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ESTIMATING THE GENERALIZED EULER-MASCHERONI CONSTANT

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The aim of this paper is to present some results related to the classical sequence converging to generalized Euler-Mascheroni constant. Some associated asymptotic series are constructed and some inequalities are established.

1. INTRODUCTION

The Euler-Mascheroni constant, denoted as $\gamma \approx 0.57721...$., is a fundamental mathematical constant that arises in various branches of number theory and analysis. Defined as the limiting difference between the harmonic series and the natural logarithm, it is expressed as the limit of the sequence:

$$\gamma_n = H_n - \ln n, \quad n \geq 1,$$

where:

$$H_n = 1 + \frac{1}{2} + ... + \frac{1}{n}, \quad n \geq 1,$$

is the harmonic sequence.

First introduced by the Swiss mathematician Leonhard Euler in the 18th century, the constant γ has since been named in honor of both Euler and the Italian mathematician Lorenzo Mascheroni, who extensively studied its properties. The Euler-Mascheroni constant appears in numerous mathematical contexts, including integrals, series, and special functions, and has intriguing connections to the gamma function and the Riemann aeta function.

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The generalized Euler-Mascheroni constant, denoted as $\gamma(s)$, extends the classical Euler-Mascheroni constant to a broader context. It is defined as the limit of the sequence:

$$\gamma_n(s) = \frac{1}{s} + \frac{1}{s+1} + ... + \frac{1}{s+n-1} - \ln \frac{s+n-1}{s}, \quad n \geq 1.$$

Here, s is any positive real number and the classical Euler-Mascheroni constant is a special case where $s = 1$.

This generalization retains the essence of the original constant but allows for a wider range of applications and deeper insights into harmonic series and logarithmic functions.

The generalized Euler-Mascheroni constant appears in various mathematical contexts, including integrals, series, and special functions, similar to its classical counterpart. It also has applications in number theory and analysis, providing a richer framework for exploring the properties of harmonic sums and their relationships with logarithmic functions.

2. MOTIVATION

It seems that Batir [**3**, p. 7], being motivated by the elementary limit:

$$\lim_{n\to\infty} \left(\ln \frac{1}{n} - \ln \left(e^{1/n} - 1 \right) \right) = 0$$

has changed the logaritm term in the sequence $(\gamma_n)_{n\geq 1}$ to introduce the sequence:

$$\beta_n = H_n + \ln \left(e^{1/(n+1)} - 1 \right), \quad n \geq 1 \tag{1}$$

(evidently, the sequence $(\beta_n)_{n\geq 1}$ converges to γ). He proved [**3**, Cor. 2.2] the following inequality, for every integer $n \geq 1$:

$$a - \ln \left(e^{1/(n+1)} - 1 \right) \leq H_n < b - \ln \left(e^{1/(n+1)} - 1 \right), \tag{2}$$

where $a = \ln \left(\pi^2/6 \right) = 0.4977...$ and $b = \gamma = 0.5772...$.

Later, Alzer [**2**, Rel. 3] showed that the inequality (2), can be improved to $a = 1 + \ln(\sqrt{e} - 1) = 0.5672...$ (with $b = \gamma = 0.5772...$). He also proved that these last constants are sharp, in the sense that a cannot be replaced by a larger constant and b cannot be replaced by a smaller constant.

Mortici [**8**] considered the following family of sequences, for every positive parameters α, β:

$$\mu_n = H_n + \ln \left(e^{\alpha/(n+\beta)} - 1 \right) - \ln \alpha$$

and he proved that $(\mu_n)_{n\geq 1}$ converges to γ with the fastest speed when $\alpha = 1/\sqrt{2}$ and $\beta = (2+\sqrt{2})/4$. The method uses a lemma first stated in [**13**] for measuring the speed of convergence of a sequence. This lemma is proven to be a strong tool for obtaining some estimates for functions involving the gamma function and related functions. See, e.g., [**7**]-[**18**].

We will call the sequence $(\beta_n)_{n\geq 1}$ given by (1) the Batir-Alzer sequence.

3. THE EXTENDED BATIR-ALZER SEQUENCE

We have seen that the Batir-Alzer sequence is associated to the sequence $(\gamma_n)_{n\geq 1}$ converging to the Euler-Mascheroni constant.

We introduce in this paper the following sequence associated to the sequence $(\gamma_n(s))_{n\geq 1}$ converging to the generalized Euler-Mascheroni constant $\gamma(s)$, by the formula:

(3) $$\beta_n(s) = \frac{1}{s} + \frac{1}{s+1} + ... + \frac{1}{s+n-1} + \ln\left(e^{s/(s+n)} - 1\right), \quad n \geq 1.$$

Note that the sequence $\beta_n(s)$ is an extension of the sequence $(\beta_n)_{n\geq 1}$ studied by Batir, Alzer and Mortici. Indeed, for $s = 1$, we have:

$$\beta_n = \beta_n(1), \quad n \geq 1.$$

The harmonic sum $(H_n)_{n\geq 1}$ is closely related to the digamma function. The digamma function $\psi(x)$, $x > 0$, is the logaritmic derivative of the gamma function:

$$\psi(x) = \frac{d}{dx}\ln\Gamma(x) = \frac{\Gamma'(x)}{\Gamma(x)}.$$

Here,

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt, \quad x > 0,$$

is the Euler-gamma function. The harmonic sequence $(H_n)_{n\geq 1}$ admits a kind of continuous extension to the digamma function, as we have:

$$\psi(n+1) = -\gamma + H_n, \quad n = 1, 2, 3, \ ...,$$

with $\psi(1) = -\gamma$. The digamma function satisfies the following recurrence relation:

$$\psi(1+s) = \psi(s) + \frac{1}{s}$$

and consequently, by induction with respect to n, we deduce that:

(4) $$\psi(n+s) - \psi(s) = \frac{1}{s} + \frac{1}{s+1} + ... + \frac{1}{s+n-2} + \frac{1}{s+n-1}.$$

The digamma function admits the following asymptotic expansion in terms of Bernoulli's numbers:

(5) $$\psi(x) \sim \ln x - \sum_{k=1}^{\infty} \frac{B_k}{kx^k}, \quad x \to \infty.$$

Bernoulli numbers $(B_k)_{k\in\mathbb{N}}$ can be defined using the exponential generating function

$$\frac{t}{e^t-1}=\sum_{k=0}^{\infty}B_k\frac{t^k}{k!},\quad |t|<2\pi. \tag{6}$$

For $k>1$ and odd, $B_k=0$, and for even k, B_k alternates the sign. The first few nonzero Bernoulli numbers are: $B_0=1$, $B_1=-1/2$, $B_2=1/6$, $B_4=-1/30$, $B_6=1/42$, $B_8=-1/30$, $B_{10}=5/66$ and so on. By replacing these values in (5)-(6), we obtain the extended forms:

$$\psi(x)\sim\ln x-\frac{1}{2x}-\frac{1}{12x^2}+\frac{1}{120x^4}-\frac{1}{252x^6}+...,\quad x\to\infty,$$

$$\frac{t}{e^t-1}=1-\frac{t}{2}+\frac{t^2}{12}-\frac{t^4}{720}+\frac{t^6}{30240}-\frac{t^8}{1209600}+...,\quad |t|<2\pi,$$

respectively. For further details, see, e.g., [**1**, Ch. 6.3].

More general, see, e.g. [**6**, p. 33], the following asymptotic series is valid:

$$\psi(n+s)\sim\ln n-\sum_{k=1}^{\infty}(-1)^k\frac{B_k(s)}{kn^k},\quad n\to\infty, \tag{7}$$

where $B_k(s)$ are the Bernoulli polynomials, defined using the next generating function:

$$\frac{te^{st}}{e^t-1}=\sum_{k=0}^{\infty}B_k(s)\frac{t^k}{k!}.$$

In case $s=0$, it obtains (6) and consequently, $B_k(0)=B_k$ are the Bernoulli numbers. The first Bernoulli polynomials are:

$$B_0(s)=1,\quad B_1(s)=s-\frac{1}{2},\quad B_2(s)=s^2-s+\frac{1}{6}\quad\text{etc.}$$

4. THE ASYMPTOTIC EXPANSION OF EXTENDED BATIR-ALZER SEQUENCE

By replacing (4) in (3), we obtain:

$$\beta_n(s)=\psi(n+s)-\psi(s)+\ln\left(e^{s/(n+s)}-1\right). \tag{8}$$

In order to construct an asymptotic series for (8), we give a method for constructing an asymptotic series as $n\to\infty$, for the function

$$C_s(n):=\frac{\frac{s}{n+s}}{e^{s/(n+s)}-1}. \tag{9}$$

By replacing t by $s/(n+s)$ in (6), we obtain:

$$C_s(n) = \sum_{k=0}^{\infty} B_k \frac{s^k}{k!\,(n+s)^k} = \sum_{k=0}^{\infty} B_k \frac{s^k}{k!} \left(1+\frac{s}{n}\right)^{-k} \frac{1}{n^k}.$$

Further, by using the classical series for the power-function:

$$(1+z)^{-k} = 1 - kz + \frac{k(k+1)}{2!} z^2 - \frac{k(k+1)(k+2)}{3!} z^3 + \ldots,$$

we can write:

$$\begin{aligned} C_s(n) \quad \sim \quad & \sum_{k=0}^{\infty} B_k \frac{s^k}{k!} \left\{ 1 - k\left(\frac{s}{n}\right) + \frac{k(k+1)}{2!}\left(\frac{s}{n}\right)^2 \right. \\ & \left. - \frac{k(k+1)(k+2)}{3!}\left(\frac{s}{n}\right)^3 + \ldots \right\} \frac{1}{n^k}, \end{aligned}$$

or

$$(10) \qquad \begin{aligned} C_s(n) \quad \sim \quad & \sum_{k=0}^{\infty} \frac{B_k}{k!} \left\{ \left(\frac{s}{n}\right)^k - k\left(\frac{s}{n}\right)^{k+1} + \frac{k(k+1)}{2!}\left(\frac{s}{n}\right)^{k+2} \right. \\ & \left. - \frac{k(k+1)(k+2)}{3!}\left(\frac{s}{n}\right)^{k+3} + \ldots \right\}. \end{aligned}$$

The coefficients of the requested asymptotic series can now be obtained by identification the coefficients of n^{-j}, $j \in \mathbb{N}$. Let us assume that the following asymptotic series is valid:

$$C_s(n) \sim \sum_{k=0}^{\infty} \frac{c_k(s)}{n^k}, \quad n \to \infty,$$

where the coefficients $c_k(s)$ are determined by (10), using the above presented method.

Now we use Lemma 4 stated by Chen et al. [**4**] by replacing their notation $C(n)$ by our function $C_s(n)$ given by (9).

Theorem 1 ([**4**, Lemma 4])**.** *Let us assume that the following asymptotic expansion is valid:*

$$\frac{\frac{s}{n+s}}{e^{s/(n+s)} - 1} \sim \sum_{k=0}^{\infty} \frac{c_k(s)}{n^k}, \quad n \to \infty,$$

where $c_0(s) = 1$*. Then the composition* $\ln C_s(n)$ *has asymptotic expansion of the following form:*

$$(11) \qquad \ln \frac{\frac{s}{n+s}}{e^{s/(n+s)} - 1} \sim \sum_{k=1}^{\infty} \frac{a_k(s)}{n^k}, \quad n \to \infty,$$

where

$$a_k(s) = c_k(s) - \frac{1}{k} \sum_{j=1}^{k-1} j a_j(s) c_{k-j}(s) \;, \quad k \geq 1.$$

From (11), we deduce:

$$\ln\left(e^{s/(n+s)}-1\right)\sim\ln\frac{s}{n+s}-\sum_{k=1}^{\infty}\frac{a_k(s)}{n^k},\quad n\to\infty. \tag{12}$$

More precisely, the first terms are the following:

$$\ln\left(e^{s/(n+s)}-1\right)\sim\ln\frac{s}{n}-\frac{s}{2n}+\frac{s^2}{24n^2}+\frac{s^3}{12n^3}-\frac{361s^4}{2880n^4}+\dots\ . \tag{13}$$

Now, we can replace (7) and (12) in (8), to obtain the requested asymptotic series of the sequence $(\beta_n(s))_{n\geq 1}$. We will also use the classical series of the logarithm function:

$$\ln\left(1+\frac{s}{n}\right)=\sum_{k=1}^{\infty}(-1)^{k-1}\frac{s^k}{kn^k}.$$

More precisely, we are in a position to give the next:

Theorem 2. *The following asymptotic expansion is valid, as $n\to\infty$:*

$$\beta_n(s)\sim-\psi(s)+\ln s+\sum_{k=1}^{\infty}\left\{(-1)^k\frac{s^k-B_k(s)}{k}-a_k(s)\right\}\frac{1}{n^k}.$$

More precisely, as $n\to\infty$, we have:

$$\beta_n(s)\sim-\psi(s)+\ln s+\frac{s-1}{2n}-\frac{11s^2-12s+2}{24n^2}+\frac{5s^3-6s^2+2s}{12n^3}\dots\ . \tag{14}$$

Proof. According to (8), we have:

$$\beta_n(s)=\psi(n+s)-\psi(s)+\ln\left(e^{s/(s+n)}-1\right),$$

so, by using (7) and (12), we get:

$$\begin{aligned}
\beta_n(s) &\sim -\psi(s)+\left\{\ln n-\sum_{k=1}^{\infty}(-1)^k\frac{B_k(s)}{kn^k}\right\}+\left\{\ln\frac{s}{n+s}-\sum_{k=1}^{\infty}\frac{a_k(s)}{n^k}\right\}\\
&= -\psi(s)+\ln s-\ln\left(1+\frac{s}{n}\right)-\sum_{k=1}^{\infty}\left\{(-1)^k\frac{B_k(s)}{k}+a_k(s)\right\}\frac{1}{n^k}\\
&= -\psi(s)+\ln s+\sum_{k=1}^{\infty}(-1)^k\frac{s^k}{kn^k}-\sum_{k=1}^{\infty}\left\{(-1)^k\frac{B_k(s)}{k}+a_k(s)\right\}\frac{1}{n^k}\\
&= -\psi(s)+\ln s+\sum_{k=1}^{\infty}\left\{(-1)^k\frac{s^k}{k}-(-1)^k\frac{B_k(s)}{k}-a_k(s)\right\}\frac{1}{n^k}\\
&= -\psi(s)+\ln s+\sum_{k=1}^{\infty}\left\{(-1)^k\frac{s^k-B_k(s)}{k}-a_k(s)\right\}\frac{1}{n^k}.
\end{aligned}$$

Now, (14) follows by replacing (7) and (13) in the expression of $\beta_n(s)$. □

Related to the limit of the sequence $\beta_n(s)$, we give the following:

Theorem 3. *a) We have:*

$$\lim_{n\to\infty} \beta_n(s) = -\psi(s) + \ln s. \tag{15}$$

b) If $0 < s < 1$, *then there exists* $N_1 > 0$ *such that*

$$\beta_n(s) < -\psi(s) + \ln s, \quad \textit{for every } n > N_1. \tag{16}$$

c) If $s > 1$, *then there exists* $N_2 > 0$ *such that*

$$\beta_n(s) > -\psi(s) + \ln s, \quad \textit{for every } n > N_2. \tag{17}$$

Proof. a) The limit (15) follows from (14).
In particular, for $s = 1$ in (15), we deduce the limit :

$$\lim_{n\to\infty} \left(H_n + \ln\left(e^{1/(n+1)} - 1\right)\right) = \gamma.$$

b) and c) By using the equality:

$$\beta_{n+1}(s) - \beta_n(s) = \frac{1}{n+s} + \ln\left(e^{s/(n+1+s)} - 1\right) - \ln\left(e^{s/(n+s)} - 1\right),$$

we get:

$$\lim_{n\to\infty} n^2 \left(\beta_{n+1}(s) - \beta_n(s)\right) = -\frac{1}{2}(s-1) < 0. \tag{18}$$

If $0 < s < 1$, this limit is positive, so the quantity $\beta_{n+1}(s) - \beta_n(s)$ is positive, for large values of n, say for every $n > N_1$. In consequence, the sequence $(\beta_n(s))_{n\geq N_1}$ is strictly increasing to its limit $-\psi(s) + \ln s$. The inequality (16) is proven.
If $s > 1$, then the limit (18) negative, so the quantity $\beta_{n+1}(s) - \beta_n(s)$ is negative, for large values of n, say for every $n > N_2$. In consequence, the sequence $(\beta_n(s))_{n\geq N_2}$ is strictly decreasing to its limit $-\psi(s) + \ln s$. The inequality (17) is proven.
□

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