BA = \begin{bmatrix} F & EG \\ I & 0 \end{bmatrix},$$

by Theorem 6 we obtain

$$(BA)^d = \begin{bmatrix} n_1 & n_2 \\ n_3 & n_4 \end{bmatrix},$$

where

$$\begin{aligned}
n_1 &= \sum_{k=0}^{s-1} F^{(2k+1)d}(EG)^k(EG)^\pi + \sum_{k=0}^{[\frac{r}{2}]} F^\pi F^{2k+1}(EG)^{(k+1)d} \\
&+ \sum_{k=2}^{s+1} F^{(2k)d}EGF(EG)^{k-2}(EG)^\pi + \sum_{k=1}^{[\frac{r}{2}]+1} F^\pi F^{2k-2}EGF(EG)^{(k+1)d} \\
&- F^{2d}EGF(EG)^d, \\
n_2 &= \sum_{k=1}^{s} F^{(2k+1)d}EGF(EG)^{k-1}(EG)^\pi + \sum_{k=1}^{s-1} F^{2kd}(EG)^k(EG)^\pi \\
&+ \sum_{k=0}^{[\frac{r}{2}]} F^\pi F^{2k}(EG)^{kd} + \sum_{k=0}^{[\frac{r}{2}]} F^\pi F^{2k+1}EGF(EG)^{(k+2)d} \\
&- F^\pi(EG)^\pi - F^dEGF(EG)^d - (EG)^{3d}EGF, \\
n_3 &= \sum_{k=0}^{s-1} F^{(2k+2)d}(EG)^k(EG)^\pi + \sum_{k=0}^{[\frac{r}{2}]} F^\pi F^{2k}(EG)^{(k+1)d} \\
&+ \sum_{k=2}^{s+1} F^{(2k+1)d}EGF(EG)^{k-2}(EG)^\pi \\
&+ \sum_{k=2}^{[\frac{r}{2}]+2} F^\pi F^{2k-3}EGF(EG)^{(k+1)d} - F^{3d}EGF(EG)^d - F^dEGF(EG)^{2d}, \\
n_4 &= \sum_{k=1}^{s} F^{(2k+2)d}EGF(EG)^{k-1}(EG)^\pi + \sum_{k=0}^{[\frac{r}{2}]} F^\pi F^{2k+1}(EG)^{(k+1)d} \\
&+ \sum_{k=0}^{s-1} F^{(2k+1)d}(EG)^k(EG)^\pi + \sum_{k=0}^{[\frac{r}{2}]} F^\pi F^{2k}EGF(EG)^{(2k+2)d} \\
&- F^d - F^{2d}EGF(EG)^d - F^{4d}EGF.
\end{aligned}$$

By Lemma 1, we deduce

$$M^d = A(BA)^{2d}B = \begin{bmatrix} n_1^2F + n_2n_3F + n_1n_2 + n_2n_4 & n_1^2E + n_2n_3E \\ Gn_3n_1F + Gn_4n_3F + Gn_3n_2 + Gn_4^2 & Gn_3n_1E + Gn_4n_3E \end{bmatrix}. \tag{6}$$

We finish this proof substituting the next expressions

$$\begin{aligned}
n_1^2 &= \sum_{k=0}^{s-1}(F^d)^{2k+2}(EG)^k(EG)^\pi + \sum_{k=2}^{s+1}(F^d)^{2k+1}EGF(EG)^{k-2}(EG)^\pi,\\
n_4^2 &= F^{3d}EGF(EG)^d,\\
n_2n_4 &= F^{2d}EGF(EG)^d,\\
n_3n_1 &= \sum_{k=0}^{s-1}(F^d)^{2k+3}(EG)^k(EG)^\pi + \sum_{k=2}^{s+1}(F^d)^{2k+2}EGF(EG)^{k-2}(EG)^\pi,\\
n_2n_3 &= F^\pi(EG)^d - F^dEGF(EG)^{2d} - F^{3d}EGF(EG)^d\\
&\quad + \sum_{k=0}^{[\frac{r}{2}]}F^\pi F^{2k+1}EGF(EG)^{(k+3)d} + \sum_{k=0}^{[\frac{r}{2}]}F^\pi F^{2k+2}(EG)^{(k+2)d},\\
n_1n_2 &= -2F^{2d}EGF(EG)^d - F^d(EG)^e - F^{4d}EGF(EG)^e\\
&\quad + \sum_{k=2}^{s}(F^d)^{2k+2}EGF(EG)^{k-1}(EG)^\pi + \sum_{k=1}^{s}(F^d)^{2k+1}(EG)^k(EG)^\pi\\
&\quad + \sum_{k=0}^{[\frac{r}{2}]}F^\pi F^{2k+1}(EG)^{(k+1)d} + \sum_{k=1}^{[\frac{r}{2}]+1}F^\pi F^{2k-2}EGF(EG)^{(k+1)d},\\
n_4n_3 &= -F^d(EG)^d - F^{2d}EGF(EG)^{2d} - F^{4d}EGF(EG)^d\\
&\quad + \sum_{k=0}^{[\frac{r}{2}]}F^\pi F^{2k}EGF(EG)^{(2k+3)d} + \sum_{k=0}^{[\frac{r}{2}]}F^\pi F^{2k+1}(EG)^{(k+2)d},\\
n_3n_2 &= -2F^{3d}EGF(EG)^d - F^dEGF(EG)^{2d} - F^{2d}(EG)^e - F^{5d}EGF(EG)^e\\
&\quad + \sum_{k=0}^{[\frac{r}{2}]}F^\pi F^{2k}(EG)^{(k+1)d} + \sum_{k=2}^{s}(F^d)^{2k+3}EGF(EG)^{k-1}(EG)^\pi\\
&\quad + \sum_{k=1}^{s-1}(F^d)^{2k+2}(EG)^k(EG)^\pi + \sum_{k=2}^{[\frac{r}{2}]+1}F^\pi F^{2k-3}EGF(EG)^{(k+1)d}
\end{aligned}$$

into (6). □

The following results are obtained as consequences of Theorem 8.

Corollary 9. [**14**, Theorem 3.4] *For square matrices F and EG of an identical dimension, let $EGF = 0$. Then*

$$M^d = \begin{bmatrix} FU & UE \\ GU & G\left[F^dU + (FU - F^d)(EG)^d\right]E \end{bmatrix},$$

where M is represented by (2), $\operatorname{ind}(F) = r$, $\operatorname{ind}(EG) = s$ *and*

$$(7) \qquad U = \sum_{k=0}^{s-1} F^{(2k+2)d}(EG)^k(EG)^\pi + \sum_{k=0}^{[\frac{r}{2}]} F^\pi F^{2k}(EG)^{(k+1)d}.$$

We strengthen the conditions in Corollary 9 to obtain the following formula.

Corollary 10. *For square matrices F and EG of an identical dimension, let $EGF = 0$ and $GEGE = 0$. Then*

$$M^d = \begin{bmatrix} F^d + F^{3d}EG + F^{5d}EGEG & F^{2d}E + F^{4d}EGE \\ GF^{2d} + GF^{4d}EG + GF^{6d}EGEG & GF^{3d}E + GF^{5d}EGE \end{bmatrix},$$

where M is represented by (2)*.*

Corollary 11. *For square matrices F, E and EG of identical dimensions, let us assume $EGF = 0$ and $EGEG = 0$. Under these conditions, it follows*

$$M^d = \begin{bmatrix} F^d + F^{3d}EG & F^{2d}E + F^{4d}EGE \\ GF^{2d} + GF^{4d}EG & GF^{3d}E + GF^{5d}EGE \end{bmatrix},$$

where M is represented by (2)*.*

Remark 12. *Let M be represented by* (2), $\operatorname{ind}(F) = r$, $\operatorname{ind}(EG) = s$*. Subsequently, under the assumptions used in Corollary 9, it is possible to derive another results presented in* [**14**]*:*

$$M^d = \begin{bmatrix} FU & UE \\ GU & GFU(EG)^dE \end{bmatrix} \text{ if } EGF = 0 \text{ and } F \text{ is nilpotent},$$

where

$$U = \sum_{k=0}^{[\frac{r}{2}]} F^{2k}(EG)^{(k+1)d};$$

$$M^d = \begin{bmatrix} F^d & F^{2d}E \\ GF^{2d} & GF^{3d}E \end{bmatrix} \text{ if } EG = 0;$$

$$M^d = \begin{bmatrix} F\bar{U} & \bar{U}E + F^\pi(EG)^dE \\ G\bar{U} + GF^\pi(EG)^d & GF^d\bar{U}E - GF^d(EG)^dE \end{bmatrix} \text{ if } F^\pi FE = 0 \text{ and } EGF = 0,$$

where

$$\bar{U} = \sum_{k=0}^{s-1} F^{(2k+2)d}(EG)^k(EG)^\pi.$$

Example 13 illustrates the computation of the Drazin inverse of an adequate ATB matrix utilizing the results of Theorem 8.

Example 13. *Consider 2×2 complex blocks E' and F' as in Example 7 in conjunction with block matrix*

$$G' = \begin{bmatrix} G_1 & 0 \\ G_3 & G_4 \end{bmatrix}$$

and 4×4 block matrix

$$M = \begin{bmatrix} F' & E' \\ G' & 0 \end{bmatrix} = \left[\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ 0 & H & HQ & 0 \\ \hline G_1 & 0 & 0 & 0 \\ G_3 & G_4 & 0 & 0 \end{array}\right],$$

with the blocks H and Q of proper dimensions. Since $(E'G')^2 = 0$ and $E'G'F' = 0$, it follows that requirements of Theorem 8 are satisfied. As a consequence,

$$M^d = \begin{bmatrix} F' & E' \\ G' & 0 \end{bmatrix}^d = \left[\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ 0 & H & HQ & 0 \\ \hline G_1 & 0 & 0 & 0 \\ G_3 & G_4 & 0 & 0 \end{array}\right]^d = \begin{bmatrix} N_1 & N_2 \\ N_3 & N_4 \end{bmatrix},$$

in which the blocks $N_i, i = 1,2,3,4$ are defined as in Theorem 8. Basic calculation gives $s = \operatorname{ind}(E'G') = 2$, $r = \operatorname{ind}(F') = \operatorname{ind}(H)$, $(E'G')^d = 0$, $(E'G')^\pi = I$, and $(F')^d = \begin{bmatrix} 0 & 0 \\ 0 & H^d \end{bmatrix}$. In addition, the following calculation follows

$$\begin{aligned}
N_1 &= \sum_{k=0}^{1} (F')^{(2k+1)d}(E'G')^k(E'G')^\pi = (F')^d + (F')^{3d}E'G' \\
&= \begin{bmatrix} 0 & 0 \\ 0 & H^d \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ H^{3d}HQG_1 & 0 \end{bmatrix}, \\
N_2 &= \sum_{k=0}^{1} (F')^{(2k+2)d}(E'G')^k(E'G')^\pi E' = (F')^{2d}E' + (F')^{4d}E'G'E' = (F')^{2d}E' \\
&= \begin{bmatrix} 0 & 0 \\ 0 & H^{2d} \end{bmatrix} \begin{bmatrix} 0 & 0 \\ HQ & 0 \end{bmatrix}, \\
N_3 &= \sum_{k=0}^{1} G'(F')^{(2k+2)d}(E'G')^k(E'G')^\pi = G'(F')^{2d} + G'(F')^{4d}E'G' \\
&= \begin{bmatrix} 0 & 0 \\ 0 & G_4H^{2d} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ G_4H^{4d}HQG_1 & 0 \end{bmatrix}, \\
N_4 &= \sum_{k=0}^{1} G'(F')^{(2k+3)d}(E'G')^k(E'G')^\pi E' = G'(F')^{3d}E' + G'(F')^{5d}E'G'E' \\
&= \begin{bmatrix} 0 & 0 \\ G_4H^{3d}HQ & 0 \end{bmatrix}.
\end{aligned}$$

Following the results of Theorem 8, it can be obtained

$$\begin{bmatrix} F' & E' \\ G' & 0 \end{bmatrix}^d = \left[\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ H^{2d}QG_1 & H^d & H^dQ & 0 \\ \hline 0 & 0 & 0 & 0 \\ G_4H^{3d}QG_1 & G_4H^{2d} & G_4H^{2d}Q & 0 \end{array}\right].$$

4. APPLICATIONS

The results proposed in the previous section are applied to obtain certain characterizations and representations of the Drazin inverse of the partitioned matrix N specified as in (3).

Theorem 14. *For square matrices F, H and EG which are such that F and EG are of identical dimensions, let*

$$EGF^2 = 0, \quad (EG)^2F = 0, \quad (EGF)^2 = 0, \quad EHG = 0 \quad \text{and} \quad EH^2 = 0.$$

Under these assumptions, it follows

$$\begin{aligned} N^d = &\begin{bmatrix} I & 0 \\ 0 & H^\pi \end{bmatrix} \sum_{i=0}^{s-1} \begin{bmatrix} 0 & 0 \\ 0 & H \end{bmatrix}^i (M^d)^{i+1} \\ &+ \sum_{i=0}^{r-1} \begin{bmatrix} 0 & 0 \\ 0 & H^{(i+1)d} \end{bmatrix} M^i \begin{bmatrix} F^\pi - F^{3d}EGF - N_2G & -N_1E \\ -GF^d - GF^{4d}EGF - N_4G & I - N_3E \end{bmatrix} \\ &+ \begin{bmatrix} I & 0 \\ 0 & H^\pi \end{bmatrix} \sum_{i=0}^{s-1} \begin{bmatrix} 0 & 0 \\ 0 & H \end{bmatrix}^i (M^d)^{i+2} \begin{bmatrix} 0 & 0 \\ 0 & H \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & H^dN_4H + H^{2d}N_3EH \end{bmatrix} \\ &+ \sum_{i=0}^{r-2} \begin{bmatrix} 0 & 0 \\ 0 & H^{(i+3)d} \end{bmatrix} M^{i+1} \begin{bmatrix} 0 & -N_1EH \\ 0 & H - N_3EH \end{bmatrix}, \end{aligned}$$

where N, M and M^d are given by (3), (2) *and* (5), *respectively, and* $\text{ind}(M) = r$, $\text{ind}(H) = s$.

Proof. The splitting $N = M + B$ composed of

$$M = \begin{bmatrix} F & E \\ G & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & 0 \\ 0 & H \end{bmatrix},$$

satisfies $MB^2 = 0$ and $MBM = 0$. According to Theorem 8, M^d is expressed as in (5) and so

$$M^\pi = \begin{bmatrix} I - N_1F - N_2G & -N_1E \\ -N_3F - N_4G & I - N_3E \end{bmatrix}.$$

Since

$$\begin{aligned} N_1F &= F^e + F^{3d}EGF, \\ N_3F &= GF^d + GF^{4d}EGF, \end{aligned}$$

it is evident that

$$M^\pi = \begin{bmatrix} F^\pi - F^{3d}EGF - N_2G & -N_1E \\ -GF^d - GF^{4d}EGF - N_4G & I - N_3E \end{bmatrix}.$$

The proof can be finished using Lemma 4 in conjunction with

$$B^d = \begin{bmatrix} 0 & 0 \\ 0 & H^d \end{bmatrix} \quad \text{and} \quad B^\pi = \begin{bmatrix} I & 0 \\ 0 & H^\pi \end{bmatrix}.$$

□

By strengthening the extra restrictions $EGF = 0$ and $GEGE = 0$, we simplify Theorem 14 to enable the verification of the following formula for N^d.

Corollary 15. *For square matrices F, H and EG such that F and EG are of the same size, let*

$$EGF = 0, \quad GEGE = 0, \quad EHG = 0 \quad \text{and} \quad EH^2 = 0.$$

Then

$$\begin{aligned} N^d = &\begin{bmatrix} I & 0 \\ 0 & H^\pi \end{bmatrix} \sum_{i=0}^{s-1} \begin{bmatrix} 0 & 0 \\ 0 & H \end{bmatrix}^i (M^d)^{i+1} \\ &+ \sum_{i=0}^{r-1} \begin{bmatrix} 0 & 0 \\ 0 & H^{(i+1)d} \end{bmatrix} M^i \\ &\times \begin{bmatrix} F^\pi - F^{2d}EG - F^{4d}EGEG & -F^dE - F^{3d}EGE \\ -GF^d - GF^{3d}EG - GF^{5d}EGEG & I - GF^{2d}E - GF^{4d}EGE \end{bmatrix} \\ &+ \begin{bmatrix} I & 0 \\ 0 & H^\pi \end{bmatrix} \sum_{i=0}^{s-1} \begin{bmatrix} 0 & 0 \\ 0 & H \end{bmatrix}^i (M^d)^{i+2} \begin{bmatrix} 0 & 0 \\ 0 & H \end{bmatrix} \\ &+ \sum_{i=0}^{r-2} \begin{bmatrix} 0 & 0 \\ 0 & H^{(i+3)d} \end{bmatrix} M^{i+1} \begin{bmatrix} 0 & -F^dEH - F^{3d}EGEH \\ 0 & H - GF^{2d}EH - GF^{4d}EGEH \end{bmatrix} \\ &- \begin{bmatrix} 0 & 0 \\ 0 & H^dGF^{3d}EH + H^dGF^{5d}EGEH + H^{2d}GF^{2d}EH + H^{2d}GF^{4d}EGEH \end{bmatrix}, \end{aligned}$$

where N and M are specified by (3) *and* (2)*, respectively. Corollary 10 gives the representation of M^d, and* $\operatorname{ind}(M) = r$, $\operatorname{ind}(H) = s$.

Theorem 14 extends analogous expressions under the following restrictions:
1. $EG = 0$, $EHG = 0$ and $EH^2 = 0$ (see [**16**, Theorem 2.2]);
2. $EGF = 0$, $GEGE = 0$, $F^{\pi}EGE = 0$, $EHG = 0$ and $EH^2 = 0$ (see [**1**, Theorem 3.6]).

Combining Lemma 5 with $BMB = 0$ and $M^2B = 0$, where M is represented by (2) and $B = \begin{bmatrix} 0 & 0 \\ 0 & H \end{bmatrix}$, we obtain the following formula.

Theorem 16. *For square matrices F, H and EG, such that F and EG are of the same size, let*

$$EGF^2 = 0, \quad (EG)^2F = 0, \quad (EGF)^2 = 0, \quad FEH = 0 \quad \text{and} \quad GEH = 0.$$

Then

$$\begin{aligned} N^d = &\begin{bmatrix} I & 0 \\ 0 & H^{\pi} \end{bmatrix} \sum_{i=0}^{s-1} \begin{bmatrix} 0 & 0 \\ 0 & H \end{bmatrix}^i M^{(i+1)d} \\ &+ \sum_{i=0}^{r-1} \begin{bmatrix} 0 & EH^{(i+2)d} \\ 0 & H^{(i+1)d} \end{bmatrix} M^i \begin{bmatrix} F^{\pi} - F^{3d}EGF - N_2G & -N_1E \\ -GF^d - GF^{4d}EGF - N_4G & I - N_3E \end{bmatrix} \\ &+ \sum_{i=0}^{s-2} \begin{bmatrix} 0 & EH^{\pi}H^{i+1} \\ 0 & 0 \end{bmatrix} M^{(i+3)d} - \begin{bmatrix} 0 & EHH^d \\ 0 & 0 \end{bmatrix} M^{2d} - \begin{bmatrix} EH^dN_3 & EH^dN_4 \\ 0 & 0 \end{bmatrix}, \end{aligned}$$

where N, M and M^d are given by (3), (2) *and* (5), *respectively,* $\text{ind}(M) = r$ *and* $\text{ind}(H) = s$.

Consequently, we show the subsequent representations for the Drazin inverse of N.

Corollary 17. *For square matrices F, H and EG such that F and EG are of the same size, let*

$$FEH = 0, \quad GEH = 0, \quad EGF = 0 \quad \text{and} \quad EGEG = 0.$$

Then

$$\begin{aligned} N^d = &\begin{bmatrix} I & 0 \\ 0 & H^{\pi} \end{bmatrix} \sum_{i=0}^{s-1} \begin{bmatrix} 0 & 0 \\ 0 & H \end{bmatrix}^i M^{(i+1)d} \\ &+ \sum_{i=0}^{r-1} \begin{bmatrix} 0 & EH^{(i+2)d} \\ 0 & H^{(i+1)d} \end{bmatrix} M^i \begin{bmatrix} F^{\pi} - F^{2d}EG & -F^dE - F^{3d}EGE \\ -GF^d - GF^{3d}EG & I - GF^{2d}E - GF^{4d}EGE \end{bmatrix} \\ &+ \sum_{i=0}^{s-2} \begin{bmatrix} 0 & EH^{\pi}H^{i+1} \\ 0 & 0 \end{bmatrix} M^{(i+3)d} - \begin{bmatrix} 0 & EHH^d \\ 0 & 0 \end{bmatrix} M^{2d} \\ &- \begin{bmatrix} EH^d(GF^{2d} + GF^{4d}EG) & EH^d(GF^{3d}E + GF^{5d}EGE) \\ 0 & 0 \end{bmatrix}, \end{aligned}$$

where N and M are specified as in (3) *and* (2), *respectively, M^d is expressed by Corollary 11, and* $\mathrm{ind}(M) = r$, $\mathrm{ind}(H) = s$.

Corollary 18. *For square matrices F, H and EG such that F and EG are of the same size, let*

$$EGF = 0, \quad GEH = 0, \quad FEH = 0 \quad \text{and} \quad GEGE = 0.$$

Then

$$\begin{aligned} N^d = &\begin{bmatrix} I & 0 \\ 0 & H^\pi \end{bmatrix} \sum_{i=0}^{s-1} \begin{bmatrix} 0 & 0 \\ 0 & H \end{bmatrix}^i M^{(i+1)d} \\ &+ \sum_{i=0}^{r-1} \begin{bmatrix} 0 & EH^{(i+2)d} \\ 0 & H^{(i+1)d} \end{bmatrix} M^i \\ &\times \begin{bmatrix} F^\pi - F^{2d}EG - F^{4d}EGEG & -F^dE - F^{3d}EGE \\ -GF^d - GF^{3d}EG - GF^{5d}EGEG & I - GF^{2d}E - GF^{4d}EGE \end{bmatrix} \\ &+ \sum_{i=0}^{s-2} \begin{bmatrix} 0 & EH^\pi H^{i+1} \\ 0 & 0 \end{bmatrix} M^{(i+3)d} - \begin{bmatrix} 0 & EHH^d \\ 0 & 0 \end{bmatrix} M^{2d} \\ &- \begin{bmatrix} EH^d(GF^{2d} + GF^{4d}EG + GF^{6d}EGEG) & EH^d(GF^{3d}E + GF^{5d}EGE) \\ 0 & 0 \end{bmatrix}, \end{aligned}$$

where N and M are specified as in (3) *and* (2), *respectively, M^d is expressed by Corollary 10,* $\mathrm{ind}(M) = r$ *and* $\mathrm{ind}(H) = s$.

Remark 19. *Theorem 16 generalizes some of known results under the next assumptions:*

1. $EGF = 0$, $FEH = 0$ and $GEH = 0$ (see [**24**, Theorem 3.2]*);*

2. $FEH = 0$, $GEH = 0$, $EGF = 0$, $HGF = 0$, $EGEG = 0$ and $H^\pi GEG = 0$ (see [**1**, Theorem 3.1]*);*

3. $HGF = 0$, $EGF = 0$, $GEH = 0$, $FEH = 0$, $GEGE = 0$ and $F^\pi EGE = 0$ (see [**1**, Theorem 3.3]*).*
Specifically, Theorem 14 and Theorem 16 separately extend the subsequent results:

1. $EG = 0$ and $EH = 0$ (see [**16**, Corollary 2.3]*);*

2. $EGF = 0$, $EH = 0$ and EG is nilpotent (see [**10**, Theorem 4.2];

3. $EG = 0$, $EH = 0$ and H is nilpotent (see [**18**, Corollary 2.3].

Next we consider that $BMB = 0$ and $BM^2 = 0$, where M is as in (2) and $B = \begin{bmatrix} 0 & 0 \\ 0 & H \end{bmatrix}$, and utilize Lemma 4 to give the following formula.

Theorem 20. *For square matrices F, H and EG such that F and EG are of the same size, let*

$$EGF^2 = 0, \quad (EG)^2F = 0, \quad (EGF)^2 = 0, \quad HGF = 0 \quad \text{and} \quad HGE = 0.$$

Then

$$N^d = \begin{bmatrix} F^\pi - F^{3d}EGF - N_2G & -N_1E \\ -GF^d - GF^{4d}EGF - N_4G & I - N_3E \end{bmatrix} \sum_{i=0}^{r-1} M^i \begin{bmatrix} 0 & 0 \\ H^{(i+2)d}G & H^{(i+1)d} \end{bmatrix}$$
$$+ \sum_{i=0}^{s-1}(M^d)^{i+1} \begin{bmatrix} 0 & 0 \\ 0 & H \end{bmatrix}^i \begin{bmatrix} I & 0 \\ 0 & H^\pi \end{bmatrix} + \sum_{i=0}^{s-2}(M^d)^{i+3} \begin{bmatrix} 0 & 0 \\ H^{i+1}H^\pi G & 0 \end{bmatrix}$$
$$- \begin{bmatrix} N_2H^dG & 0 \\ N_4H^dG & 0 \end{bmatrix} - (M^d)^2 \begin{bmatrix} 0 & 0 \\ HH^dG & 0 \end{bmatrix},$$

where N, M and M^d are given by (3)*,* (2) *and* (5)*, respectively,* $\operatorname{ind}(M) = r$ *and* $\operatorname{ind}(H) = s$.

Remark 21. *Theorem 20 generalizes the following results:*

1. $EGF = 0$, $HGF = 0$ and $HGE = 0$ (see [**24**, Theorem 3.1]*);*

2. $EGF = 0$, $EGE = 0$, $HGF = 0$ and $HGE = 0$ (see [**30**, Theorem 3.1].

Applying Lemma 5 with $MBM = 0$ and $B^2M = 0$, where M is as in (2), and $B = \begin{bmatrix} 0 & 0 \\ 0 & H \end{bmatrix}$, we get the next representation in Theorem 22.

Theorem 22. *For square matrices F, H, and EG such that F and EG are of the same size, let*

$$EGF^2 = 0, \quad (EG)^2F = 0, \quad (EGF)^2 = 0, \quad EHG = 0 \quad \text{and} \quad H^2G = 0.$$

Then

$$N^d = \begin{bmatrix} F^\pi - F^{3d}EGF - N_2G & -N_1E \\ -GF^d - GF^{4d}EGF - N_4G & I - N_3E \end{bmatrix} \sum_{i=0}^{r-1} M^i \begin{bmatrix} 0 & 0 \\ 0 & H^{(i+1)d} \end{bmatrix}$$
$$+ \sum_{i=0}^{s-1} M^{(i+1)d} \begin{bmatrix} 0 & 0 \\ 0 & H \end{bmatrix}^i \begin{bmatrix} I & 0 \\ 0 & H^\pi \end{bmatrix}$$
$$+ \sum_{i=0}^{s-1} \begin{bmatrix} 0 & 0 \\ 0 & H \end{bmatrix} M^{(i+2)d} \begin{bmatrix} 0 & 0 \\ 0 & H \end{bmatrix}^i \begin{bmatrix} I & 0 \\ 0 & H^\pi \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & HGN_2H^{2d} + HN_4H^d \end{bmatrix}$$
$$+ \sum_{i=0}^{r-2} \begin{bmatrix} 0 & 0 \\ -H(GF^d + GF^{4d}EGF + N_4G) & H(I - N_3E) \end{bmatrix} M^{i+1} \begin{bmatrix} 0 & 0 \\ 0 & H^{(i+3)d} \end{bmatrix},$$

where N, M and M^d are given by (3)*, and* (2) *and* (5)*, respectively,* $\operatorname{ind}(M) = r$ *and* $\operatorname{ind}(H) = s$.

Remark 23. *Theorem 20 and Theorem 22, each separately, extend the following result:*

1. $EGF = 0, \quad HG = 0$ *and* H *is nilpotent (see* [**10**, Theorem 4.5]*).*
Moreover, Theorem 14, Theorem 16, Theorem 20 and Theorem 22 separately generalize the next results:

1. $F = 0$ *and* $H = 0$ *(see* [**5**, Theorem 2.1];
2. $EGF = 0$, $HG = 0$ and $EH = 0$ (see [**10**, Theorem 4.4]);
3. $EG = 0$, $HG = 0$ and $EH = 0$ (see [**15**, Theorem 5.3]).

5. CONCLUSION

Inspired by the significant utility of the Drazin inverse of ATB matrices, explicit expressions are derived under new conditions, which are weaker than those already used in this topic. In particular, we firstly establish a novel representation for the Drazin inverse of the ATB matrix $\bar{M}$, defined by the block structure (1), under the constraints $EF^2 = (EF)^2 = E^2F = 0$. Using this result, we present the expression for the Drazin inverse of the ATB matrix M, represented in the form (2), under the new assumptions $EGF^2 = (EGF)^2 = (EG)^2F = 0$. Consequently, we obtain some well-known and new representations and characterizations of the Drazin inverse of ATB matrices. Applying our representations for the Drazin inverse of ATB matrices, we prove several explicit representations of the Drazin inverse on a block matrix N stated in (3), generalizing some of the existing results in this topic.

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