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EXTENSIONS OF THE ERDÖS-LAX POLYNOMIAL INEQUALITY

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In this paper, we establish several new inequalities in the sup-norm of the derivative and polar derivative of a polynomial on the unit disk in the plane. The obtained results take into account the placement (absolute value) of the zeros and the extremal coefficients of the polynomial. Besides several consequences, our results extend considerably some results of Milovanović, Mir, Breaz, Aziz, Ahmad and others.

1. INTRODUCTION

Let $P(z) := \sum_{v=0}^{n} a_v z^v$ be an algebraic polynomial of degree n and $P'(z)$ its derivative. A classical area of analysis is the study of extremal problems of the functions of a complex variable, particularly those involving polynomial inequalities. One fundamental result in this domain is due to Bernstein [**2**], which relates the size of the derivative and the polynomial for the sup-norm on the unit disk in the plane: namely, if $P(z)$ is a polynomial of degree n, then

$$\max_{|z|=1} |P'(z)| \leq n \max_{|z|=1} |P(z)|. \tag{1}$$

Equality holds in (1) if and only if $P(z)$ has all its zeros at the origin. The above inequality (1) was established by Bernstein in 1912 and has been the source

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of a substantial literature in polynomial approximations. It is easy to observe that the restriction on the zeros of $P(z)$ implies the improvement in (1). It turns out that tightening the upper bound of $|P'(z)|$ in (1) requires some control over the location of zeros of $P(z)$. Many variants and generalizations across time of this inequality were produced by introducing limits on the multiplicity of zero at $z = 0$, the modulus of largest root of $P(z)$, restrictions on coefficients etc. In fact, Erdös conjectured and later Lax [**6**] proved that, if $P(z) \neq 0$ in $|z| < 1$, then

$$\max_{|z|=1} |P'(z)| \leq \frac{n}{2} \max_{|z|=1} |P(z)|. \tag{2}$$

The inequality (2) is sharp and equality holds if $P(z)$ has all of its zeros on $|z| = 1$. As an extension of (2), Malik [**7**] proved that, if $P(z) \neq 0$ in $|z| < k,\ k \geq 1$, then

$$\max_{|z|=1} |P'(z)| \leq \frac{n}{1+k} \max_{|z|=1} |P(z)|. \tag{3}$$

The result is sharp and equality in (3) holds for $P(z) = (z+k)^n$. On the other hand, if $P(z) \neq 0$ in $|z| < k,\ k \leq 1$, the precise estimate of maximum $|P'(z)|$ on $|z| = 1$ does not seem to be easily obtainable and this problem is still open. However, some special cases in this direction have been considered by many people where some partial extensions of (2) are established. In 1980, it was shown by Govil [**4**] that if $P(z)$ is a polynomial of degree n and $P(z) \neq 0$ in $|z| < k,\ k \leq 1$, then

$$\max_{|z|=1} |P'(z)| \leq \frac{n}{1+k^n} \max_{|z|=1} |P(z)|, \tag{4}$$

provided $|P'(z)|$ and $|Q'(z)|$ attain maximum at the same point on $|z| = 1$, where $Q(z) = z^n \overline{P\left(\frac{1}{\bar{z}}\right)}$. Under the same hypothesis as in (4), Aziz and Ahmad [**1**] established an improved inequality in the following form:

$$\max_{|z|=1} |P'(z)| \leq \frac{n}{1+k^n} \left(\max_{|z|=1} |P(z)| - \min_{|z|=k} |P(z)| \right). \tag{5}$$

For a polynomial $P(z)$ of degree n, we define

$$D_\beta P(z) := nP(z) + (\beta - z)P'(z),$$

the polar derivative of $P(z)$ with respect to the point β. The polynomial $D_\beta P(z)$ is of degree at most $n-1$ and it generalizes the ordinary derivative in the sense that

$$\lim_{\beta \to \infty} \left\{ \frac{D_\beta P(z)}{\beta} \right\} = P'(z),$$

uniformly with respect to z for $|z| \leq R$, $R > 0$.

There is a considerable body of research in the literature about the polar derivative of a polynomial, and we refer the interested reader to the comprehensive books of

Gardner et al. [**3**], Marden [**8**], Milovanović et al. [**9**] and Rahman and Schmeisser [**20**]. These sources provide valuable insights into the various aspects of polar derivatives and their applications in polynomial inequalities. Recently, several works appeared in the literature concerning the estimates for the polar derivative of a polynomial on the unit disk. To name a few, we refer the interested reader to [**10**]-[**19**], where an extensive number of variants and generalizations of the aforementioned inequalities arise from various authors comparing the polar derivative $D_\beta P(z)$ with different selections of $P(z)$, β and other parameters.

In 2021, Mir and Breaz [**15**] extended (4) and (5) by replacing the derivative with the polar derivative of the polynomial. In fact, they proved the following results:

Theorem A. Let $P(z)$ be a polynomial of degree n having no zeros in $|z| < k,\ k \leq 1$, and let $Q(z) = z^n\overline{P\left(\frac{1}{\bar{z}}\right)}$. If $|P'(z)|$ and $|Q'(z)|$ attain maximum at the same point on $|z| = 1$, then for every complex number β with $|\beta| \geq 1$,

$$\max_{|z|=1}|D_\beta P(z)| \leq n\left(\frac{|\beta| + k^n}{1 + k^n}\right)\max_{|z|=1}|P(z)|. \tag{6}$$

Theorem B. Let $P(z)$ be a polynomial of degree n having no zeros in $|z| < k,\ k \leq 1$, and let $Q(z) = z^n\overline{P\left(\frac{1}{\bar{z}}\right)}$. If $|P'(z)|$ and $|Q'(z)|$ attain maximum at the same point on $|z| = 1$, then for every complex number β with $|\beta| \geq 1$,

$$\max_{|z|=1}|D_\beta P(z)| \leq \frac{n}{1 + k^n}\left((|\beta| + k^n)\max_{|z|=1}|P(z)| - (|\beta| - 1)\min_{|z|=k}|P(z)|\right). \tag{7}$$

Equality in (6) and (7) holds for $P(z) = z^n + k^n$, with real $\beta \geq 1$.
In the same paper, they further refined (6) and (7) by proving the following results:

Theorem C. Let $P(z)$ be a polynomial of degree n having no zeros in $|z| < k,\ k \leq 1$ and let $Q(z) = z^n\overline{P\left(\frac{1}{\bar{z}}\right)}$. If $|P'(z)|$ and $|Q'(z)|$ attain maximum at the same point on $|z| = 1$, then for every complex number β with $|\beta| \geq 1$,

$$\max_{|z|=1}|D_\beta P(z)| \leq n\left(\frac{|\beta| + k^{n-1}A_n(k)}{1 + k^{n-1}A_n(k)}\right)\max_{|z|=1}|P(z)|, \tag{8}$$

where

$$A_n(k) = \frac{|a_n|k^{n+1} + |a_0|}{|a_n|k^{n-1} + |a_0|}.$$

Theorem D. Let $P(z)$ be a polynomial of degree n having no zeros in $|z| < k,\ k \leq 1$ and let $Q(z) = z^n\overline{P\left(\frac{1}{\bar{z}}\right)}$. If $|P'(z)|$ and $|Q'(z)|$ attain maximum at the same point on $|z| = 1$, then for every complex number β with $|\beta| \geq 1$,

$$\begin{aligned}&\max_{|z|=1}|D_\beta P(z)|\\ &\leq \left(\frac{n\left(|\beta| + k^n A_n^*(k)\right)}{1 + k^n A_n^*(k)}\right)\max_{|z|=1}|P(z)| - \left(\frac{n\left(|\beta| - 1\right)A_n^*(k)}{1 + k^n A_n^*(k)}\right)\min_{|z|=k}|P(z)|,\end{aligned} \tag{9}$$

where

$$A_n^*(k) = \frac{|a_0| + |a_n|k^{n+1} + km_k}{|a_0|k + |a_n|k^n + m_k} \quad \text{and} \quad m_k = \min_{|z|=k} |P(z)|.$$

Equality in (8) and (9) holds for $P(z) = z^n + k^n$, with real $\beta \geq 1$.
Note: We obtain refinements of (4) and (5), respectively, by dividing both sides of (8) and (9) by $|\beta|$ and allowing $|\beta| \to \infty$. Milovanović and Mir (see, [**10**], [**11**]) also established these enhanced versions of (4) and (5).
The inequalities (6) and (7) were also refined by Mir and Ahmad [**14**], yielding the subsequent outcomes:
Theorem E. Let $P(z)$ be a polynomial of degree n having no zeros in $|z| < k,\ k \leq 1$, and let $Q(z) = z^n \overline{P\left(\frac{1}{\bar{z}}\right)}$. If $|P'(z)|$ and $|Q'(z)|$ attain maximum at the same point on $|z| = 1$, then for every complex number β with $|\beta| \geq 1$,

$$\max_{|z|=1} |D_\beta P(z)| \leq \left(\frac{n(|\beta| + k^n)}{1 + k^n} - \frac{k^n(|\beta| - 1)(|a_0| - k^n|a_n|)}{(1 + k^n)(|a_0| + k^n|a_n|)} \right) \max_{|z|=1} |P(z)|. \tag{10}$$

Theorem F. Let $P(z)$ be a polynomial of degree n having no zeros in $|z| < k,\ k \leq 1$, and let $Q(z) = z^n \overline{P\left(\frac{1}{\bar{z}}\right)}$. If $|P'(z)|$ and $|Q'(z)|$ attain maximum at the same point on $|z| = 1$, then for every complex number β with $|\beta| \geq 1$,

$$\begin{aligned} \max_{|z|=1} |D_\beta P(z)| \leq{}& \frac{n}{1 + k^n} \left((|\beta| + k^n) \max_{|z|=1} |P(z)| - (|\beta| - 1) \min_{|z|=k} |P(z)| \right) \\ & - \frac{k^n(|\beta| - 1)(|a_0| - k^n|a_n| - m_k)}{(1 + k^n)(|a_0| + k^n|a_n| - m_k)} \left(\max_{|z|=1} |P(z)| - \min_{|z|=k} |P(z)| \right), \end{aligned} \tag{11}$$

where $m_k = \min_{|z|=k} |P(z)|$.
Equality in (10) and (11) holds for $P(z) = z^n + k^n$, with real $\beta \geq 1$.
Recently, Mir et al. [**17**] further strengthened (10) and (11) in the following form:
Theorem G. Let $P(z)$ be a polynomial of degree n having no zeros in $|z| < k,\ k \leq 1$ and let $Q(z) = z^n \overline{P\left(\frac{1}{\bar{z}}\right)}$. If $|P'(z)|$ and $|Q'(z)|$ attain maximum at the same point on $|z| = 1$, then for every complex number β with $|\beta| \geq 1$ and $0 \leq \zeta \leq 1$,

$$\begin{aligned} |D_\beta P(z)| \leq{}& n|\beta| \max_{|z|=1} |P(z)| - \frac{nk^n(|\beta| - 1)}{1 + k^n} \\ & \times \left[\left(1 + \frac{1-k}{2k} E^*(\zeta) \right) \max_{|z|=1} |P(z)| + \left(\frac{1}{k^n} - \frac{1-k}{2k} E^*(\zeta) \right) \zeta \min_{|z|=k} |P(z)| \right] \\ & - (|\beta| - 1)\, F^*(\zeta) \left(1 + \frac{1-k}{2k} E^*(\zeta) \right) \left(\max_{|z|=1} |P(z)| - \zeta \min_{|z|=k} |P(z)| \right), \end{aligned} \tag{12}$$

where

$$E^*(\zeta) = \frac{|a_0| - k^n|a_n| - \zeta m_k}{|a_0| + k^{n-1}|a_n| - \zeta m_k}, \quad F^*(\zeta) = \frac{k^n(|a_0| - k^n|a_n| - \zeta m_k)}{(1 + k^n)(|a_0| + k^n|a_n| - \zeta m_k)},$$

and $m_k = \min_{|z|=k} P(z)|$.
Equality in (12) holds for $P(z) = z^n + k^n$, with real $\beta \geq 1$.
In this work, we further investigate these kinds of inequalities for a particular class of polynomials with a zero of order $s \geq 0$ at the origin and derive some new upper bounds for the derivative and polar derivative of a polynomial on the unit disk. We shall use the extremal coefficients and the location (absolute value) of the remaining $n - s$ zeros of the underlying polynomial in our findings.

2. MAIN RESULTS

In this section, we present our main results. Their proofs are provided in the next section. We commence by introducing the following generalization of (8) and some related results.

Theorem 1. Let $P(z) = z^s \left(\sum_{v=0}^{n-s} a_v z^v\right)$ be a polynomial of degree n having no zeros in $|z| < k$, $k \leq 1$, except a zero of order s, where $0 \leq s < n$, at the origin, and let $Q(z) = z^n \overline{P\left(\frac{1}{\bar{z}}\right)}$. If $|P'(z)|$ and $|Q'(z)|$ attain maximum at the same point on $|z| = 1$, then for every complex number β with $|\beta| \geq 1$,

$$\max_{|z|=1} |D_\beta P(z)| \leq \left(\frac{k^{n-s} E_k(s)(n-s) + |\beta| \left(n + s k^{n-s} E_k(s)\right)}{1 + k^{n-s} E_k(s)}\right) \max_{|z|=1} |P(z)|, \tag{13}$$

where

$$E_k(s) = \frac{|a_{n-s}| k^{n-s+1} + |a_0|}{|a_{n-s}| k^{n-s} + |a_0| k}.$$

Equality in (13) holds for $P(z) = z^s(z^{n-s} + k^{n-s})$, with real $\beta \geq 1$.

Remark 2. We get the inequality (8) from (13) by taking $s = 0$.

Remark 3. Let $P(z) = z^s H(z)$ where $H(z) = \sum_{v=0}^{n-s} a_v z^v$ is a polynomial of degree $n - s$ having no zeros in $|z| < k$, $k \leq 1$. If $z_1, z_2, ...z_{n-s}$, are the zeros of $H(z)$, then

$$\left|\frac{a_0}{a_{n-s}}\right| = |z_1 z_2 ... z_{n-s}| \geq k^{n-s}. \tag{14}$$

Using this it is easy to verify that

$$E_k(s) = \frac{|a_{n-s}| k^{n-s+1} + |a_0|}{|a_{n-s}| k^{n-s} + |a_0| k} \geq 1.$$

Also, the function

$$x \mapsto \frac{k^{n-s} x(n-s) + |\beta| \left(n + s k^{n-s} x\right)}{1 + k^{n-s} x}, \ (x \geq 0)$$

is non-increasing, so it follows that

$$\frac{k^{n-s}E_k(s)(n-s)+|\beta|\left(n+sk^{n-s}E_k(s)\right)}{1+k^{n-s}E_k(s)} \le \frac{k^{n-s}(n-s)+|\beta|\left(n+sk^{n-s}\right)}{1+k^{n-s}}.$$

Therefore, based on Theorem 1, we derive the subsequent result which gives the polar derivative generalization of (4) as a consequence.

Corollary 4. Let $P(z)=z^s\left(\sum_{v=0}^{n-s}a_v z^v\right)$ be a polynomial of degree n having no zeros in $|z|<k$, $k\le 1$, except a zero of order s, where $0\le s<n$, at the origin, and let $Q(z)=z^n\overline{P\left(\frac{1}{\bar z}\right)}$. If $|P'(z)|$ and $|Q'(z)|$ attain maximum at the same point on $|z|=1$, then for every complex number β with $|\beta|\ge 1$,

$$\max_{|z|=1}|D_\beta P(z)| \le \left(\frac{k^{n-s}(n-s)+|\beta|(n+sk^{n-s})}{1+k^{n-s}}\right)\max_{|z|=1}|P(z)|. \tag{15}$$

Equality in (15) holds for $P(z)=z^s(z^{n-s}+k^{n-s})$, with real $\beta\ge 1$.

Remark 5. If we divide both sides of (15) by $|\beta|$ and let $|\beta|\to\infty$, we obtain

$$\max_{|z|=1}|P'(z)| \le \left(\frac{n+sk^{n-s}}{1+k^{n-s}}\right)\max_{|z|=1}|P(z)|. \tag{16}$$

Equality in (16) holds for $P(z)=z^s(z^{n-s}+k^{n-s})$.

Remark 6. It is evident that (16) reduces to (4) when $s=0$. Once we divide both sides of (13) by $|\beta|$ and allow $|\beta|\to\infty$, we obtain the following:

$$\max_{|z|=1}|P'(z)| \le \left(\frac{n+sk^{n-s}E_k(s)}{1+k^{n-s}E_k(s)}\right)\max_{|z|=1}|P(z)|. \tag{17}$$

Equality in (17) holds for $P(z)=z^s(z^{n-s}+k^{n-s})$.
In fact, excepting the case when $P(z)$ has the form $z^s(z^{n-s}+k^{n-s})$, the bound obtained in (17) is always sharper than the bound obtained in (16) as $E_k(s)\ge 1$.

Next, we prove the following extension of Theorem 1. As a consequence, the obtained result provides the polar derivative generalization of (5) as well.

Theorem 7. Let $P(z)=z^s\left(\sum_{v=0}^{n-s}a_v z^v\right)$ be a polynomial of degree n having no zeros in $|z|<k$, $k\le 1$, except a zero of order s, where $0\le s<n$, at the origin, and let $Q(z)=z^n\overline{P\left(\frac{1}{\bar z}\right)}$. If $|P'(z)|$ and $|Q'(z)|$ attain maximum at the same point on $|z|=1$, then for every complex number β with $|\beta|\ge 1$,

$$\begin{aligned}\max_{|z|=1}|D_\beta P(z)| \le{}& \left(\frac{k^{n-s}F_k(s)\,(n-s)+|\beta|\left(n+sk^{n-s}F_k(s)\right)}{1+k^{n-s}F_k(s)}\right)\max_{|z|=1}|P(z)| \\ &-\left(\frac{(|\beta|-1)\,F_k(s)\,(n-s)}{1+k^{n-s}F_k(s)}\right)\min_{|z|=k}|P(z)|,\end{aligned} \tag{18}$$

where

$$F_k(s) = \frac{|a_0| + |a_{n-s}|k^{n-s+1} + km_k}{|a_0|k + |a_{n-s}|k^{n-s} + m_k},$$

and $m_k = \min_{|z|=k} |P(z)|$.
Equality in (18) holds for $P(z) = z^s(z^{n-s} + k^{n-s})$, with real $\beta \geq 1$.

Remark 8. Let $P(z) = z^s H(z)$ where $H(z) = \sum_{v=0}^{n-s} a_v z^v \neq 0$ in $|z| < k,\ k \leq 1$. We first show that

$$k^{n-s}|a_{n-s}| \leq |a_0| - m_k. \tag{19}$$

We can assume, without loss of generality, that $H(z)$ has no zeros on $|z| = k$. Otherwise, inequality (19) holds trivially by (14). Now, following the proof of Lemma 3 (provided in the next section), for every λ with $|\lambda| < 1$, we consider the polynomial

$$H(z) + \frac{\lambda m_k z^{n-s}}{k^{n-s}} = \left(a_{n-s} + \frac{\lambda m_k}{k^{n-s}}\right) z^{n-s} + \sum_{v=0}^{n-s-1} a_v z^v,$$

which does not vanish in $|z| < k,\ k \leq 1$, hence

$$\left| \frac{a_0}{a_{n-s} + \frac{\lambda m_k}{k^{n-s}}} \right| \geq k^{n-s}. \tag{20}$$

If, in (20), we appropriately select the argument of λ such that

$$\left| a_{n-s} + \frac{\lambda m_k}{k^{n-s}} \right| = |a_{n-s}| + \frac{|\lambda| m_k}{k^{n-s}},$$

we get

$$k^{n-s}|a_{n-s}| + |\lambda| m_k \leq |a_0|. \tag{21}$$

The inequality (19) follows by letting $|\lambda| \to 1$ in (21). By using the inequality (21), one can see that $F_k(s) \geq 1$ for every $k \leq 1$. Also, it is easy to verify that the function

$$x \mapsto \left(\frac{k^{n-s}x\,(n-s) + |\beta|\,(n + sk^{n-s}x)}{1 + k^{n-s}x} \right) \max_{|z|=1} |P(z)| - \left(\frac{(|\beta| - 1)\,x\,(n-s)}{1 + k^{n-s}x} \right) \min_{|z|=k} |P(z)|,\ (x \geq 0)$$

is non-increasing. Therefore, from Theorem 7, we obtain the following result which as a special case gives the polar derivative generalization of (5).

Corollary 9. Let $P(z) = z^s \left(\sum_{v=0}^{n-s} a_v z^v\right)$ be a polynomial of degree n having no zeros in $|z| < k$, $k \leq 1$, except a zero of order s, where $0 \leq s < n$, at the origin, and let $Q(z) = z^n \overline{P\left(\frac{1}{\bar{z}}\right)}$. If $|P'(z)|$ and $|Q'(z)|$ attain maximum at the same point on $|z| = 1$, then for every complex number β with $|\beta| \geq 1$,

$$\begin{aligned}\max_{|z|=1} |D_\beta P(z)| \leq & \left(\frac{k^{n-s}(n-s) + |\beta|(n + sk^{n-s})}{1 + k^{n-s}}\right) \max_{|z|=1} |P(z)| \\ & - \left(\frac{(|\beta| - 1)(n-s)}{1 + k^{n-s}}\right) \min_{|z|=k} |P(z)|. \end{aligned} \tag{22}$$

Equality in (22) holds for $P(z) = z^s(z^{n-s} + k^{n-s})$, with real $\beta \geq 1$.

Remark 10. If we take $s = 0$ in (22), we get the polar derivative generalization of (5). Dividing both sides of (18) by $|\beta|$ and let $|\beta| \to \infty$, we obtain the following generalization of (17):

$$\max_{|z|=1} |P'(z)| \leq \left(\frac{n + sk^{n-s}F_k(s)}{1 + k^{n-s}F_k(s)}\right) \max_{|z|=1} |P(z)| - \left(\frac{(n-s)F_k(s)}{1 + k^{n-s}F_k(s)}\right) \min_{|z|=k} |P(z)|. \tag{23}$$

Equality in (23) holds for $P(z) = z^s(z^{n-s} + k^{n-s})$, with real $\beta \geq 1$.

Finally, we shall prove the following result, which generalizes the Theorem G and several other related results.

Theorem 11. Let $P(z) = z^s \left(\sum_{v=0}^{n-s} a_v z^v\right)$ be a polynomial of degree n having no zeros in $|z| < k$, $k \leq 1$, except a zero of order s, where $0 \leq s < n$, at the origin, and let $Q(z) = z^n \overline{P\left(\frac{1}{\bar{z}}\right)}$. If $|P'(z)|$ and $|Q'(z)|$ attain maximum at the same point on $|z| = 1$, then for every complex number β with $|\beta| \geq 1$ and $0 \leq \zeta \leq 1$,

$$\begin{aligned}\max_{|z|=1} |D_\beta P(z)| \leq & \; n|\beta| \max_{|z|=1} |P(z)| - \frac{(n-s)k^{n-s}(|\beta| - 1)}{1 + k^{n-s}} \\ & \times \left[\left(1 + \frac{1-k}{2k} E^*(\zeta, s)\right) \max_{|z|=1} |P(z)| + \left(\frac{1}{k^n} - \frac{1-k}{2k^{s+1}} E^*(\zeta, s)\right) \zeta m_k\right] \\ & - (|\beta| - 1) F^*(\zeta, s) \left(1 + \frac{1-k}{2k} E^*(\zeta, s)\right) \left(\max_{|z|=1} |P(z)| - \frac{1}{k^s} \zeta m_k\right), \end{aligned} \tag{24}$$

where

$$E^*(\zeta, s) = \frac{k^s|a_0| - k^n|a_{n-s}| - \zeta m_k}{k^s|a_0| + k^{n-1}|a_{n-s}| - \zeta m_k},$$

$$F^*(\zeta, s) = \frac{k^{n-s}(k^s|a_0| - k^n|a_{n-s}| - \zeta m_k)}{(1 + k^{n-s})(k^s|a_0| + k^n|a_{n-s}| - \zeta m_k)}$$

and $m_k = \min_{|z|=k} |P(z)|$.
Equality in (24) holds for $P(z) = z^n + k^n$, with real $\beta \geq 1$.

Remark 12. We get the inequality (12) from (24) by taking $s = 0$. If we take $\zeta = 0$, we get the following result:

Corollary 13. Let $P(z) = z^s \left(\sum_{v=0}^{n-s} a_v z^v\right)$ be a polynomial of degree n having no zeros in $|z| < k$, $k \leq 1$, except a zero of order s, where $0 \leq s < n$, at the origin, and let $Q(z) = z^n \overline{P\left(\frac{1}{\bar{z}}\right)}$. If $|P'(z)|$ and $|Q'(z)|$ attain maximum at the same point on $|z| = 1$, then for every complex number β with $|\beta| \geq 1$,

$$\begin{aligned}
\max_{|z|=1} |D_\beta P(z)| \leq \Bigg[&\frac{n(|\beta| + k^{n-s}) + sk^{n-s}(|\beta| - 1)}{1 + k^{n-s}} \\
&- \frac{(n-s)(|\beta|-1)k^{n-s}(k^s|a_0| - k^n|a_{n-s}|)(1-k)}{2(1+k^{n-s})(|a_0|k^{s+1} + k^n|a_{n-s}|)}\Bigg] \max_{|z|=1} |P(z)| \\
- \frac{k^{n-s}(|\beta|-1)(k^s|a_0| - k^n|a_{n-s}|)}{(1+k^{n-s})(k^s|a_0| + k^n|a_{n-s}|)} &\left[1 + \frac{(k^s|a_0| - k^n|a_{n-s}|)(1-k)}{2(k^{s+1}|a_0| + k^n|a_{n-s}|)}\right] \max_{|z|=1} |P(z)|.
\end{aligned} \tag{25}$$

Equality in (25) holds for $P(z) = z^s(z^{n-s} + k^{n-s})$, with real $\beta \geq 1$.

Dividing both sides of (24) and (25) by $|\beta|$ and let $|\beta| \to \infty$, we get the following two results:

Corollary 14. Let $P(z) = z^s \left(\sum_{v=0}^{n-s} a_v z^v\right)$ be a polynomial of degree n having no zeros in $|z| < k$, $k \leq 1$, except a zero of order s, where $0 \leq s < n$, at the origin, and let $Q(z) = z^n \overline{P\left(\frac{1}{\bar{z}}\right)}$. If $|P'(z)|$ and $|Q'(z)|$ attain maximum at the same point on $|z| = 1$, then for $0 \leq \zeta \leq 1$,

$$\begin{aligned}
\max_{|z|=1} |P'(z)| \leq n \max_{|z|=1} |P(z)| - &\frac{(n-s)k^{n-s}}{1+k^{n-s}} \\
&\times \left[\left(1 + \frac{1-k}{2k} E^*(\zeta, s)\right) \max_{|z|=1} |P(z)| + \left(\frac{1}{k^n} - \frac{1-k}{2k^{s+1}} E^*(\zeta, s)\right)\zeta m_k\right] \\
&- F^*(\zeta, s)\left(1 + \frac{1-k}{2k} E^*(\zeta, s)\right)\left(\max_{|z|=1} |P(z)| - \frac{1}{k^s}\zeta m_k\right),
\end{aligned} \tag{26}$$

where $E_k^*(\zeta, s)$, $F_k^*(\zeta, s)$ and m_k are as defined in Theorem 11.
Equality in (26) holds for $P(z) = z^s(z^{n-s} + k^{n-s})$.

Corollary 15. Let $P(z) = z^s \left(\sum_{v=0}^{n-s} a_v z^v\right)$ be a polynomial of degree n having no zeros in $|z| < k$, $k \leq 1$, except a zero of order s, where $0 \leq s < n$, at the origin, and let $Q(z) = z^n\overline{P\left(\frac{1}{\bar{z}}\right)}$. If $|P'(z)|$ and $|Q'(z)|$ attain maximum at the same point on $|z| = 1$, then for $0 \leq \zeta \leq 1$,

$$\max_{|z|=1}|P'(z)| \leq \left[\frac{n + sk^{n-s}}{1 + k^{n-s}} - \frac{(n-s)k^{n-s}(k^s|a_0| - k^n|a_{n-s}|)(1-k)}{2(1+k^{n-s})(|a_0|k^{s+1} + k^n|a_{n-s}|)}\right]\max_{|z|=1}|P(z)|$$

$$- \frac{k^{n-s}(|\beta|-1)(k^s|a_0| - k^n|a_{n-s}|)}{(1+k^{n-s})(k^s|a_0| + k^n|a_{n-s}|)}\left[1 + \frac{(k^s|a_0| - k^n|a_{n-s}|)(1-k)}{2(k^{s+1}|a_0| + k^n|a_{n-s}|)}\right]\max_{|z|=1}|P(z)|. \tag{27}$$

Equality in (27) holds for $P(z) = z^s(z^{n-s} + k^{n-s})$, with real $\beta \geq 1$.

3. LEMMAS

We require the following lemmas for the proofs of our results. The first lemma is due to Mir and Ahmad [**16**].

Lemma 1. If $P(z) = \sum_{v=0}^{n} a_v z^v$ is a polynomial of degree n having no zeros in $|z| < k$, $k \leq 1$, and $Q(z) = z^n\overline{P(\frac{1}{\bar{z}})}$, then

$$k^n A_n(k)\max_{|z|=1}|P'(z)| \leq \max_{|z|=1}|Q'(z)|,$$

where

$$A_n(k) = \frac{|a_n|k^{n+1} + |a_0|}{|a_n|k^n + |a_0|k}.$$

Lemma 2. Let $P(z) = z^s H(z)$, where $H(z) = \sum_{v=0}^{n-s} a_v z^v$ is a polynomial of degree $n - s$ having no zeros in $|z| < k$, $k \leq 1$. If $Q(z) = z^n\overline{P(\frac{1}{\bar{z}})}$, then for each point z on $|z| = 1$, we have

$$k^{n-s}E_k(s)(|P'(z)| - s|H(z)|) \leq \max_{|z|=1}|Q'(z)|,$$

where $E_k(s)$ is as defined in Theorem 1.

Proof. Recall that $P(z) = z^s H(z)$, where $H(z) = \sum_{v=0}^{n-s} a_v z^v$ is a polynomial of degree $n - s$ having no zeros in $|z| < k$, $k \leq 1$. Also, $Q(z) = z^n\overline{P\left(\frac{1}{\bar{z}}\right)} = z^{n-s}\overline{H\left(\frac{1}{\bar{z}}\right)}$.
On applying Lemma 1 to $H(z)$, we get

$$k^{n-s}E_k(s)\max_{|z|=1}|H'(z)| \leq \max_{|z|=1}|Q'(z)|. \tag{28}$$

Since $P(z)=z^sH(z)$ which gives for $|z|=1$, that

$$|P'(z)|\le s|H(z)|+|H'(z)|.$$

This implies for $|z|=1$, that

$$|P'(z)|-s|H(z)|\le |H'(z)|\le \max_{|z|=1}|H'(z)|. \tag{29}$$

Inequality (29) in conjunction with inequality (28) gives for $|z|=1$,

$$k^{n-s}E_k(s)(|P'(z)|-s|H(z)|)\le \max_{|z|=1}|Q'(z)|.$$

Hence, Lemma 2 is proved. □

Lemma 3. Let $P(z)=z^sH(z)$ where, $H(z)=\sum_{v=0}^{n-s}a_vz^v$ is a polynomial of degree $n-s$ having no zeros in $|z|<k$, $k\le 1$. If $Q(z)=z^n\overline{P(\frac{1}{\bar z})}$, then for each point z on $|z|=1$, we have

$$k^{n-s}F_k(s)(|P'(z)|-s|H(z)|)+m_k(n-s)F_k(s)\le \max_{|z|=1}|Q'(z)|,$$

where $F_k(s)$ and m_k are as defined in Theorem 7.

Proof. By hypothesis $H(z)$ has all its zeros in $|z|\ge k$, $k\le 1$. If $H(z)$ has a zero on $|z|=k$, then $P(z)$ has a zero on $|z|=k$, and hence

$$m_k=\min_{|z|=k}|P(z)|=0,$$

and Lemma 3 follows from Lemma 2 in this case. Henceforth, we assume that $H(z)$ does not vanish on $|z|=k$, $k\le 1$. It follows by Rouché's theorem that for each λ with $|\lambda|<1$, the polynomials $F(z)=H(z)+\frac{z^{n-s}\lambda m_k}{k^{n-s}}$, and $H(z)$ have same number of zeros in $|z|<k$, $k\le 1$.
Let $G(z)=z^{n-s}\overline{F\left(\frac{1}{\bar z}\right)}=z^{n-s}\overline{H\left(\frac{1}{\bar z}\right)}+\bar\lambda\frac{m_k}{k^{n-s}}=Q(z)+\bar\lambda\frac{m_k}{k^{n-s}}$.
On applying inequality (28) to the polynomial $F(z)$, we get for every $|\lambda|<1$,

$$k^{n-s}\left(\frac{|a_0|+|a_{n-s}+\frac{\lambda m}{k^{n-s}}|k^{n-s+1}}{|a_0|k+|a_{n-s}+\frac{\lambda m}{k^{n-s}}|k^{n-s}}\right)\max_{|z|=1}|F'(z)|\le \max_{|z|=1}|G'(z)|,$$

which is equivalent to

$$k^{n-s}\left(\frac{|a_0|+|a_{n-s}+\frac{\lambda m_k}{k^{n-s}}|k^{n-s+1}}{|a_0|k+|a_{n-s}+\frac{\lambda m_k}{k^{n-s}}|k^{n-s}}\right)\max_{|z|=1}\left|H'(z)+\frac{\lambda m_k(n-s)z^{n-s-1}}{k^{n-s}}\right| \le \max_{|z|=1}|Q'(z)|, \tag{30}$$

for $|\lambda| < 1$.
Also, for every $\lambda \in \mathbb{C}$, we have

$$\left| a_{n-s} + \frac{\lambda m}{k^{n-s}} \right| \leq |a_{n-s}| + \frac{|\lambda| m}{k^{n-s}},$$

and since the function

$$x \mapsto \frac{|a_0| + k^{n-s+1}x}{|a_0| + k^{n-s}x}, \ (x \geq 0)$$

is non-increasing for $k \leq 1$, it follows from (30) that for every $|\lambda| < 1$,

$$k^{n-s} \left(\frac{|a_0| + \left(|a_{n-s}| + \frac{|\lambda| m_k}{k^{n-s}}\right) k^{n-s+1}}{|a_0|k + \left(|a_{n-s}| + \frac{|\lambda| m_k}{k^{n-s}}\right) k^{n-s}} \right) \max_{|z|=1} \left| H'(z) + \frac{\lambda m_k (n-s) z^{n-s-1}}{k^{n-s}} \right| \leq \max_{|z|=1} |Q'(z)|. \tag{31}$$

Choosing the argument of λ on the left hand side of (31) suitably, we get

$$k^{n-s} \left(\frac{|a_0| + |a_{n-s}|k^{n-s+1} + |\lambda| k m_k}{|a_0|k + |a_{n-s}|k^{n-s} + |\lambda| m_k} \right) \left(\max_{|z|=1} |H'(z)| + \frac{|\lambda| m_k (n-s)}{k^{n-s}} \right) \leq \max_{|z|=1} |Q'(z)|. \tag{32}$$

Finally, letting $|\lambda| \to 1$ in (32) and using (29), we get for $|z| = 1$,

$$k^{n-s} \left(\frac{|a_0| + |a_{n-s}|k^{n-s+1} + k m_k}{|a_0|k + |a_{n-s}|k^{n-s} + m_k} \right) \left(|P'(z)| - s|H(z)| + \frac{m_k (n-s)}{k^{n-s}} \right) \leq \max_{|z|=1} |Q'(z)|.$$

Hence, Lemma 3 is proved. □

Lemma 4. If $P(z)$ is a polynomial of degree n and $Q(z) = z^n \overline{P\left(\frac{1}{\bar{z}}\right)}$, then for $|z| = 1$,

$$|P'(z)| + |Q'(z)| \leq n \max_{|z|=1} |P(z)|.$$

The above lemma is a special case of a result due to Govil and Rahman [**5**].

Lemma 5. If $P(z) = \sum_{v=0}^{n} a_v z^v$ is a polynomial of degree n having all zeros in

$|z| \leq k,\ k \geq 1$, then for $0 \leq \zeta \leq 1$,

$$\max_{|z|=1} |P'(z)| \geq \frac{n}{1+k^n}\left[\left(1+\frac{k-1}{2}E_k(\zeta)\right)\max_{|z|=1}|P(z)| + \left(1-\frac{k-1}{2k^n}E_k(\zeta)\right)\zeta m_k\right] + F_k(\zeta)\left(1+\frac{k-1}{2}E_k(\zeta)\right)\left(\max_{|z|=1}|P(z)| - \frac{1}{k^n}\zeta m_k\right). \tag{33}$$

where

$$E_k(\zeta) = \frac{k^n|a_n| - |a_0| - \zeta m_k}{k^n|a_n| + k|a_0| - \zeta m_k},$$

$$F_k(\zeta) = \frac{k^n|a_n| - |a_0| - \zeta m_k}{(1+k^n)(k^n|a_n| + |a_0| - \zeta m_k)}$$

and $\quad m_k = \min_{|z|=k} |P(z)|$.

The above lemma is a special case of a result due to Mir et al. ([**17**], Lemma 5).

4. PROOFS OF THE MAIN RESULTS

Proof of Theorem 1. Recall that $P(z) = z^s\left(\sum_{v=0}^{n-s} a_v z^v\right)$ is a polynomial of degree n having no zeros in $|z| < k,\ k \leq 1$, except a zero of order $s,\ 0 \leq s < n$, at the origin. On using Lemma 2 and noting that $H(z) = \sum_{v=0}^{n-s} a_v z^v$, we have for $|z| = 1$,

$$k^{n-s}E_k(s)\left(|P'(z)| - s|H(z)|\right) \leq \max_{|z|=1}|Q'(z)|, \tag{34}$$

where $Q(z) = z^n\overline{P(\frac{1}{\bar{z}})}$ and $E_k(s)$ is as defined in Theorem 1. By the given hypothesis, $|P'(z)|$ and $|Q'(z)|$ attain maximum at the same point on $|z| = 1$, we have

$$\max_{|z|=1}(|P'(z)| + |Q'(z)|) = \max_{|z|=1}|P'(z)| + \max_{|z|=1}|Q'(z)|. \tag{35}$$

Therefore, by Lemma 4, we get

$$\max_{|z|=1}|P'(z)| + \max_{|z|=1}|Q'(z)| \leq n\max_{|z|=1}|P(z)|. \tag{36}$$

This gives for $|z| = 1$ with the help of (34) and (36), that

$$|P'(z)| + k^{n-s}E_k(s)(|P'(z)| - s|H(z)|) \leq n \max_{|z|=1} |P(z)|,$$

which implies

$$|P'(z)| \left(1 + k^{n-s}E_k(s)\right) \leq n \max_{|z|=1} |P(z)| + sk^{n-s}E_k(s)|H(z)|. \tag{37}$$

Since $P(z) = z^s H(z)$, then for $|z| = 1$, we have

$$\max_{|z|=1} |P(z)| = \max_{|z|=1} |H(z)|. \tag{38}$$

Using (38) in (37), we get

$$\max_{|z|=1} |P'(z)| \leq \frac{n + sk^{n-s}E_k(s)}{1 + k^{n-s}E_k(s)} \max_{|z|=1} |P(z)|. \tag{39}$$

Using the definition of polar derivative of polynomial $P(z)$ with respect to the complex number β, we have for $|z| = 1$,

$$\begin{aligned}
|D_\beta P(z)| &\leq |nP(z) - zP'(z)| + |\beta||P'(z)| \\
&= |Q'(z)| + |\beta||P'(z)| \\
&= |Q'(z)| + |P'(z)| - |P'(z)| + |\beta||P'(z)| \\
&\leq n \max_{|z|=1} |P(z)| + (|\beta| - 1)|P'(z)| \qquad \text{(by Lemma 4)} \\
&\leq n \max_{|z|=1} |P(z)| + (|\beta| - 1) \max_{|z|=1} |P'(z)|. \qquad (40)
\end{aligned}$$

Inequality (40) in conjunction with inequality (39), gives

$$\max_{|z|=1} |D_\beta P(z)| \leq \left(\frac{k^{n-s}E_k(s)(n-s) + |\beta| \left(n + sk^{n-s}E_k(s)\right)}{1 + k^{n-s}E_k(s)} \right) \max_{|z|=1} |P(z)|.$$

This completes the proof of Theorem 1.

Proof of Theorem 7. Since $P(z) = z^s \left(\sum_{v=0}^{n-s} a_v z^v \right)$ is a polynomial of degree n having no zeros in $|z| < k$, $k \leq 1$, except a zero of order s, $0 \leq s < n$, at the origin. By Lemma 3 and noting that $H(z) = \sum_{v=0}^{n-s} a_v z^v$, we have for $|z| = 1$,

$$k^{n-s}F_k(s) \left(|P'(z)| - s|H(z)|\right) + m_k(n-s)F_k(s) \leq \max_{|z|=1} |Q'(z)|, \tag{41}$$

where $Q(z) = z^n \overline{P(\frac{1}{\bar{z}})}$. By the given hypothesis, $|P'(z)|$ and $|Q'(z)|$ attain maximum at the same point on $|z| = 1$, we get with the help of (35), (36) and (41) for $|z| = 1$, that

$$|P'(z)| + k^{n-s}F_k(s) \left(|P'(z)| - s|H(z)|\right) + m_k(n-s)F_k(s) \leq n \max_{|z|=1} |P(z)|,$$

which implies

$$|P'(z)|\left(1+k^{n-s}F_k(s)\right) \le n\max_{|z|=1}|P(z)| + \left(sk^{n-s}|H(z)| - m_k(n-s)\right)F_k(s), \tag{42}$$

for $|z|=1$. The above inequality (42) gives with the help of (38), that

$$\max_{|z|=1}|P'(z)| \le \frac{1}{1+k^{n-s}F_k(s)}\left((n+sk^{n-s}F_k(s))\max_{|z|=1}|P(z)| - m_k(n-s)F_k(s)\right). \tag{43}$$

Using the definition of polar derivative, we have from (40) that for every β with $|\beta|\ge 1$ and for $|z|=1$,

$$|D_\beta P(z)| \le n\max_{|z|=1}|P(z)| + (|\beta|-1)\max_{|z|=1}|P'(z)|. \tag{44}$$

Inequality (44) in conjunction with (43), gives for $|\beta|\ge 1$ and $|z|=1$, that

$$|D_\beta P(z)| \le \left(\frac{k^{n-s}F_k(s)\,(n-s) + |\beta|\,(n+sk^{n-s}F_k(s))}{1+k^{n-s}F_k(s)}\right)\max_{|z|=1}|P(z)| - \left(\frac{(|\beta|-1)\,F_k(s)\,(n-s)}{1+k^{n-s}F_k(s)}\right)\min_{|z|=k}|P(z)|,$$

which is equivalent to (18) and this completes the proof of Theorem 7.

Proof of Theorem 11. By hypothesis, we have $P(z)=z^sH(z)$, where $H(z)=\sum_{v=0}^{n-s}a_vz^v$ and $H(z)\ne 0$ in $|z|<k, k\le 1$. If $Q(z)=z^n\overline{P\left(\frac{1}{\bar z}\right)}$, then $Q(z)$ is a polynomial of degree $n-s$ having all its zeros in $|z|\le \frac{1}{k},\ \frac{1}{k}\ge 1$. On applying Lemma 5 to $Q(z)$, we get for $0\le\zeta\le 1$,

$$\begin{aligned}\max_{|z|=1}|Q'(z)| &\ge \frac{n-s}{1+\frac{1}{k^{n-s}}}\left[\left(1+\frac{\frac{1}{k}-1}{2}E_{\frac{1}{k}}(\zeta,s)\right)\max_{|z|=1}|Q(z)| + \left(1-\frac{(\frac{1}{k}-1)}{\frac{2}{k^{n-s}}}E_{\frac{1}{k}}(\zeta,s)\right)\zeta m_k^*\right]\\ &\quad + F_{\frac{1}{k}}(\zeta,s)\left(1+\frac{\frac{1}{k}-1}{2}E_{\frac{1}{k}}(\zeta,s)\right)\left(\max_{|z|=1}|Q(z)| - k^{n-s}\zeta m_k^*\right).\end{aligned} \tag{45}$$

Now

$$m_k^* = \min_{|z|=\frac{1}{k}}|Q(z)| = \min_{|z|=\frac{1}{k}}\left|z^n\overline{P\left(\frac{1}{\bar z}\right)}\right| = \frac{1}{k^n}\min_{|z|=k}|P(z)| = \frac{m_k}{k^n},$$

$$E_{\frac{1}{k}}(\zeta) = \frac{\frac{1}{k^{n-s}}|a_0| - |a_{n-s}| - \zeta m_k^*}{\frac{1}{k^{n-s}}|a_0| + \frac{1}{k}|a_{n-s}| - \zeta m_k^*} = \frac{k^s|a_0| - k^n|a_{n-s}| - \zeta m_k}{k^s|a_0| + k^{n-1}|a_{n-s}| - \zeta m_k} = E^*(\zeta,s)$$

and

$$\begin{aligned} F_{\frac{1}{k}}(\zeta) &= \frac{\frac{1}{k^{n-s}}|a_0| - |a_{n-s}| - \zeta m_k^*}{(1+\frac{1}{k^{n-s}})(\frac{1}{k^{n-s}}|a_0| + |a_{n-s}| - \zeta m_k^*)} \\ &= \frac{k^{n-s}(k^s|a_0| - k^n|a_{n-s}| - \zeta m_k)}{(1+k^{n-s})(k^s|a_0| + k^n|a_{n-s}| - \zeta m_k)} = F^*(\zeta, s). \end{aligned}$$

Using these observations and the fact that $\max_{|z|=1}|P(z)| = \max_{|z|=1}|Q(z)|$ in (45), we get

$$\begin{aligned} &\max_{|z|=1}|Q'(z)| \\ &\geq \frac{(n-s)k^{n-s}}{1+k^{n-s}}\left[\left(1+\frac{1-k}{2k}E^*(\zeta,s)\right)\max_{|z|=1}|P(z)| + \left(\frac{1}{k^n} - \frac{1-k}{2k^{s+1}}E^*(\zeta,s)\right)\zeta m_k\right] \\ &\quad + F^*(\zeta,s)\left(1+\frac{1-k}{2k}E^*(\zeta,s)\right)\left(\max_{|z|=1}|P(z)| - \frac{1}{k^s}\zeta m_k\right). \end{aligned} \tag{46}$$

Since $|P'(z)|$ and $|Q'(z)|$ attain maximum at the same point on $|z| = 1$, we have

$$\max_{|z|=1}(|P'(z)| + |Q'(z)|) = \max_{|z|=1}|P'(z)| + \max_{|z|=1}|Q'(z)|. \tag{47}$$

On combining (46), (47) and Lemma 4, we get

$$\begin{aligned} &n\max_{|z|=1}|P(z)| \geq \max_{|z|=1}|P'(z)| \\ &+ \frac{(n-s)k^{n-s}}{1+k^{n-s}}\left[\left(1+\frac{1-k}{2k}E^*(\zeta,s)\right)\max_{|z|=1}|P(z)| + \left(\frac{1}{k^n} - \frac{1-k}{2k^{s+1}}E^*(\zeta,s)\right)\zeta m_k\right] \\ &+ F^*(\zeta,s)\left(1+\frac{1-k}{2k}E^*(\zeta,s)\right)\left(\max_{|z|=1}|P(z)| - \frac{1}{k^s}\zeta m_k\right), \end{aligned}$$

which gives

$$\begin{aligned} &\max_{|z|=1}|P'(z)| \leq n\max_{|z|=1}|P(z)| \\ &- \frac{(n-s)k^{n-s}}{1+k^{n-s}}\left[\left(1+\frac{1-k}{2k}E^*(\zeta,s)\right)\max_{|z|=1}|P(z)| + \left(\frac{1}{k^n} - \frac{1-k}{2k^{s+1}}E^*(\zeta,s)\right)\zeta m_k\right] \\ &\quad - F^*(\zeta,s)\left(1+\frac{1-k}{2k}E^*(\zeta,s)\right)\left(\max_{|z|=1}|P(z)| - \frac{1}{k^s}\zeta m_k\right). \end{aligned} \tag{48}$$

Also, it is easy to verify that for $|z| = 1$,

$$|Q'(z)| = |nP(z) - zP'(z)|. \tag{49}$$

Note that for any complex number β, and $|z|=1$, we have

$$\begin{aligned}|D_\beta P(z)| &= |nP(z)-(\beta-z)P'(z)|\\ &\le |nP(z)-zP'(z)|+|\beta||P'(z)|,\end{aligned}$$

which gives on using (49) and $|\beta|\ge 1$, that

$$\begin{aligned}|D_\beta P(z)| &\le |Q'(z)|+|\beta||P'(z)|\\ &= |Q'(z)|+|P'(z)|-|P'(z)|+|\beta||P'(z)|\\ &\le n\max_{|z|=1}|P(z)|+(|\beta|-1)|P'(z)| \quad \text{(by Lemma 4)}\\ &\le n\max_{|z|=1}|P(z)|+(|\beta|-1)\max_{|z|=1}|P'(z)|. \qquad (50)\end{aligned}$$

Inequality (50) in conjunction with (48) gives for $|z|=1$,

$$\begin{aligned}|D_\beta P(z)| \le n|\beta|\max_{|z|=1}|P(z)| &- \frac{(n-s)k^{n-s}(|\beta|-1)}{1+k^{n-s}}\\ &\times\left[\left(1+\frac{1-k}{2k}E^*(\zeta,s)\right)\max_{|z|=1}|P(z)|+\left(\frac{1}{k^n}-\frac{1-k}{2k^{s+1}}E^*(\zeta,s)\right)\zeta m_k\right]\\ &-(|\beta|-1)\,F^*(\zeta,s)\left(1+\frac{1-k}{2k}E^*(\zeta,s)\right)\left(\max_{|z|=1}|P(z)|-\frac{1}{k^s}\zeta m_k\right),\end{aligned}$$

which is equivalent to (24) and this completes the proof of Theorem 11.

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