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BY ANALYSIS OF MOMENTS OF GEOMETRIC DISTRIBUTION: NEW FORMULAS INVOLVING EULERIAN AND FUBINI NUMBERS

Buket Simsek and Neslihan Kilar*

The aim of this paper is to find new moment formulas for geometric distribution by using moments of the geometric distribution in terms of Apostol-type and Bernstein polynomials, and Fubini and Eulerian numbers, etc. A new generating function for moments of the geometric distribution is constructed. New sequences of special numbers with their recurrence relations are given. In order to compute values of these sequences and moments, codes in Wolfram language are given.

1. INTRODUCTION

Moments are specific quantitative measurements of the graph of a probability distribution function. If the function is representative of mass density, it is well known that the zeroth moment corresponds to the total mass. The first moment corresponds to the center of mass. The second moment is known as the moment of inertia. Additionally, the first moment is also called the expected value. Variance is defined by the first and second moments. The third standardized moment is known as skewness, and the fourth standardized moment is known as kurtosis. Moments have inspired the work of many researchers. Probability theory was one of the scientific developments that reached their peak especially in the 1700s. For

*Corresponding author. Buket Simsek
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example, with the development of applied statistics, the theory of errors and the method of least squares, probability and its applications to the statistical theory of thermodynamics, applications to genetics, etc. developed rapidly after 1700. Considering the historical developments, it is known that by the middle of the nineteenth century, Russian mathematician Pafnuty Lvovich Chebyshev (16 May 1821-8 December 1894) was the first researcher to conduct systematic research in terms of moments of random variables. As a result of Chebyshev's studies and experimental studies, moment calculation problems in the theory of probability distribution functions have many important applications (*cf.* [**2**], [**3**], [**14**]). Similarly, generating functions and moment generating functions have a variety of applications in many areas, ranging from probability and statistics, many branches of mathematics, engineering, and computer science. Likewise, characteristic functions associated with moment generating functions have also many applications in mathematics, especially in probability theory, analysis, combinatorics, and other applied sciences. Recently, by using moment generating function and characteristic function of the geometric distribution, Simsek [**19**] gave new formulas for moments of the geometric distribution in terms of the Apostol-Bernoulli polynomials and numbers. The motivation of this paper is to construct a new generating function for the moment of the geometric distribution. By using this generating function and its derivative and functional equations, we give new formulas and recurrence relations for the moments involving the Apostol-Bernoulli numbers and polynomials, the Bernstein polynomials, the Apostol-Euler numbers and polynomials, the Fubini numbers, and the Eulerian numbers and polynomials. These new results are related to special numbers with their generating functions, which are used in combinatorics to solve problems involving partitions, permutations, combinations, and recurrence relations. For example, the Eulerian numbers and the Fubini numbers are used to enumerate many different combinatorial situations.

The following notations, relations, and formulas associated with the results of this paper are inspired by the references (see [**1**]-[**25**]).

Let

$$\mathbb{N} = \{1, 2, 3, \ldots\}, \qquad \mathbb{N}_0 := \mathbb{N} \cup \{0\},$$

and also $\mathbb{Z}$, $\mathbb{R}$, $\mathbb{C}$ denote the set of integers, the set of real numbers, the set of complex numbers, respectively. Moreover, we assume that $\exp(t) := e^t$, $i^2 = -1$,

$$0^r = \begin{cases} 1, & r = 0 \\ 0, & r \in \mathbb{N} \end{cases}$$

and

$$(y)_r = \prod_{s=0}^{r-1} (y - s)$$

with $(y)_0 = 1$ and $r \in \mathbb{N}$ (*cf.* [**1**]-[**25**]).

A random variable X has the geometric distributions with parameter p $(0 < p < 1)$ if X has discrete distribution for which the geometric probability distribution function is defined as follows:

$$P(X = r) = pq^{r-1},$$

where $r \in \mathbb{N}$ and $p + q = 1$ (*cf.* [**2**], [**3**], [**5**]).

The moment generating function and the characteristic function of the geometric distribution with success probability p are defined as follows, respectively:

$$M_X(u) = \mathbf{E}(\exp(Xu))$$

and

$$K_X(u) \quad = \quad \mathbf{E}(\exp(iXu))$$

(*cf.* [**2**], [**5**], [**12**]).

The classical method, which involves quite complex operations to calculate each moment of the geometric distribution, is generally based on derivatives of the geometric series. In order to solve these difficulties, Simsek [**19**] gave novel relations between the $M_X(u)$ and the $K_X(u)$ in terms of the Apostol-Bernoulli numbers and polynomials. These numbers and polynomials are defined as follows:

The Apostol-Bernoulli polynomials are defined by

$$\frac{t\exp(ty)}{\beta\exp(t)-1} = \sum_{n=0}^{\infty} \mathcal{B}_n(y;\beta)\frac{t^n}{n!} \tag{1}$$

(*cf.* [**1**]).

Putting $y = 0$ in (1), $\mathcal{B}_n(0;\beta)$ reduces to the Apostol-Bernoulli numbers $\mathcal{B}_n(\beta)$, which are given by the following generating function:

$$\frac{t}{\beta\exp(t)-1} = \sum_{n=0}^{\infty} \mathcal{B}_n(\beta)\frac{t^n}{n!} \tag{2}$$

(*cf.* [**1**]).

Substituting $\beta = 1$ into (1) and (2), one has the Bernoulli polynomials $B_n(x)$ and the Bernoulli numbers B_n, respectively.

With the aid of (1) and (2), the Apostol-Bernoulli numbers and polynomials satisfies the following relations:

$$\beta\mathcal{B}_n(y+1;\beta) - \mathcal{B}_n(y;\beta) = ny^{n-1},$$

where $n \in \mathbb{N}$ (*cf.* [**1**]). When $y = 0$, the above equation reduce to the following formulas

$$\beta\mathcal{B}_1(1;\beta) = 1 + \mathcal{B}_1(\beta)$$

and for $n \geq 2$

$$\beta\mathcal{B}_n(1;\beta) = \mathcal{B}_n(\beta) \tag{3}$$

(*cf.* [**1**]).

The Apostol-Euler polynomials are defined by

$$\frac{2\exp(ty)}{\beta\exp(t)+1}=\sum_{n=0}^{\infty}\mathcal{E}_n(y;\beta)\frac{t^n}{n!} \tag{4}$$

(*cf.* [**25**]).

Putting $y=0$ and $\beta=1$ in (4), one obtains the Apostol-Euler numbers and the Euler polynomials, respectively:

$$\mathcal{E}_n(0;\beta)=\mathcal{E}_n(\beta)$$

and

$$\mathcal{E}_n(y;1)=E_n(y).$$

The Stirling numbers of the second kind are defined by

$$\frac{(\exp(t)-1)^r}{r!}=\sum_{n=0}^{\infty}S_2(n,r)\frac{t^n}{n!} \tag{5}$$

and their computational formula

$$S_2(n,r)=\frac{1}{r!}\sum_{v=0}^{r}(-1)^{r-v}\binom{r}{v}v^n$$

with $S_2(0,0)=1$, $S_2(n,r)=0$ when $r>n$ (*cf.* [**4**], [**9**], [**19**], [**21**], [**25**]).

Using (2) and (5), one has the following relation:

$$\mathcal{B}_n(\beta)=\frac{n\beta}{(\beta-1)^n}\sum_{s=1}^{n-1}(-1)^s s!\beta^{s-1}(\beta-1)^{n-1-s}S_2(n-1,s),$$

where $n\in\mathbb{N}$ and $\beta\neq1$ (*cf.* [**1**]).

The Eulerian polynomials are defined by

$$\frac{k-1}{k-\exp(t(k-1))}=\sum_{n=0}^{\infty}A_n(k)\frac{t^n}{n!} \tag{6}$$

(*cf.* [**4**], [**17**]).

Using (6), we have

$$A_n(k)=\sum_{s=0}^{n}A_{n,s}k^s, \tag{7}$$

where $A_{n,s}$ denotes the Eulerian numbers, the number of permutations of the numbers 1 to n in which exactly s elements, are given by the following alternating sum formula

$$A_{n,s}=\sum_{v=0}^{s}(-1)^v\binom{n+1}{v}(s-v+1)^n \tag{8}$$

(*cf.* [**4**], [**17**]).

The Bernstein basis functions $B_r^n(w)$ are defined by

$$B_r^n(w) = \binom{n}{r} w^r (1-w)^{n-r},$$

where $0 \leq r \leq n$, and $n, r \in \mathbb{N}_0$ and also for $r < 0$ or $r > n$,

$$B_r^n(w) = 0$$

(*cf.* [**6**], [**13**], [**20**]).

The Fubini numbers are defined by

$$\frac{1}{2-\exp(t)} = \sum_{n=0}^{\infty} w_g(n) \frac{t^n}{n!}$$

(*cf.* [**4**]).

The Fubini numbers, also known as the ordered Bell numbers, count the number of weak orderings of a set of n elements. Thereby, these numbers have many important applications in number theory, probability theory, and combinatorics. The numbers $w_g(n)$ are also related to the many special numbers. Some of them are given as follows:

$$\mathcal{B}_n\left(\frac{1}{2}\right) = -2n w_g(n-1), \tag{9}$$

$$w_g(n) = \sum_{r=0}^{n} r! S_2(n,r),$$

$$w_g(n) = \sum_{r=0}^{n-1} 2^r A_{n,r}$$

and

$$w_g(n) = A_n(2)$$

(*cf.* [**4**], [**8**], [**9**]).

An outline of the content presented in each section of this paper is given as follows:

In Section 2, we give a solution to the open problem given by Simsek. This problem is related to the Eulerian numbers and polynomials. Using some properties of some special numbers and polynomials, we give some finite combinatorial sums and identities, including not only these numbers and polynomials but also the Bernstein polynomials and the Stirling numbers. Moreover, for the purpose of calculating the values of the moments of the geometric random variable, we implement these formulas in the Wolfram language. With the help of these implementations, we give some values of these moments. In Section 3, using generating function methods, we give some recurrence relations for the moments of the geometric random variable. Further, we obtain some relations among this function and the Fubini numbers, the Stirling numbers of the second kind, and the Apostol-Euler numbers and polynomials. Finally, we give conclusion section.

2. FORMULAS AND RELATIONS FOR CHARACTERISTIC FUNCTION OF THE GEOMETRIC DISTRIBUTION IN TERMS OF THE EULERIAN NUMBERS AND POLYNOMIALS

In this section, using the properties of the Eulerian numbers, we solve the open problem given by Simsek [**19**]. We give some formulas for the characteristic function of the geometric random variable in terms of the Eulerian numbers and polynomials. We also give some finite combinatorial sums.

Recently Simsek [**19**] gave the explicit formulas for the mth moment of the geometric random variable (with parameter p) in terms of the Apostol-Bernoulli polynomials:

$$\mu_{2m}(p,q) = -\frac{p\mathcal{B}_{2m+1}(1;q)}{2m+1} \tag{10}$$

and

$$\mu_{2m+1}(p,q) = -\frac{p\mathcal{B}_{2m+2}(1;q)}{2m+2}, \tag{11}$$

where $p+q=1$ with $0<p<1$ and $m \in \mathbb{N}_0$.

Simsek [**19**] gave the following result:

"When the coefficients of each numerator of the Apostol-Bernoulli numbers are carefully examined respectively", the following sequence was given

$$\{1,-2,3,-4,\ldots\} = \left\{(-1)^{n+1}n\right\}_{n=1}^{\infty}.$$

We set

$$b_n = (-1)^{n+1}n,$$

where a positive integer n is an index of the Apostol-Bernoulli numbers. That is, each of member of this sequence corresponds to the index of the relevant Apostol-Bernoulli numbers. Generating function for the numbers b_n is given as follows:

$$\frac{t}{(1+t)^2} = \sum_{n=0}^{\infty} b_n t^n. \tag{12}$$

Applying binomial theorem to Eq. (12), we get

$$\sum_{n=0}^{\infty} b_n t^n = \sum_{n=1}^{\infty} \binom{-2}{n-1} t^n.$$

Thus we get another formula for the numbers b_n:

$$b_n = \frac{(-2)_{n-1}}{(n-1)!},$$

where $n \in \mathbb{N}$.

Simsek [**19**] also came up with the following open problem:

Is there any relationship between moments and the triangle of Eulerian numbers? How can the polynomial of q in the numerators of the moments be expressed in terms of the triangle of Eulerian numbers?

Now, let's investigate the answer to this open problem. There are relations among moment of the geometric random variable with parameter p, the Eulerian numbers and polynomials. Using (3), (10) and (11), for $m \in \mathbb{N}$, we have

$$\mu_{2m}(p,q) = -\frac{p}{q}\frac{\mathcal{B}_{2m+1}(q)}{2m+1} \tag{13}$$

and

$$\mu_{2m+2}(p,q) = -\frac{p}{q}\frac{\mathcal{B}_{2m+2}(q)}{2m+2}. \tag{14}$$

Combining (13) and (14) with the following formula

$$A_{m-1}(\beta) = -\frac{(1-\beta)^m}{m\beta}\mathcal{B}_m(\beta), \tag{15}$$

where $m \in \mathbb{N}$ (*cf.* [**21**, Corollary 4]), we have

$$\mu_{2m}(p,q) = \frac{1}{p^{2m}}A_{2m}(q) \tag{16}$$

and

$$\mu_{2m+1}(p,q) = \frac{1}{p^{2m+1}}A_{2m+1}(q). \tag{17}$$

From (16) and (17), we obtain the following result:

Theorem 1. *Let $p+q=1$ with $0<p<1$, $n \in \mathbb{N}_0$. Let $\mu_n(p,q)$ be nth moment of the geometric random variable with parameter p. We have*

$$\mu_n(p,q) = \frac{1}{p^n}A_n(q). \tag{18}$$

Combining (18) with (7), we also obtain the following result involving the Eulerian numbers:

Corollary 2. *Let $p+q=1$ with $0<p<1$ and $n \in \mathbb{N}_0$. We have*

$$\mu_n(p,q) = \frac{1}{p^n}\sum_{r=0}^{n} A_{n,r}q^r. \tag{19}$$

Combining (19) with (8) yields the following formula:

Theorem 3. *Let $p+q=1$ with $0<p<1$ and $n \in \mathbb{N}_0$. Let $\mu_n(p,q)$ be nth moment of the geometric random variable with parameter p. Then we have*

$$\mu_n(p,q) = \sum_{r=0}^{n} \frac{q^r}{p^n} \sum_{v=0}^{r} (-1)^v \binom{n+1}{v} (r-v+1)^n . \tag{20}$$

Joining the following formula

$$A_n(\beta) = (1-\beta)^{n+1} \sum_{k=0}^{\infty} (k+1)^n \beta^k,$$

(*cf.* [**17**]) with (18), we arrive at the following infinite series representation of the $\mu_n(p,q)$:

Theorem 4. *Let $p+q=1$ with $0<p<1$ and $n \in \mathbb{N}_0$. Let $\mu_n(p,q)$ be nth moment of the geometric random variable with parameter p. Then we have*

$$\mu_n(p,q) = p \sum_{k=0}^{\infty} (k+1)^n q^k . \tag{21}$$

Here we note that the Eq. (21) is related to moment generating function and probability generating function for geometric distribution. This equation gives us the following definition:

$$\mu_n(p,q) = \sum_{k=1}^{\infty} k^n p q^{k-1} .$$

That is,

$$\mu_n(p,q) = \sum_{k=1}^{\infty} k^n P(X=k) .$$

To calculate the values of $\mu_n(p,q)$, we implement (20) in the Wolfram language as follows:

Implementation 1: Using (20), the following code is written in the Wolfram language for the $\mu_n(p,q)$.

```
Unprotect[Power];
Power[0,0] = 1;
Protect[Power];
Table[(1/p^n)*Sum[(−1)^v * q^r Binomial[n+1,v]*(r−v+1)^n,{r,0,n},{v,0,r}], {n,0,10}]
```

From the above implementation, a few values of the $\mu_n(p,q)$ are given as follows:

$$
\begin{aligned}
\mu_0(p,q) &= 1,\\
\mu_1(p,q) &= \frac{1}{p},\\
\mu_2(p,q) &= \frac{1+q}{p^2},\\
\mu_3(p,q) &= \frac{1+4q+q^2}{p^3},\\
\mu_4(p,q) &= \frac{1+11q+11q^2+q^3}{p^4},\\
\mu_5(p,q) &= \frac{1+26q+66q^2+26q^3+q^4}{p^5},\\
\mu_6(p,q) &= \frac{1+57q+302q^2+302q^3+57q^4+q^5}{p^6},\\
\mu_7(p,q) &= \frac{1+120q+1191q^2+2416q^3+1191q^4+120q^5+q^6}{p^7},\\
\mu_8(p,q) &= \frac{1+247q+4293q^2+15619q^3+15619q^4+4293q^5+247q^6+q^7}{p^8},\\
\mu_9(p,q) &= \frac{1+502q+14608q^2+88243q^3+156190q^4+88263q^5}{p^9}\\
&\quad+\frac{14608q^6+502q^7+q^8}{p^9},\\
\mu_{10}(p,q) &= \frac{1+1013q+47840q^2+455192q^3+1310354q^4+1310354q^5}{p^{10}}\\
&\quad+\frac{455192q^6+47840q^7+1013q^8+q^9}{p^{10}}.
\end{aligned}
$$

2.1 Some relations for the moments of the geometric random variable

Here, we give some formulas for special values of p and q on rational numbers. It can be easily seen that the moments found in the previous section with the help of values of n from 1 to 10 are rational functions. It can be seen that the moment values for $p = \frac{1}{k}$ and $q = \frac{k-1}{k}$ are polynomials with integer coefficients. In this section, these are detailed with examples. These formulas include the Bernstein polynomials, the Eulerian numbers and polynomials, and the Stirling numbers of the second kind.

Substituting $p = \frac{1}{k}$ and $q = \frac{k-1}{k}$ into (18), we obtain the following result:

$$\mu_j\left(\frac{1}{k}, \frac{k-1}{k}\right) = k^j A_j\left(\frac{k-1}{k}\right), \tag{22}$$

where $k \in \mathbb{N}$ with $k > 1$ and $j \in \mathbb{N}_0$.

Combining the following formula, for $n \in \mathbb{N}$,

$$A_n(\beta) = \sum_{r=1}^{n} \frac{r!}{\binom{n}{r}} B_r^n(\beta) S_2(n, r),$$

(*cf.* [**21**, Theorem 4]) with (22), we have the following relation including jth moments of the geometric random variable with parameter $\frac{1}{k}$, the Bernstein polynomials, and the Stirling numbers of the second kind:

Theorem 5. *Let $j \in \mathbb{N}$. Then we have*

$$\mu_j\left(\frac{1}{k}, \frac{k-1}{k}\right) = k^j \sum_{r=1}^{j} \frac{r!}{\binom{j}{r}} B_r^j\left(\frac{k-1}{k}\right) S_2(j, r).$$

To calculate the values of $\mu_j\left(\frac{1}{k}, \frac{k-1}{k}\right)$, we implement (22) in the Wolfram language as follows:

Implementation 2: Using (7), (8) and (22), the following code is written in the Wolfram language for the $\mu_j\left(\frac{1}{k}, \frac{k-1}{k}\right)$.

```
A[n_, s_]:=Sum[(−1)^v *Binomial[n+1,v]*(s−v+1)^n,{v,0,s}];
EulerianPoly [n_, k_]:=Sum[A[n,s]*k^s,{s ,0, n }];
Simplify [Table[k^j*EulerianPoly [ j ,( k−1)/k],{j ,0,10}]]
```

From the Implementation 2, a few values of the $\mu_j\left(\frac{1}{k}, \frac{k-1}{k}\right)$ are given as follows:

$$\begin{aligned}
\mu_0\left(\frac{1}{k},\frac{k-1}{k}\right) &= 1,\\
\mu_1\left(\frac{1}{k},\frac{k-1}{k}\right) &= k,\\
\mu_2\left(\frac{1}{k},\frac{k-1}{k}\right) &= k(-1+2k),\\
\mu_3\left(\frac{1}{k},\frac{k-1}{k}\right) &= k\left(1-6k+6k^2\right),\\
\mu_4\left(\frac{1}{k},\frac{k-1}{k}\right) &= k\left(-1+14k-36k^2+24k^3\right),\\
\mu_5\left(\frac{1}{k},\frac{k-1}{k}\right) &= k\left(1-30k+150k^2-240k^3+120k^4\right),\\
\mu_6\left(\frac{1}{k},\frac{k-1}{k}\right) &= k\left(-1+62k-540k^2+1560k^3-1800k^4+720k^5\right),\\
\mu_7\left(\frac{1}{k},\frac{k-1}{k}\right) &= k\left(1-126k+1806k^2-8400k^3+16800k^4\right)\\
&\quad +k\left(-15120k^5+5040k^6\right),\\
\mu_8\left(\frac{1}{k},\frac{k-1}{k}\right) &= k\left(-1+254k-5796k^2+40824k^3-126000k^4\right)\\
&\quad +k\left(191520k^5-141120k^6+40320k^7\right),\\
\mu_9\left(\frac{1}{k},\frac{k-1}{k}\right) &= k\left(1-510k+18150k^2-186480k^3+834120k^4\right)\\
&\quad +k\left(-1905120k^5+2328480k^6-1451520k^7+362880k^8\right),\\
\mu_{10}\left(\frac{1}{k},\frac{k-1}{k}\right) &= k\left(-1+1022k-55980k^2+818520k^3-5103000k^4\right)\\
&\quad +k\left(16435440k^5-29635200k^6+30240000k^7\right)\\
&\quad +k\left(-16329600k^8+3628800k^9\right).
\end{aligned}$$

3. RECURRENCE RELATIONS FOR THE MOMENTS OF THE GEOMETRIC RANDOM VARIABLE

In this section, we give generating function for the moments of the geometric random variable, $\mu_j\left(\frac{1}{k},\frac{k-1}{k}\right)$. We also give some recurrence relations and formulas for the moments. These formulas are related to the Fubini numbers, the Stirling numbers of the second kind, and the Apostol-Euler numbers and polynomials. We also give some remarks and observations for the some special values of the $\mu_j\left(\frac{1}{k},\frac{k-1}{k}\right)$.

Theorem 6. *Let $j \in \mathbb{N}$. Then we have*

$$\mu_j\left(\frac{1}{k},\frac{k-1}{k}\right) = 1+(k-1)\sum_{r=0}^{j-1}\binom{j}{r}\mu_r\left(\frac{1}{k},\frac{k-1}{k}\right). \tag{23}$$

Proof. Using (10) and (11), we have

$$\mu_j\left(p,q\right) = -\frac{p\mathcal{B}_{j+1}\left(1;q\right)}{j+1}. \tag{24}$$

Substituting $p=\frac{1}{k}$ and $q=\frac{k-1}{k}$ ($k \in \mathbb{N}$ with $k>1$) into (24), we get

$$\mu_j\left(\frac{1}{k},\frac{k-1}{k}\right) = -\frac{\frac{1}{k}\mathcal{B}_{j+1}\left(1;\frac{k-1}{k}\right)}{j+1}. \tag{25}$$

Summing both sides of the Eq. (25) from $j=0$ to ∞, we have

$$\sum_{j=0}^{\infty}\mu_j\left(\frac{1}{k},\frac{k-1}{k}\right)\frac{t^j}{j!} = -\frac{1}{k}\sum_{j=0}^{\infty}\frac{\mathcal{B}_{j+1}\left(1;\frac{k-1}{k}\right)}{j+1}\frac{t^j}{j!}.$$

Combining the above equation with (1), after some calculations, we obtain the following generating function for the $\mu_j\left(\frac{1}{k},\frac{k-1}{k}\right)$:

$$\frac{\frac{1}{(k-1)}\exp\left(t\right)}{\frac{k}{k-1}-\exp\left(t\right)} = \sum_{j=0}^{\infty}\mu_j\left(\frac{1}{k},\frac{k-1}{k}\right)\frac{t^j}{j!}. \tag{26}$$

Using (26), we have

$$\frac{k}{k-1}\sum_{j=0}^{\infty}\mu_j\left(\frac{1}{k},\frac{k-1}{k}\right)\frac{t^j}{j!} - \sum_{j=0}^{\infty}\frac{t^j}{j!}\sum_{j=0}^{\infty}\mu_j\left(\frac{1}{k},\frac{k-1}{k}\right)\frac{t^j}{j!} = \frac{1}{(k-1)}\sum_{j=0}^{\infty}\frac{t^j}{j!}.$$

Thus

$$k\sum_{j=0}^{\infty}\mu_j\left(\frac{1}{k},\frac{k-1}{k}\right)\frac{t^j}{j!} - (k-1)\sum_{j=0}^{\infty}\sum_{r=0}^{j}\binom{j}{r}\mu_r\left(\frac{1}{k},\frac{k-1}{k}\right)\frac{t^j}{j!} = \sum_{j=0}^{\infty}\frac{t^j}{j!}.$$

Comparing the coefficients of $\frac{t^j}{j!}$ on both sides of the above equation, we get

$$k\mu_j\left(\frac{1}{k},\frac{k-1}{k}\right) - (k-1)\sum_{r=0}^{j}\binom{j}{r}\mu_r\left(\frac{1}{k},\frac{k-1}{k}\right) = 1.$$

From the above equation, we arrive at the desired result. □

Theorem 7. *Let $j \in \mathbb{N}$ and $\mu_0\left(\frac{1}{k},\frac{k-1}{k}\right)=1$. Then we have*

$$\mu_j\left(\frac{1}{k},\frac{k-1}{k}\right) = -k\sum_{r=1}^{j}(-1)^r\binom{j}{r}\mu_{j-r}\left(\frac{1}{k},\frac{k-1}{k}\right). \tag{27}$$

Proof. Using (26), we have

$$k\sum_{j=0}^{\infty}(-1)^j\frac{t^j}{j!}\sum_{j=0}^{\infty}\mu_j\left(\frac{1}{k},\frac{k-1}{k}\right)\frac{t^j}{j!}-(k-1)\sum_{j=0}^{\infty}\mu_j\left(\frac{1}{k},\frac{k-1}{k}\right)\frac{t^j}{j!}=1.$$

Hence,

$$k\sum_{j=0}^{\infty}\sum_{r=0}^{j}(-1)^r\binom{j}{r}\mu_{j-r}\left(\frac{1}{k},\frac{k-1}{k}\right)\frac{t^j}{j!}-(k-1)\sum_{j=0}^{\infty}\mu_j\left(\frac{1}{k},\frac{k-1}{k}\right)\frac{t^j}{j!}=1.$$

Comparing the coefficients of $\frac{t^j}{j!}$ on both sides of the above equation, we get

$$\mu_0\left(\frac{1}{k},\frac{k-1}{k}\right)=1$$

and $j\in\mathbb{N}$,

$$k\sum_{r=0}^{j}(-1)^r\binom{j}{r}\mu_{j-r}\left(\frac{1}{k},\frac{k-1}{k}\right)-(k-1)\,\mu_j\left(\frac{1}{k},\frac{k-1}{k}\right)=0.$$

After some calculations, we have

$$\mu_j\left(\frac{1}{k},\frac{k-1}{k}\right)=-k\sum_{r=1}^{j}(-1)^r\binom{j}{r}\mu_{j-r}\left(\frac{1}{k},\frac{k-1}{k}\right).$$

Thus, the proof of this theorem is completed. □

Theorem 8. *Let $j\in\mathbb{N}_0$. Then we have*

$$\mu_{j+1}\left(\frac{1}{k},\frac{k-1}{k}\right)=-\frac{1}{j+1}\sum_{r=0}^{j+1}\binom{j+1}{r}\mu_r\left(\frac{1}{k},\frac{k-1}{k}\right)\mathcal{B}_{j+1-r}\left(\frac{k-1}{k}\right).$$

Proof. By applying derivative operator with respect to t to the Eq. (26), we have

$$\sum_{j=0}^{\infty}\mu_{j+1}\left(\frac{1}{k},\frac{k-1}{k}\right)\frac{t^j}{j!}=\frac{\frac{k\exp(t)}{(k-1)^2}}{\left(\frac{k}{k-1}-\exp(t)\right)^2}.$$

Combining the above equation with (2) and (26), we get

$$\sum_{j=0}^{\infty}\mu_{j+1}\left(\frac{1}{k},\frac{k-1}{k}\right)\frac{t^j}{j!}=-\frac{1}{t}\sum_{j=0}^{\infty}\mu_j\left(\frac{1}{k},\frac{k-1}{k}\right)\frac{t^j}{j!}\sum_{j=0}^{\infty}\mathcal{B}_j\left(\frac{k-1}{k}\right)\frac{t^j}{j!}.$$

Thus

$$\sum_{j=0}^{\infty}\mu_{j+1}\left(\frac{1}{k},\frac{k-1}{k}\right)\frac{t^j}{j!}=-\sum_{j=0}^{\infty}\sum_{r=0}^{j+1}\frac{\binom{j+1}{r}}{j+1}\mu_r\left(\frac{1}{k},\frac{k-1}{k}\right)\mathcal{B}_{j+1-r}\left(\frac{k-1}{k}\right)\frac{t^j}{j!}.$$

Comparing the coefficients of $\frac{t^j}{j!}$ on both sides of the above equation, we arrive at the desired result. □

Theorem 9. *Let $j \in \mathbb{N}$. Then we have*

$$\mu_j\left(\frac{1}{k}, \frac{k-1}{k}\right) = 1 + \sum_{r=1}^{j} \binom{j}{r} \sum_{n=1}^{r} (k-1)^n\, n! S_2(r,n). \tag{28}$$

Proof. By applying Herschel's theorem (*cf.* [**1**]) to (26), we get

$$\sum_{j=0}^{\infty} \mu_j\left(\frac{1}{k}, \frac{k-1}{k}\right) \frac{t^j}{j!} = e^t + e^t \sum_{n=1}^{\infty} \left(\left(e^t-1\right)(k-1)\right)^n.$$

Combining the above equation with (5) yields

$$\sum_{j=0}^{\infty} \mu_j\left(\frac{1}{k}, \frac{k-1}{k}\right) \frac{t^j}{j!} = \sum_{j=0}^{\infty} \frac{t^j}{j!} + \sum_{j=0}^{\infty} \frac{t^j}{j!} \sum_{j=0}^{\infty} \sum_{n=1}^{j} (k-1)^n\, n! S_2(j,n) \frac{t^j}{j!}.$$

By applying the Cauchy product formula to the right-hand side of the above equation for two infinite series yields

$$\sum_{j=0}^{\infty} \mu_j\left(\frac{1}{k}, \frac{k-1}{k}\right) \frac{t^j}{j!} = \sum_{j=0}^{\infty} \frac{t^j}{j!} + \sum_{j=0}^{\infty} \sum_{r=0}^{j} \binom{j}{r} \sum_{n=1}^{r} (k-1)^n\, n! S_2(r,n) \frac{t^j}{j!}.$$

Comparing the coefficients of $\frac{t^j}{j!}$ on both sides of the above equation, we arrive at the desired result. □

Remark 10. *In* [**19**] and [**20**], *Simsek gave the following formula:*

$$\mu_j(p,q) = p \sum_{r=1}^{j} \frac{r! q^{r-1}}{(1-q)^{r+1}} S_2(j,r).$$

When $p = \frac{1}{k}$ and $q = \frac{k-1}{k}$ ($k \in \mathbb{N}$ with $k > 1$), the above formula reduces to the following result:

$$\mu_j\left(\frac{1}{k}, \frac{k-1}{k}\right) = k \sum_{r=1}^{j} r! (k-1)^{r-1} S_2(j,r).$$

Moreover, using (26)*, we also get the following formula for the $\mu_j\left(\frac{1}{k}, \frac{k-1}{k}\right)$:*

$$\mu_j\left(\frac{1}{k}, \frac{k-1}{k}\right) = \sum_{r=1}^{j} (-1)^{j-r} r! k^r S_2(j,r). \tag{29}$$

Thus, proof of formula (28) *is different that of the above formulas.*

Theorem 11. *Let $j \in \mathbb{N}$. Then we have*

$$\mu_j\left(\frac{1}{k}, \frac{k-1}{k}\right) = \frac{1}{2k}\mathcal{E}_j\left(1; \frac{1-k}{k}\right)$$

or equivalently

$$\mu_j\left(\frac{1}{k}, \frac{k-1}{k}\right) = \frac{1}{2k}\sum_{r=0}^{j}\binom{j}{r}\mathcal{E}_r\left(\frac{1-k}{k}\right).$$

Proof. Using (26), we have

$$\sum_{j=0}^{\infty}\mu_j\left(\frac{1}{k}, \frac{k-1}{k}\right)\frac{t^j}{j!} = \frac{1}{2k}\frac{2\exp(t)}{\left(1+\frac{1-k}{k}\exp(t)\right)}.$$

Combining the above equation with (4), we obtain

$$\sum_{j=0}^{\infty}\mu_j\left(\frac{1}{k}, \frac{k-1}{k}\right)\frac{t^j}{j!} = \frac{1}{2k}\sum_{j=0}^{\infty}\mathcal{E}_j\left(1; \frac{1-k}{k}\right)\frac{t^j}{j!}.$$

Comparing the coefficients of $\frac{t^j}{j!}$ on both sides of the above equation, we arrive at the desired result. □

Now, we give some examples of the $\mu_j\left(\frac{1}{k}, \frac{k-1}{k}\right)$:
When $k = 2$ in (22) and using (15), we have

$$\mu_j\left(\frac{1}{2}, \frac{1}{2}\right) = -\frac{\mathcal{B}_{j+1}\left(\frac{1}{2}\right)}{j+1}.$$

Combining the above equation with (9), we obtain the following result:

Corollary 12. *Let $j \in \mathbb{N}$. Then we have*

$$\mu_j\left(\frac{1}{2}, \frac{1}{2}\right) = 2w_g(j). \tag{30}$$

For $k = 2$ in (29), we have

$$\mu_j\left(\frac{1}{2}, \frac{1}{2}\right) = \sum_{r=1}^{j}(-1)^{j-r}r!2^r S_2(j,r).$$

Here we note that using Eqs. (28) and (29), many interesting known or new families of definite finite sums can be found. For example, the following finite sum, which includes the Bernoulli numbers and the Stirling numbers of the second kind, are among the members of the similar family of such sums:

$$\sum_{r=1}^{j}(-1)^r r!2^{-r}S_2(j,r) = \frac{2}{j+1}\left(1-2^{j+1}\right)B_{j+1}, \tag{31}$$

where B_j denotes the Bernoulli numbers (*cf.* [**24**, p. 154, Eq. (17)]). Proof of equation (31) was found by using the analytical continuation of the Riemann zeta function and properties of the finite sums. Consequently, the proof technique of this sum is definitely different from the that of ours.

When $k=2$ in (23) and (27), we have the following relations, for $j \geq 1$:

$$\mu_j\left(\frac{1}{2},\frac{1}{2}\right) = 1+\sum_{r=0}^{j-1}\binom{j}{r}\mu_r\left(\frac{1}{2},\frac{1}{2}\right)$$

and

$$\mu_j\left(\frac{1}{2},\frac{1}{2}\right) = -2\sum_{r=1}^{j}(-1)^r\binom{j}{r}\mu_{j-r}\left(\frac{1}{2},\frac{1}{2}\right),$$

where $\mu_0\left(\frac{1}{2},\frac{1}{2}\right)=1$.

Using the above equations (or (30)), some values of the $\mu_j\left(\frac{1}{2},\frac{1}{2}\right)$ are given as follows:

$$\begin{aligned}
\mu_0\left(\frac{1}{2},\frac{1}{2}\right) &= 1, \quad \mu_1\left(\frac{1}{2},\frac{1}{2}\right)=2, \quad \mu_2\left(\frac{1}{2},\frac{1}{2}\right)=6,\\
\mu_3\left(\frac{1}{2},\frac{1}{2}\right) &= 26, \quad \mu_4\left(\frac{1}{2},\frac{1}{2}\right)=150, \quad \mu_5\left(\frac{1}{2},\frac{1}{2}\right)=1082,\\
\mu_6\left(\frac{1}{2},\frac{1}{2}\right) &= 9366, \quad \mu_7\left(\frac{1}{2},\frac{1}{2}\right)=94586, \quad \mu_8\left(\frac{1}{2},\frac{1}{2}\right)=1091670,\\
\mu_9\left(\frac{1}{2},\frac{1}{2}\right) &= 14174522, \quad \mu_{10}\left(\frac{1}{2},\frac{1}{2}\right)=204495126,\ldots
\end{aligned}$$

see, for detail A000629 in the On-Line Encyclopedia of Integer Sequences [**16**].

When $k=3$ in (29), we get

$$\mu_j\left(\frac{1}{3},\frac{2}{3}\right) = \sum_{r=0}^{j}(-1)^{j-r}3^r S_2(j,r)r!$$

see, for detail A201339 [**16**].

When $k=3$ in (26), we obtain the following generating function of the $\mu_j\left(\frac{1}{3},\frac{2}{3}\right)$:

$$\sum_{j=0}^{\infty}\mu_j\left(\frac{1}{3},\frac{2}{3}\right)\frac{t^j}{j!} = \frac{\exp(t)}{3-2\exp(t)}$$

for detail A201339 [**16**].

When $k=3$ in (23) and (27), we have

$$\mu_j\left(\frac{1}{3},\frac{2}{3}\right) = -3\sum_{r=1}^{j}(-1)^{j-r}\binom{j}{r}\mu_{j-r}\left(\frac{1}{3},\frac{2}{3}\right)$$

and

$$\mu_j\left(\frac{1}{3},\frac{2}{3}\right)=1+2\sum_{r=0}^{j-1}\binom{j}{r}\mu_r\left(\frac{1}{3},\frac{2}{3}\right)$$

see, for detail A201339 [**16**].

Using the above equations (or (22)), some values of the $\mu_j\left(\frac{1}{3},\frac{2}{3}\right)$ are given as follows:

$$\begin{aligned}
\mu_0\left(\frac{1}{3},\frac{2}{3}\right) &= 1, \quad \mu_1\left(\frac{1}{3},\frac{2}{3}\right)=3, \quad \mu_2\left(\frac{1}{3},\frac{2}{3}\right)=15,\\
\mu_3\left(\frac{1}{3},\frac{2}{3}\right) &= 111, \quad \mu_4\left(\frac{1}{3},\frac{2}{3}\right)=1095, \quad \mu_5\left(\frac{1}{3},\frac{2}{3}\right)=13503,\\
\mu_6\left(\frac{1}{3},\frac{2}{3}\right) &= 199815, \quad \mu_7\left(\frac{1}{3},\frac{2}{3}\right)=3449631, \quad \mu_8\left(\frac{1}{3},\frac{2}{3}\right)=68062695,\\
\mu_9\left(\frac{1}{3},\frac{2}{3}\right) &= 1510769343, \quad \mu_{10}\left(\frac{1}{3},\frac{2}{3}\right)=37260156615,\ldots
\end{aligned}$$

see, for detail A201339 [**16**].

For $k=4$ in (29), we get

$$\mu_j\left(\frac{1}{4},\frac{3}{4}\right)=\sum_{r=0}^{j}(-1)^{j-r}4^r S_2(j,r)r!$$

see, for detail A201354 [**16**].

When $k=4$ in (26), we have the following generating function of the $\mu_j\left(\frac{1}{4},\frac{3}{4}\right)$

$$\sum_{j=0}^{\infty}\mu_j\left(\frac{1}{4},\frac{3}{4}\right)\frac{t^j}{j!}=\frac{\exp(t)}{4-3\exp(t)}$$

see, for detail A201354 [**16**].

When $k=4$ in (23) and (27), we have the recurrence relations of the $\mu_j\left(\frac{1}{4},\frac{3}{4}\right)$ are given by

$$\mu_j\left(\frac{1}{4},\frac{3}{4}\right)=-4\sum_{r=1}^{j}(-1)^r\binom{j}{r}\mu_{j-r}\left(\frac{1}{4},\frac{3}{4}\right)$$

and

$$\mu_j\left(\frac{1}{4},\frac{3}{4}\right)=1+3\sum_{r=0}^{j-1}\binom{j}{r}\mu_r\left(\frac{1}{4},\frac{3}{4}\right)$$

for detail A201354.

When $k=4$ in the above equations (or (22)), some values of the $\mu_j\left(\frac{1}{4},\frac{3}{4}\right)$

are given as follows:

$$\begin{aligned}
\mu_0\left(\frac{1}{4},\frac{3}{4}\right) &= 1, \quad \mu_1\left(\frac{1}{4},\frac{3}{4}\right)=4, \quad \mu_2\left(\frac{1}{4},\frac{3}{4}\right)=28,\\
\mu_3\left(\frac{1}{4},\frac{3}{4}\right) &= 292, \quad \mu_4\left(\frac{1}{4},\frac{3}{4}\right)=4060, \quad \mu_5\left(\frac{1}{4},\frac{3}{4}\right)=70564,\\
\mu_6\left(\frac{1}{4},\frac{3}{4}\right) &= 1471708, \quad \mu_7\left(\frac{1}{4},\frac{3}{4}\right)=35810212, \quad \mu_8\left(\frac{1}{4},\frac{3}{4}\right)=995827420,\\
\mu_9\left(\frac{1}{4},\frac{3}{4}\right) &= 31153998244, \quad \mu_{10}\left(\frac{1}{4},\frac{3}{4}\right)=1082931514588
\end{aligned}$$

see, for detail A201354 [**16**].

4. CONCLUSION

In general, in order to calculate each moment of the geometric distribution, applying higher order derivatives to the geometric series is known as the classical method. It is known that this method requires really time-consuming operations to find the moments after the 2nd moment value. With the aid of new moments formulas for moments of the geometric distribution in terms of the Apostol-Bernoulli polynomials and numbers which was proved by Simsek [**19**], we gave new moments formulas and their generating functions. We showed that these formulas can be given in terms of the Bernstein polynomials, the Apostol-Euler polynomials, the Fubini numbers and the Eulerian numbers and polynomials. Some recurrence relations and identities for these moments were found. Using these formulas, some applications of the moments and associated special numbers sequences were given. Furthermore, some codes in the Wolfram language and some numerical values for these moments were given. The formulas of this paper may also be related to other the special numbers, polynomials, and their generating functions. Generating functions have a variety of applications in many areas, ranging from probability and statistics, many branches of mathematics, engineering, and computer science. Perhaps researching these formulas will provide significant contributions to the discovery of newer formulas with asymptotic expansions and their use in applied sciences.

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Buket Simsek
Department of Electrical-Electronics Engineering,
Faculty of Engineering,
Akdeniz University,
TR-07058 Antalya, Turkey
E-mail: *buketsimsek@akdeniz.edu.tr; simsekbukett@gmail.com*

Neslihan Kilar
Department of Computer Technologies,
Bor Vocational School,
Niğde Ömer Halisdemir University,
TR-51700 Niğde, Turkey
E-mail: *neslihankilar@ohu.edu.tr; neslihankilar@gmail.com*