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ABSOLUTE MONOTONICITY OF A FAMILY OF FUNCTIONS RELATED TO CHEN-MALEŠEVIĆ CONJECTURE

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In this paper, using the recurrence method and the absolute monotonicity rule, it is proved that the function F_q for $q \geq 1/2$ defined on $(0,1)$ by

$$F_q(x) = \frac{x \arcsin x}{(1 - x^2/(2q))^q (\operatorname{arctanh} x)^2}$$

is absolutely monotonic on $(0,1)$ if and only if $q = 1/2$, and $-F_q'$ is absolutely monotonic on $(0,1)$ if and only if $q \geq 45/82$. These give positive answers to Wang and Chen-Malešević conjectures and yield several elegant results, such as, $\ln F_{1/2}$ and $1/F_q$ ($q \geq 45/82$) are also absolutely monotonic on $(0,1)$.

1. INTRODUCTION

In 2010, by applying the generalized Cauchy-Schwarz inequality, an interesting inequality for inverse tangent function and inverse hyperbolic sine function that

$$(\arctan x)^2 \leq \frac{x \operatorname{arcsinh} x}{\sqrt{1+x^2}}, \quad x \in (-1,1) \tag{1}$$

was firstly established by Masjed-Jamei [**13**], which is also known as Masjed-Jamei's inequality. After that, the topic related to (1) attracted much attention, which has

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been generalized in various different ways, see [**4, 7, 9, 31**]. In the beginning, Zhu and Malešević [**28**] affirmed Masjed-Jamei's guess, that is, the range of x such that inequality (1) holds can be extended to $(-\infty, \infty)$. They also presented some refinements of (1) as follows: the inequalities

$$-\frac{1}{45}x^6 \le (\arctan x)^2 - \frac{x \operatorname{arcsinh} x}{\sqrt{1+x^2}} \le -\frac{1}{45}x^6 + \frac{4}{105}x^8, \tag{2}$$

$$\begin{aligned} -\frac{1}{45}x^6 + \frac{4}{105}x^8 - \frac{11}{225}x^{10} &\le (\arctan x)^2 - \frac{x \operatorname{arcsinh} x}{\sqrt{1+x^2}} \\ &\le -\frac{1}{45}x^6 + \frac{4}{105}x^8 - \frac{11}{225}x^{10} + \frac{586}{10395}x^{12} \end{aligned} \tag{3}$$

hold for $x \in (-\infty, \infty)$. Inspired by (2) and (3), they posed a conjecture in the end of [**28**], that is, the double inequality

$$\sum_{n=3}^{2m+1} (-1)^n v_n x^{2n} \le (\arctan x)^2 - \frac{x \operatorname{arcsinh} x}{\sqrt{1+x^2}} \le \sum_{n=3}^{2m+2} (-1)^n v_n x^{2n}$$

holds for $x \in (-\infty, \infty)$ and $m \in \mathbb{N}$, where

$$v_n = \frac{1}{n}\left[\frac{n!2^{n-1}}{(2n-1)!!} - \sum_{i=1}^{n}\frac{1}{2i-1}\right] > 0, \qquad n \ge 3.$$

This conjecture has been solved by Zhu and Malešević in [**31**].

In the same paper [**28**], the authors also proved an analogue inequality of (1), which can be called *Masjed-Jamei type* inequality, that is, the inequality

$$(\operatorname{arctanh} x)^2 \le \frac{x \arcsin x}{\sqrt{1-x^2}} \tag{4}$$

holds for all $x \in (-1, 1)$ and the power number 2 is the best. Moreover, they refined the upper bound of (4) and proved the following inequality

$$\frac{x \arcsin x}{\sqrt{1-x^2}} - (\operatorname{arctanh} x)^2 \ge \sum_{k=3}^{n} v_k x^{2k}$$

for all $x \in (-1, 1)$ and $n \in \mathbb{N}$ with $n \ge 3$. In order to strengthen (4), by investigating the power series expansion of the following function

$$\frac{(\operatorname{arctanh} x)^2}{\frac{\arcsin x}{\sqrt{1-x^2}}} = x - \frac{1}{45}x^5 - \frac{22}{945}x^7 - \frac{61}{2835}x^9 + o(x^9), \tag{5}$$

Zhu [**30**] presented a new Masjed-Jamei type inequality

$$\frac{(x - x^5)\arcsin x}{\sqrt{1-x^2}} < (\operatorname{arctanh} x)^2 < \frac{(x - \frac{1}{45}x^5)\arcsin x}{\sqrt{1-x^2}} \tag{6}$$

for all $x \in (-1, 1)$, where the coefficients of x^5, -1 and $-1/45$, are the best. Very recently, Wang [**17**] provided a new and elementary proof of inequality (6), and posed the following conjecture.

Conjecture 1. *The function*

$$x \mapsto x - \frac{\sqrt{1-x^2}(\operatorname{arctanh} x)^2}{\arcsin x}$$

is absolutely monotonicity on $(0, 1)$. *Our main result is stated.*

In [**7**], Chen and Malešević found a new way to produce a lower bound for the function $(\operatorname{arctanh} x)^2$, that is,

$$\frac{x \arcsin x}{1 - \frac{1}{2}x^2} < (\operatorname{arctanh} x)^2, \tag{7}$$

for $0 < x < 1$ with the best constant $1/2$. To achieve higher accuracy near the origin than (4) and (7), Zhu [**29**, Theorem 2] presented a new upper bound of $(\operatorname{arctanh} x)^2$, that is,

$$(\operatorname{arctanh} x)^2 \leq \frac{(\arcsin x)^2}{(1-x^2)^{1/3}} \tag{8}$$

for $x \in (-1, 1)$. We remark that the statement of [**29**, Theorem 2] is incorrect, and the inequality (8) does not keep up with the structure of Masjed-Jamei type inequality since the power of $\arcsin x$ is 2. In fact, Chen and Malešević have raised a conjecture [**7**, Conjecture 2.1] that $(\operatorname{arctanh} x)^2$ has a better lower bound with the same accuracy as (8), that is,

$$\frac{x \arcsin x}{(1 - \frac{41}{45}x^2)^{45/82}} < (\operatorname{arctanh} x)^2, \quad -1 < x < 1, \tag{9}$$

which has been solved in [**8**] (also see [**19**]). As in (5), expanding in power series gives

$$G(x) = \frac{x \arcsin x}{(1 - \frac{41}{45}x^2)^{45/82}(\operatorname{arctanh} x)^2} = 1 - \frac{214}{42525}x^6 - \frac{5546}{637875}x^8 + o(x^8). \tag{10}$$

From this, we make a stronger conjecture.

Conjecture 2. *The function* $-G'(x)$ *defined by* (10) *is absolutely monotonic on* $(0, 1)$.

Now let the function F_q be defined on $(0, 1)$ by

$$F_q(x) = \frac{x \arcsin x}{(1 - x^2/(2q))^q\,(\operatorname{arctanh} x)^2} \tag{11}$$

for $q \geq 1/2$. Clearly, this family of functions contains two members, $F_{1/2}$ and $F_{45/82}$, which are related to Conjectures 1 and 2. The key motivation of this study arises from the potential benefits of Conjectures 1 and 2–if they are proved to be valid. These would not only re-solve the inequalities (4) and (9), but also greatly improve (6) and (9). This is also the main purpose of this paper to study the necessary and sufficient conditions that make F_q and $-F_q'$ to be absolutely monotonic on $(0,1)$, which is stated as follows.

Theorem 1. *Let the function F_q be defined by* (11) *with $q \geq 1/2$. Then*

(i) $F_q(x)$ is absolutely monotonic on $(0,1)$ if and only if $q = 1/2$.

(ii) $-F_q'(x)$ is absolutely monotonic on $(0,1)$ if and only if $q \geq 45/82$.

Remark 1. Theorem 1(ii) gives a positive answer to Conjecture 2, while Corollary 1 shows that Conjecture 1 is valid.

The rest of this paper is as follows. In this section, we give an introduction and highlight the relevant previous results together with the statement of main result in the form of theorem. Section 2 consists of two tools and several technical lemmas, which will be used in the following section. In Section 3, we give a proof of the main theorem stated in Section 1. As applications, Section 4 closes the paper by establishing several inequalities to solve the conjectures 1 and 2, and give further improvements.

2. PRELIMINARIES

2.1 Basic knowledge

Given $a, b, c \in \mathbb{R}$ with $c \neq 0, -1, -2, \cdots$, the Gaussian hypergeometric function is defined by

$$F(a,b;c;x) := {}_2F_1(a,b;c;x) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{x^n}{n!} \quad \text{for} \quad |x| < 1,$$

where $(a)_0 = 1$ and $(a)_n = a(a+1)(a+2)\cdots(a+n-1) = \Gamma(a+n)/\Gamma(a)$ is the shifted factorial function or the Pochhammer symbol for $n \in \mathbb{N}$. Here $\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt$ $(\mathrm{Re}(x) > 0)$ is the classical Euler gamma function [**23, 24**].

Recall that $\arcsin x$ and $\operatorname{arctanh} x$ are elliptic integrals and also hypergeometric functions. Taylor series of $\arcsin x$ and $\operatorname{arctanh} x$ had been listed in [**1**, 15.1.4 and 15.1.6], which are, in terms of hypergeometric functions, presented as follows:

$$\arcsin x = xF\left(\frac{1}{2}, \frac{1}{2}; \frac{3}{2}; x^2\right) = \sum_{n=0}^{\infty} \frac{W_n}{2n+1} x^{2n+1}, \tag{12}$$

$$\operatorname{arctanh} x = xF\left(\frac{1}{2}, 1; \frac{3}{2}; x^2\right) = \sum_{n=0}^{\infty} \frac{1}{2n+1} x^{2n+1}, \tag{13}$$

where

$$W_n = \frac{(1/2)_n}{n!} = \frac{(2n-1)!!}{(2n)!!}$$

is the classical Wallis' ratio.

2.2 Two tools

The first tool is the recurrence relation of maclaurin's coefficients for the product of power function and hypergeometric function. In [**21**], it has been proved by Yang that the coefficients of the function $x \mapsto (1-\theta x)^p F(a,b;c;x)$ satisfy a 3-order recurrence relation for $\theta \in [-1,1]$, and in particular they satisfy a 2-order recurrence relation for $\theta = 1$. We now state it in the following proposition.

Proposition 1. ([**21**, Theorem 2.1]). *Let $a, b, p \in \mathbb{R}$, $-c \notin \mathbb{N} \cup \{0\}$ and $\theta \in [-1,1]$. Then we have*

$$U_\theta(x) = (1-\theta x)^p F(a,b;c;x) = \sum_{n=0}^{\infty} u_n x^n, \tag{14}$$

with $u_0 = 1$, $u_1 = ab/c - p\theta$,

$$u_2 = \frac{1}{2}\theta^2 p(p-1) - \theta p \frac{ab}{c} + \frac{1}{2}\frac{ab(b+1)(a+1)}{c(c+1)}, \tag{15}$$

and for $n \geq 2$,

$$u_{n+1} = \frac{\xi_{n,p,\theta}(a,b,c)}{(n+1)(n+c)} u_n - \theta \frac{\eta_{n,p,\theta}(a,b,c)}{(n+1)(n+c)} u_{n-1} + \theta^2 \frac{\lambda_{n,p}(a,b)}{(n+1)(n+c)} u_{n-2}, \tag{16}$$

where

$$\begin{aligned} \xi_{n,p,\theta}(a,b,c) &= (n+a)(n+b) + \theta\left[2n^2 - 2n(p-c+1) - cp\right], \\ \eta_{n,p,\theta}(a,b,c) &= 2n^2 + 2(a+b-p-2)n - (a+b-1)p + 2(a-1)(b-1) \\ &\quad +\theta(n-p-1)(n-p+c-2), \\ \lambda_{n,p}(a,b) &= (n+a-p-2)(n+b-p-2). \end{aligned}$$

Substituting $a = b = 1/2, c = 3/2$, $\theta = 1/(2q)$ and $p = -q$ into Proposition 1, we obtain the following recurrence formula.

Lemma 1. *Let*

$$\left(1 - \frac{x^2}{2q}\right)^{-q} \frac{\arcsin x}{x} = \sum_{n=0}^{\infty} u_n x^{2n}.$$

Then $u_0 = 1$, $u_1 = 2/3$, $u_2 = (34q+15)/(120q)$ and for $n \geq 2$,

$$u_{n+1} = \alpha_n u_n - \beta_n u_{n-1} + \gamma_n u_{n-2}, \tag{17}$$

where

$$\alpha_n = \frac{1}{q}\frac{2(q+1)n^2+(4q+1)n+2q}{(n+1)(2n+3)}, \tag{18}$$

$$\beta_n = \frac{1}{4q^2}\frac{2(4q+1)n^2+\left(8q^2-4q-3\right)n+2q^2-q+1}{(n+1)(2n+3)}, \tag{19}$$

$$\gamma_n = \frac{1}{8q^2}\frac{(2n+2q-3)^2}{(n+1)(2n+3)}. \tag{20}$$

The second tool is the absolute monotonicity rule for the ratio of two power series, which was firstly established by Jurkat [**11**].

Proposition 2. *Let $f(x)=\sum_{n=0}^{\infty} a_n x^n$ and $g(x)=\sum_{n=0}^{\infty} b_n x^n$ be two power series converging on $(0,r)$ with $b_n>0$ for all $n\geq 0$. Suppose that the sequence $\{b_{n+1}/b_n\}_{n\geq 0}$ is increasing. If the sequence $\{a_n/b_n\}_{n\geq 0}$ is increasing (resp. decreasing), then the first derivatives $(f/g)'$ (resp. $-(f/g)'$) is absolutely monotonic on $(0,r)$.*

2.3 Lemmas

In this section, we will establish several lemmas to satisfy the conditions of Proposition 2.

Lemma 2. *For $q\geq 1/2$, let*

$$u_n' = u_n - \frac{2n+2q-1}{2q(2n+1)}u_{n-1}, \tag{21}$$

where u_n is defined by (17). Then $u_1'=(2q-1)/(6q)$, $u_2'=3(2q-1)/(40q)$ and for $n\geq 2$,

$$u_{n+1}' = \alpha_n' u_n' - \beta_n' u_{n-1}', \tag{22}$$

where

$$\alpha_n' = \frac{1}{2q}\frac{(2n+1)((2q+1)n+2q-1)}{(n+1)(2n+3)}, \tag{23a}$$

$$\beta_n' = \frac{1}{4q}\frac{(2n-1)(2n+2q-3)}{(n+1)(2n+3)}. \tag{23b}$$

Moreover, we have $u_n'\geq 0$ for all $n\geq 1$. In particular, $u_n'=0$ for $q=1/2$, that is, $u_n/u_{n-1}=2n/(2n+1)$ for all $n\geq 1$.

Proof. (i) Let

$$\tau_n = \frac{2n+2q-1}{2q(2n+1)}. \tag{24}$$

Then

$$u_n' = u_n - \tau_n u_{n-1}.$$

The relation (17) can be written as

$$\begin{aligned} u_{n+1} - \tau_{n+1} u_n = &\left(\alpha_n - \tau_{n+1}\right)\left(u_n - \tau_n u_{n-1}\right) - \left[\beta_n - \left(\alpha_n - \tau_{n+1}\right)\tau_n\right]\left(u_{n-1} - \tau_{n-1} u_{n-2}\right) \\ &+ \left[\gamma_n - \left(\beta_n - \left(\alpha_n - \tau_{n+1}\right)\tau_n\right)\tau_{n-1}\right] u_{n-2}. \end{aligned}$$

Straightforward computations give

$$\begin{aligned} \alpha_n' :&= \alpha_n - \tau_{n+1} \\ &= \frac{1}{2q}\frac{2\left(q+1\right)n^2 + \left(4q+1\right)n + 2q}{\left(n+1\right)\left(n+3/2\right)} - \frac{2n+2q+1}{2q\left(2n+3\right)} \\ &= \frac{1}{2q}\frac{\left(2n+1\right)\left(\left(2q+1\right)n + 2q - 1\right)}{\left(n+1\right)\left(2n+3\right)}, \\ \beta_n' :&= \beta_n - \left(\alpha_n - \tau_{n+1}\right)\tau_n \\ &= \frac{1}{8q^2}\frac{2\left(4q+1\right)n^2 + \left(8q^2 - 4q - 3\right)n + 2q^2 - q + 1}{\left(n+1\right)\left(n+3/2\right)} \\ &\qquad - \frac{1}{2q}\frac{\left(2n+1\right)\left(\left(2q+1\right)n + 2q - 1\right)}{\left(2n+3\right)\left(n+1\right)}\frac{2n+2q-1}{2q\left(2n+1\right)} \\ &= \frac{1}{4q}\frac{\left(2n-1\right)\left(2n+2q-3\right)}{\left(n+1\right)\left(2n+3\right)}, \\ \gamma_n' :&= \gamma_n - \left[\beta_n - \left(\alpha_n - \tau_{n+1}\right)\tau_n\right]\tau_{n-1} \\ &= \frac{1}{16q^2}\frac{\left(2n+2q-3\right)^2}{\left(n+1\right)\left(n+3/2\right)} - \frac{1}{4q}\frac{\left(2n-1\right)\left(2n+2q-3\right)}{\left(n+1\right)\left(2n+3\right)}\frac{2n+2q-3}{2q\left(2n-1\right)} = 0. \end{aligned}$$

Thereby, the recurrence relation (22) is established.

(ii) Let

$$u_n'' = u_n' - \frac{2n-3}{2n} u_{n-1}'. \tag{25}$$

We prove that $u_n'' \geq 0$ for $n \geq 2$. A direct computation gives

$$\begin{aligned} u_1' &= \frac{2q-1}{6q} \geq 0, \quad u_2' = \frac{3\left(2q-1\right)}{40q} \geq 0, \quad u_3' = \frac{\left(2q-1\right)\left(14q+1\right)}{336q^2} \geq 0, \\ u_4' &= \frac{\left(2q-1\right)\left(94q^2 + 9q + 2\right)}{3456q^3} \geq 0, \\ u_5' &= \frac{\left(2q-1\right)\left(6q+1\right)\left(46q^2 - 3q + 2\right)}{14\,080q^4} \geq 0, \\ u_6' &= \frac{\left(2q-1\right)\left(8996q^4 + 940q^3 + 325q^2 + 110q + 24\right)}{599\,040q^5} \geq 0. \end{aligned}$$

Then

$$u_2'' = \frac{2q-1}{30q} \geq 0, \quad u_3'' = \frac{(2q-1)(7q+5)}{1680q^2} \geq 0,$$
$$u_4'' = \frac{(2q-1)\left(14q^2+9q+7\right)}{12\,096q^3} \geq 0,$$
$$u_5'' = \frac{(2q-1)\left(214q^3+63q^2+89q+54\right)}{380\,160q^4} \geq 0. \tag{26}$$

By the relation (22), we have

$$\begin{aligned} u_{n+1}' - \frac{2n-1}{2(n+1)}u_n' &= \frac{1}{2q}\frac{(2q-1)n+2n^2+5q-1}{(2n+3)(n+1)}\left(u_n' - \frac{2n-3}{2n}u_{n-1}'\right) \\ &\quad + \frac{1}{4q}\frac{(6q-2)n-15q+3}{n(2n+3)(n+1)}u_{n-1}', \end{aligned}$$

that is,

$$u_{n+1}'' = \alpha_n'' u_n'' + \beta_n'' u_{n-1}',$$

where

$$\begin{aligned} \alpha_n'' &= \frac{1}{2q}\frac{(2q-1)n+2n^2+5q-1}{(2n+3)(n+1)}, \\ \beta_n'' &= \frac{1}{4q}\frac{(6q-2)n-15q+3}{n(2n+3)(n+1)}. \end{aligned}$$

It has been shown that $u_n' > 0$ for $1 \leq n \leq 5$. Suppose that $u_k' > 0$ for $4 \leq k \leq n$. We prove that $u_{n+1}' > 0$ for $n \geq 5$. It is easy to check that $\alpha_n'' > 0$,

$$\beta_n'' \geq \frac{1}{4q}\frac{(6q-2)5-15q+3}{n(2n+3)(n+1)} = \frac{1}{4q}\frac{15q-7}{n(2n+3)(n+1)} > 0$$

for $n \geq 5$, which in combination with $u_{n-1}' > 0$ gives

$$u_{n+1}'' > \alpha_n'' u_n'' \text{ for } n \geq 5.$$

This results in that

$$u_{n+1}'' > \alpha_n'' u_n'' > (\alpha_n'' \alpha_{n-1}'')u_{n-1}'' > \cdots > \left(\alpha_n'' \alpha_{n-1}'' ... \alpha_5''\right) u_5'' > 0,$$

where the last inequality follows from (26). Then $u_{n+1}'' > 0$, which, by (25), implies that

$$u_{n+1}' > \frac{2n-1}{2(n+1)}u_n' > 0$$

by the inductive assumption. We thus arrive at $u_n' > 0$ for all $n \geq 1$, which completes the proof of $u_n' > 0$ for all $n \geq 1$. In particular, it is easy to see that $u_1' = u_2' = 0$ for $q = 1/2$, which in conjunction with (22) and mathematical induction implies that $u_n' = 0$ for all $n \geq 1$. □

Remark 2. For $q = 1/2$, it follows from Lemma 2 that $u_n/u_{n-1} = 2n/(2n+1)$, which gives

$$u_n = \frac{u_n}{u_{n-1}}\frac{u_{n-1}}{u_{n-2}}\cdots\frac{u_2}{u_1}u_1 = \frac{(2n)\cdot(2n-2)\cdots 4\cdot 2}{(2n+1)(2n-1)\cdots 5\cdot 3} = \frac{(2n)!!}{(2n+1)!!} = \frac{1}{(2n+1)W_n}.$$

In fact, it has been given in [**28**, Lemma 7.1] that

$$\frac{2x\arcsin x}{\sqrt{1-x^2}} = \sum_{n=1}^{\infty}\frac{(2x)^{2n}}{n\binom{2n}{n}} \quad\Longleftrightarrow\quad \frac{\arcsin x}{x\sqrt{1-x^2}} = \sum_{n=0}^{\infty}\frac{(2n)!!}{(2n+1)!!}x^{2n}. \tag{27}$$

More information for (27), we refer to see [**2, 5, 12, 14, 15**].

Lemma 3. *Let*

$$\left(\frac{\operatorname{arctanh} x}{x}\right)^2 = \sum_{n=0}^{\infty} v_n x^{2n}.$$

Then we have

$$v_n = \frac{1}{n+1}\sum_{k=0}^{n}\frac{1}{2k+1} = \frac{\psi(n+3/2)+\gamma+2\ln 2}{2(n+1)}. \tag{28}$$

Proof. In [**28**, (7.4)], it has been proved that

$$(\operatorname{arctanh} x)^2 = \sum_{n=1}^{\infty}\frac{1}{n}\left(1+\frac{1}{3}+\cdots+\frac{1}{2n-1}\right)x^{2n},$$

which gives

$$v_n = \frac{1}{n+1}\sum_{k=0}^{n}\frac{1}{2k+1} = \frac{\psi(n+3/2)+\gamma+2\ln 2}{2(n+1)},$$

due to $\psi(n+3/2) - \psi(n+1/2) = 2/(2n+1)$ and $\psi(1/2) = -\gamma - 2\ln 2$. □

2.4 Signs of certain sequences

Lemma 4. *For $n \geq 1$, let v_n be defined by* (28). *Then*

$$\frac{v_{n+1}}{v_n} - \frac{v_n}{v_{n-1}} > 0.$$

In other words, the sequence $\{v_{n+1}/v_n\}$ is increasing for $n \geq 0$.

Proof. For convenience, we denote by

$$T_n = \psi(n+3/2) + \gamma + 2\ln 2, \tag{29}$$

which satisfies $T_0 = 2$ and for $n \geq 0$,

$$T_{n+1} - T_n = \frac{2}{2n+3}. \tag{30}$$

By (29) and (30), we have

$$v_n = \frac{T_n}{2(n+1)}, \quad v_{n+1} = \frac{T_n + 2/(2n+3)}{2(n+2)}, \quad v_{n-1} = \frac{T_n - 2/(2n+1)}{2n}. \tag{31}$$

To prove the desired inequality, it suffices to prove that

$$\begin{aligned} v_{n-1}v_{n+1} - v_n^2 &= \frac{T_n - 2/(2n+1)}{2n}\frac{T_n + 2/(2n+3)}{2(n+2)} - \frac{T_n^2}{4(n+1)^2} \\ &= \frac{T_n'}{4n(2n+1)(2n+3)(n+1)^2(n+2)} > 0 \end{aligned}$$

for $n \geq 1$, where

$$T_n' = (2n+3)(2n+1)T_n^2 - 4(n+1)^2 T_n - 4(n+1)^2.$$

Since $\{T_n\}_{n\geq 0}$ is increasing with $T_0 = 2$, T_n' can be written as

$$T_n' = (2n+1)(2n+3)(T_n - 2)^2 + 4\left(3n^2 + 6n + 2\right)(T_n - 2) + 4n(n+2) > 0$$

for $n \geq 1$, which completes the proof. □

Lemma 5. *Let τ_n and v_n be defined by (24) and (28), respectively. Then for $q \geq 45/82$,*

$$s_n = \frac{v_{n+1}}{v_n}\tau_n - \frac{v_n}{v_{n-1}}\tau_{n+1} > 0 \quad \text{for } n \geq 2. \tag{32}$$

Proof. If $q \geq 1$, then since $v_{n+1}/v_n > v_n/v_{n-1}$, we have

$$\begin{aligned} s_n &> \frac{v_n}{v_{n-1}}(\tau_n - \tau_{n+1}) = \frac{v_n}{v_{n-1}}\left[\frac{2n+2q-1}{2q(2n+1)} - \frac{2n+2q+1}{2q(2n+3)}\right] \\ &= 2\frac{v_n}{v_{n-1}}\frac{q-1}{q(2n+3)(2n+1)} \geq 0. \end{aligned}$$

It remains to prove that $s_n > 0$ for $q \in [45/82, 1)$. We have

$$\begin{aligned} v_n v_{n-1} s_n &= \\ &= \frac{T_n - 2/(2n+1)}{2n}\frac{T_n + 2/(2n+3)}{2(n+2)}\frac{2n+2q-1}{2q(2n+1)} - \frac{T_n^2}{4(n+1)^2}\frac{2n+2q+1}{2q(2n+3)} \\ &= \frac{T_n^*}{8nq(2n+1)^2(2n+3)(n+1)^2(n+2)}, \end{aligned}$$

where

$$\begin{aligned} T_n^* = {} & (2n+1)\left(4n^2 q + 4(3q-1)n + 6q - 3\right)T_n^2 \\ & - 4(n+1)^2(2n+2q-1)T_n - 4(n+1)^2(2n+2q-1). \end{aligned}$$

A direct verification gives

$$T_2^* = \frac{32}{9}\left(526q - 269\right) > 0.$$

To show that $T_n^* > 0$ for $n \geq 3$, we write T_n^* as

$$T_n^* = s_n^* T_n - 4\left(n+1\right)^2\left(2n + 2q - 1\right),$$

where

$$s_n^* = \left(2n+1\right)\left(4n^2 q + 4\left(3q-1\right)n + 6q - 3\right)T_n - 4\left(n+1\right)^2\left(2n+2q-1\right).$$

Since $\{T_n\}_{n\geq 3}$ is increasing, we have

$$T_n \geq T_3 = \frac{352}{105} = 3.352... > \frac{10}{3},$$

which leads to

$$\begin{aligned}
s_n^* &\geq \left(2n+1\right)\left[4n^2 q + 4\left(3q-1\right)n + 6q - 3\right]\frac{10}{3} - 4\left(n+1\right)^2\left(2n+2q-1\right) \\
&= \left(\frac{80}{3}n^3 + \frac{256}{3}n^2 + 64n + 12\right)q - \left(8n^3 + \frac{116}{3}n^2 + \frac{100}{3}n + 6\right) \\
&> \left(\frac{80}{3}n^3 + \frac{256}{3}n^2 + 64n + 12\right)\frac{1}{2} - \left(8n^3 + \frac{116}{3}n^2 + \frac{100}{3}n + 6\right) \\
&= \frac{4}{3}n\left(n+1\right)\left(4n-1\right) > 0.
\end{aligned}$$

Thus, due to $q \in [45/82, 1]$, we have

$$\begin{aligned}
\frac{T_n^*}{n+1} &= \frac{s_n^*}{n+1}T_n - 4\left(n+1\right)\left(2n+2q-1\right) \\
&> \frac{4}{3}n\left(4n-1\right)\frac{10}{3} - 4\left(n+1\right)\left(2n + 2\times 1 - 1\right) \\
&= \frac{4}{9}\left(22n^2 - 37n - 9\right) > 0
\end{aligned}$$

for $n \geq 3$. Hence, $s_n > 0$ for $n \geq 2$, thereby completing the proof. □

Lemma 6. *Let α_n' and v_n be defined by* (23a) *and* (28)*, respectively. Then the following statements hold true:*

(i) If $q \in [45/82, 115/102]$, then $\rho_n = \alpha_n' - v_{n+1}/v_n > 0$ for $n \geq 2$.

(ii) If $q > 115/102$, then there is an $n_0 \geq 3$ such that $\rho_n < 0$ for $2 \leq n \leq n_0$ and $\rho_n > 0$ for $n \geq n_0$.

Proof. To prove the requried assertion, it is considered the sign of $\alpha'_n - v_{n+1}/v_n$, which can be written as

(33)

$$\begin{aligned}\alpha'_n v_n - v_{n+1} &= \frac{1}{2q}\frac{(2n+1)\left[(2q+1)n+2q-1\right]}{(n+1)(2n+3)}\frac{T_n}{2(n+1)} - \frac{T_n+2/(2n+3)}{2(n+2)}\\ &= \frac{\left[2n^3+3n^2-(2q+3)n-(2q+2)\right]T_n - 4q(n+1)^2}{4q(2n+3)(n+1)^2(n+2)}\\ &= \frac{2(n+1)(T_n+2n+2)}{4q(2n+3)(n+1)^2(n+2)}(S_n-q),\end{aligned}$$

where

$$S_n = \frac{(n-1)(2n+1)(n+2)T_n}{2(n+1)(T_n+2n+2)}.$$

Since $T_{n+1} = T_n + 2/(2n+3)$, we have

$$\begin{aligned}S_{n+1} - S_n &= \frac{n(2n+3)(n+3)(T_n+2/(2n+3))}{2(n+2)(T_n+2/(2n+3)+2n+4)} - \frac{(n-1)(2n+1)(n+2)T_n}{2(n+1)(T_n+2n+2)}\\ &= \frac{S'_n}{2(n+1)(n+2)(2n+T_n+2)(14n+3T_n+2nT_n+4n^2+14)},\end{aligned}$$

where

$$\begin{aligned}S'_n &= (2n+3)\left(4n^3+15n^2+17n+4\right)T_n^2\\ &\quad+\left(8n^5+60n^4+196n^3+320n^2+240n+56\right)T_n\\ &\quad+4n(n+3)(2n+3)(n+1)^2 > 0\end{aligned}$$

for $n \geq 1$. In view of

$$S_2 = \frac{115}{102} \quad \text{and} \quad S_\infty = \lim_{n\to\infty} S_n = \infty,$$

we see that $\alpha'_n v_n - v_{n+1} \geq 0$ if $q \in [45/82, 115/102]$. If $q > 115/102$, then there is an $n_0 \geq 3$ such that $S_n - q < 0$ for $2 \leq n \leq n_0$ and $S_n - q > 0$ for $n \geq n_0$. This implies the second assertion of this lemma, and the proof is done. □

Lemma 7. *Let α'_n, β'_n and v_n be defined by* (23a), (23b) *and* (28), *respectively. Then*

(34) $$\sigma_n = \left(\alpha'_n - \frac{v_{n+1}}{v_n}\right)\frac{v_n}{v_{n-1}} - \beta'_n < 0$$

for $n \geq 1$.

Proof. It suffices to prove that

$$v_{n-1}\sigma_n = \alpha'_n v_n - v_{n+1} - \beta'_n v_{n-1} < 0.$$

By (31) and (33), it follows that

$$\begin{aligned} v_{n-1}\sigma_n &= \frac{1}{2q}\frac{(2n+1)\left((2q+1)n+2q-1\right)}{(n+1)(2n+3)}\frac{T_n}{2(n+1)} - \frac{T_n+2/(2n+3)}{2(n+2)} \\ &\quad - \frac{1}{4q}\frac{(2n-1)(2n+2q-3)}{(n+1)(2n+3)}\frac{T_n-2/(2n+1)}{2n} \\ &= -\frac{1}{8q}\frac{(2n+1)p_1(n)T_n+2(n+1)p_2(n)}{n(2n+3)(2n+1)(n+2)(n+1)^2}, \end{aligned}$$

where

$$\begin{aligned} p_1(n) &= 2(2q-1)n^3+7(2q-1)n^2+3(2q-1)n+2(3-2q), \\ p_2(n) &= 4(2q-1)n^3+8qn^2+(13-2q)n+2(2q-3). \end{aligned}$$

Clearly, for $q \geq 1/2$, $n \mapsto p_1(n), p_2(n)$ are increasing for $n \geq 1$, and hence

$$\begin{aligned} p_1(n) &\geq p_1(1) = 2(10q-3) > 0, \\ p_2(n) &\geq p_2(1) = 3(6q+1) > 0. \end{aligned}$$

It then follows that $\sigma_n < 0$, which completes the proof. □

2.5 The decreasing properties of u_n'/v_n and u_n/v_n

Lemma 8. *Let u_n and v_n be defined by* (17) *and* (28)*, respectively. Then the sequence $\{u_n'/v_n\}$ is decreasing for $n \geq 1$ and $q \geq 45/82$.*

Proof. Due to $v_n > 0$ for $n \geq 1$, it suffices to show that

$$w_n' = u_{n+1}' - \frac{v_{n+1}}{v_n}u_n' < 0 \tag{35}$$

for $n \geq 1$. A direct computation gives

$$w_1' = -\frac{19}{360}\frac{2q-1}{q} < 0, \qquad w_2' = -\frac{1}{38\,640}\frac{(2q-1)(766q-115)}{q^2} < 0,$$

$$w_3' = -\frac{1}{1330\,560}\frac{(2q-1)\left(11\,102q^2-87q-770\right)}{q^3} < 0,$$

$$w_4' = -\frac{(2q-1)\left(2707\,372q^3+187\,416q^2-85\,027q-91\,206\right)}{642\,090\,240q^4} < 0,$$

$$w_5' = -\frac{(2q-1)\left(185\,570\,492q^4+15\,398\,068q^3+2085\,859q^2-4269\,238q-3006\,696\right)}{75\,047\,132\,160q^5} < 0.$$

Assume that $w_{n-1}' < 0$ for $n \geq 3$ and we need to prove that $w_n' < 0$ for $n \geq 3$. To do this, we use (22) to write

$$\begin{aligned} w_n' &= u_{n+1}' - \frac{v_{n+1}}{v_n}u_n' = \left(\alpha_n' - \frac{v_{n+1}}{v_n}\right)u_n' - \beta_n' u_{n-1}' \\ &= \left(\alpha_n' - \frac{v_{n+1}}{v_n}\right)\left(u_n' - \frac{v_n}{v_{n-1}}u_{n-1}'\right) + \left[\left(\alpha_n' - \frac{v_{n+1}}{v_n}\right)\frac{v_n}{v_{n-1}} - \beta_n'\right]u_{n-1}', \end{aligned} \tag{36}$$

that is,

$$w'_n = \rho_n w'_{n-1} + \sigma_n u'_{n-1}, \tag{37}$$

where

$$\rho_n = \alpha'_n - \frac{v_{n+1}}{v_n} \quad \text{and} \quad \sigma_n = \left(\alpha'_n - \frac{v_{n+1}}{v_n}\right)\frac{v_n}{v_{n-1}} - \beta'_n. \tag{38}$$

Case 1 $q \in [45/82, 115/102]$. Then by Lemmas 6 and 7, $\rho_n > 0$ and $\sigma_n < 0$ for $n \geq 2$. Since $u'_{n-1} > 0$, by inductive assumption, we have $w'_n < 0$. Then $w'_n < 0$ for all $n \geq 1$.

Case 2 $q > 115/102$. By Lemma 6, there is an $n_0 \geq 3$ such that $\rho_n < 0$ for $2 \leq n \leq n_0$ and $\rho_n > 0$ for $n \geq n_0$. For every fixed $q > 115/102$, n_0 is fixed. We distinguish two subcases to prove that $w'_n < 0$.

Subcase 2.1 $3 \leq n \leq n_0$. Then $\rho_n < 0$. From (36) with $u'_n, \beta'_n, u'_{n-1} > 0$ for $n \geq 2$, by inductive assumption, we have $w'_n < 0$.

Subcase 2.2 $3 \leq n_0 \leq n$. Then $\rho_n > 0$. From (37) with $\rho_n, u'_{n-1} > 0$ and $\sigma_n < 0$ for $n \geq 2$, by inductive assumption, we have $w'_n < 0$.

Therefore, $w'_n < 0$ for all $n \geq 1$, and the proof is completed. □

Lemma 9. *Let u_n and v_n be defined by* (17) *and* (28)*, respectively. Then the sequence $\{u_n/v_n\}$ is decreasing for $n \geq 0$ and $q \geq 45/82$.*

Proof. To prove Lemma 9, we only need to show

$$w_n = u_{n+1} - \frac{v_{n+1}}{v_n} u_n \leq 0$$

for $n \geq 0$.

A simple computation leads to

$$w_0 = 0, \quad w_1 = -\frac{1}{360}\frac{82q-45}{q} \leq 0, \quad w_2 = -\frac{1}{19\,320}\frac{2096q^2+370q-805}{q^2} < 0.$$

We now assume that $w_{n-1} \leq 0$ for $n \geq 3$ and then we need to prove that $w_n \leq 0$. To do this, we note that

$$\begin{aligned} w_n - \tau_{n+1} w_{n-1} &= \left(u_{n+1} - \frac{v_{n+1}}{v_n} u_n\right) - \tau_{n+1}\left(u_n - \frac{v_n}{v_{n-1}} u_{n-1}\right) \\ &= u'_{n+1} - \frac{v_{n+1}}{v_n} u'_n - \left(\frac{v_{n+1}}{v_n}\tau_n - \frac{v_n}{v_{n-1}}\tau_{n+1}\right) u_{n-1} \\ &= w'_n - s_n u_{n-1}, \end{aligned}$$

where s_n and w'_n are defined by (32) and (35), respectively. By Lemmas 5 and 8, it can be obtained that

$$w_n - \tau_{n+1} w_{n-1} < 0,$$

which, by inductive hypothesis, reveals that $w_n < \tau_{n+1} w_{n-1} < 0$. By induction, $w_n \le 0$ for all $n \ge 0$, which completes the proof. $\square$

3. PROOF OF THE MAIN RESULT

In this section, we give a proof of Theorem 1 by Proposition 2.

Proof of Theorem 1. By Lemmas 1 and 3, we write F_q as

$$F_q(x) = \frac{\left(1 - \frac{x^2}{2q}\right)^{-q} \frac{\arcsin x}{x}}{\left(\frac{\operatorname{arctanh} x}{x}\right)^2} = \frac{\sum_{n=0}^{\infty} u_n x^{2n}}{\sum_{n=0}^{\infty} v_n x^{2n}}.$$

In Lemma 4, it has been proved that the sequence $\{v_{n+1}/v_n\}$ is increasing for $n \ge 0$. In order for Theorem 1 to hold, we only need to consider the monotonicity of u_n/v_n.

(i) For $q = 1/2$, we see from (27) that $u_n > 0$. To prove the increasing property of u_n/v_n, it suffices to show that

$$\frac{u_{n+1}}{u_n} v_n - v_{n+1} \ge 0 \tag{39}$$

for $n \ge 0$.

From Lemmas 2 and 3 together with (29), we see that

$$\begin{aligned} \frac{u_{n+1}}{u_n} v_n - v_{n+1} &= \frac{2n+2}{2n+3} \frac{\psi(n+3/2) + \gamma + 2\ln 2}{2(n+1)} - \frac{\psi(n+5/2) + \gamma + 2\ln 2}{2(n+2)} \\ &= \frac{T_n}{2n+3} - \frac{T_n + 2/(2n+3)}{2(n+2)} = \frac{T_n - 2}{2(n+2)(2n+3)} \\ &\ge \frac{T_0 - 2}{2(n+2)(2n+3)} = 0, \end{aligned}$$

where the inequality follows from the increasing property of T_n. This shows that the sequence $\{u_n/v_n\}$ is increasing and thereby $F'_{1/2}(x)$ is absolutely monotonic on $(0, 1)$ by Proposition 2 and so is $F_{1/2}(x)$ due to $F_{1/2}(0) = 1$.

Conversely, if $F_q(x)$ is absolutely monotonic on $(0, 1)$, then $F'_q(x) > 0$ and thereby $F_q(1^-) \ge F_q(0) = 1$, which gives $q = 1/2$. Otherwise, $F_q(1^-) = 0$ for $q > 1/2$.

(ii) For $q \ge 45/82$, by Lemma 9, we see that the sequence $\{u_n/v_n\}$ is decreasing for $n \ge 0$, and thereby, $-F'_q(x)$ is absolutely monotonic on $(0, 1)$ by Proposition 2.

Conversely, if $-F'_q(x)$ is absolutely monotonic on $(0, 1)$, then by the Maclaurin expansion of $F'_q(x)$

$$F'_q(x) = \frac{45 - 82q}{90q} x^3 + o(x^3),$$

it is required that $45 - 82q \le 0$, that is, $q \ge 45/82$. $\square$

4. CONSEQUENCES AND REMARKS

In this section, we give several direct consequences of Theorem 1, which solve the conjectures 1 and 2. To this end, we need a useful lemma to characterize a property of absolutely monotonic functions, which gives an positive answer to [**16**, Conjecture 3].

Lemma 10. *If $F(x) > 0$ and $-F'(x)$ is absolutely monotonic on the interval I, then $1/F(x)$ and $-(\ln F(x))'$ are also absolutely monotonic on I.*

Proof. We first prove that $1/F$ is absolutely monotonic. Clearly, $(1/F)' = -F'/F^2 > 0$. Assume that $(1/F)'$, $(1/F)''$, $\cdots$, $(1/F)^{(n)} \geq 0$ for $n \geq 1$. Then we can show that

$$\left(\frac{1}{F}\right)^{(n+1)} = \left(-\frac{F'}{F^2}\right)^{(n)} = \sum_{k=0}^{n} \binom{n}{k} (-F')^{(n-k)} \left(\frac{1}{F^2}\right)^{(k)}$$
$$= \sum_{k=0}^{n} \binom{n}{k} (-F')^{(n-k)} \left[\sum_{j=0}^{k} \binom{k}{j} \left(\frac{1}{F}\right)^{(k-j)} \left(\frac{1}{F}\right)^{(j)}\right] \geq 0.$$

By induction, $(1/F)^{(n)} \geq 0$ for all $n \geq 1$. which in combination with $F > 0$ means that $1/F$ is absolutely monotonic.

Second, we have

$$\left(-(\ln F)'\right)^{(n)} = \left(-\frac{F'}{F}\right)^{(n)} = \sum_{k=0}^{n} \binom{n}{k} (-F')^{(n-k)} \left(\frac{1}{F}\right)^{(k)} > 0,$$

which completes the proof. □

Note that

$$\frac{1}{F_{1/2}(x)} = \frac{\sqrt{1-x^2}(\operatorname{arctanh} x)^2}{x \arcsin x} = \frac{\sum_{n=0}^{\infty} v_n x^{2n}}{\sum_{n=0}^{\infty} u_n x^{2n}}.$$

From Lemma 2 and the proof Theorem 1, we see that $u_{n+1}/u_n = 2(n+1)/(2n+3)$ is increasing for $n \geq 0$ and v_n/u_n is decreasing for $n \geq 0$. Applying Proposition 2 and Lemma 10, we arrive at that

Proposition 3. *The function $-(1/F_{1/2}(x))'$ is absolutely monotonic on $(0, 1)$ and so are $F_{1/2}(x)$ and $(\ln F_{1/2}(x))'$.*

Remark 3. The absolute monotonicity of $F_{1/2}(x)$ has been proved in Theorem 1. However, Lemma 10 tells us that the absolute monotonicity of $-(1/F_{1/2}(x))'$ is stronger than that of $F_{1/2}(x)$.

Corollary 1. *The function*

$$x \mapsto x - \frac{\sqrt{1-x^2}(\operatorname{arctanh} x)^2}{\arcsin x}$$

is absolutely monotonic on $(0,1)$.

Proof. Clearly, $F_{1/2}(0) = 1$ and

$$x - \frac{\sqrt{1-x^2}(\operatorname{arctanh} x)^2}{\arcsin x} = x\left[1 - \frac{1}{F_{1/2}(x)}\right],$$

which, by Proposition 3, gives the required absolute monotonicity, and the proof is done. □

Remark 4. Corollary 1 gives a positive answer to Conjecture 1.

Taking into account Theorem 1(ii) and Lemma 10 gives more absolute monotonicity resluts.

Proposition 4. *The functions* $1/F_{45/82}(x)$ *and* $-(\ln F_{45/82}(x))'$ *are absolutely monotonic on* $(0,1)$.

Now let

(40)
$$F_{1/2}(x) = \frac{x \arcsin x}{\sqrt{1-x^2}(\operatorname{arctanh} x)^2} = \sum_{n=0}^{\infty} a_n x^{2n}, \qquad \frac{1}{F_{1/2}(x)} = \sum_{n=0}^{\infty} a_n^* x^{2n},$$

(41)
$$F_{45/82}(x) = \frac{x \arcsin x}{(1-\frac{41}{45}x^2)^{45/82}(\operatorname{arctanh} x)^2} = \sum_{n=0}^{\infty} b_n x^{2n}, \quad \frac{1}{F_{45/82}(x)} = \sum_{n=0}^{\infty} b_n^* x^{2n}.$$

Then the absolute monotonicity of $F'_{1/2}(x)$ and $-(1/F_{1/2}(x))'$ imply that the coefficients a_n and a_n^* satisfy the following properties:

- $a_0 = a_0^* = 1$ and $a_n \geq 0$, $a_n^* \leq 0$ for $n \geq 1$. Moreover, the first five terms of a_n and a_n^* are

$$a_0 = 1, \quad a_1 = 0, \quad a_2 = \frac{1}{45}, \quad a_3 = \frac{22}{945}, \quad a_4 = \frac{104}{4725},$$
$$a_0^* = 1, \quad a_1^* = 0, \quad a_2^* = -\frac{1}{45}, \quad a_3^* = -\frac{22}{945}, \quad a_4^* = -\frac{61}{2835},$$

which follow from the recurrence relation

$$a_n = u_n - \sum_{k=1}^{n} v_k a_{n-k}, \quad a_n^* = v_n - \sum_{k=1}^{n} u_k a_{n-k}^*$$

for $n \geq 1$ with (17) (in the case $q = 1/2$) and (28).

- Taking $x \to 1^-$ in (40) gives

$$\sum_{n=0}^{\infty} a_n = \infty \quad \text{and} \quad \sum_{n=0}^{\infty} a_n^* = 0.$$

Similarly, the coefficients b_n and b_n^* satisfy the following properties, due to the absolute monotonicity of $-F'_{45/82}(x)$ and $1/F_{45/82}(x)$,

- $b_0 = b_0^* = 1$ and $b_n \le 0$, $b_n^* \ge 0$ for $n \ge 1$. Then the coefficients b_n and b_n^* have the following recurrence relation

$$b_n = u_n - \sum_{k=1}^{n} v_k b_{n-k}, \quad b_n^* = v_n - \sum_{k=1}^{n} u_k b_{n-k}^*$$

for $n \ge 1$ with (17) (in the case $q = 45/82$) and (28), which can deduce the first five terms of b_n and b_n^* as follows

$$\begin{aligned} &b_0 = 1, \quad b_1 = 0, \quad b_2 = 0, \quad b_3 = -\frac{214}{42525}, \quad b_4 = -\frac{5546}{637875}, \\ &b_0^* = 1, \quad b_1^* = 0, \quad b_2^* = 0, \quad b_3^* = \frac{214}{42525}, \quad b_4^* = \frac{5546}{637875}. \end{aligned}$$

- Taking $x \to 1^-$ in (41) gives

$$\sum_{n=0}^{\infty} b_n = 0 \quad \text{and} \quad \sum_{n=0}^{\infty} b_n^* = \infty.$$

By Theorem 1(i) and Proposition 3, we obtain the following statements.

Proposition 5. *For $n \ge 1$, the functions*

$$\begin{aligned} \mathcal{G}_{1,n}(x) &= \frac{1}{x^{2n}} \left[\frac{x \arcsin x}{\sqrt{1-x^2}(\operatorname{arctanh} x)^2} - \sum_{k=0}^{n-1} a_k x^{2k} \right], \\ \mathcal{G}_{1,n}^*(x) &= \frac{1}{x^{2n}} \left[\sum_{k=0}^{n-1} a_k^* x^{2k} - \frac{\sqrt{1-x^2}(\operatorname{arctanh} x)^2}{x \arcsin x} \right] \end{aligned}$$

are absolutely monotonic on $(0,1)$. Consequently, the inequality

$$(\operatorname{arctanh} x)^2 < \frac{x \arcsin x}{\left(\sum_{k=0}^{n} a_k x^{2k}\right)\sqrt{1-x^2}} \tag{42}$$

holds for $x \in (0,1)$ and the double inequality

(43)

$$\frac{x\left[\sum_{k=0}^{n-1} a_k^* x^{2k} - \left(\sum_{k=0}^{n-1} a_k^*\right) x^{2n}\right] \arcsin x}{\sqrt{1-x^2}} < (\operatorname{arctanh} x)^2 < \frac{x\left(\sum_{k=0}^{n} a_k^* x^{2k}\right) \arcsin x}{\sqrt{1-x^2}}$$

holds for $x \in (0,1)$.

Remark 5. Taking $n = 2$ into (43) yields (6), which gives a new proof of (6).

Taking $n = 3$ into (42) and (43), we obtain the following corollary.

Corollary 2. *The inequalities*

$$(\operatorname{arctanh} x)^2 < \frac{x \arcsin x}{\left(1 + \frac{x^4}{45} + \frac{22x^6}{945}\right)\sqrt{1 - x^2}}, \tag{44}$$

$$\frac{x\left(1 - \frac{x^4}{45} - \frac{44x^6}{45}\right)\arcsin x}{\sqrt{1 - x^2}} < (\operatorname{arctanh} x)^2 < \frac{x\left(1 - \frac{x^4}{45} - \frac{22x^6}{945}\right)\arcsin x}{\sqrt{1 - x^2}} \tag{45}$$

hold for $x \in (0, 1)$.

Remark 6. Evidently, the double inequality (45) is sharper than (6). While (44) seems to be new.

Proposition 6. *For* $n \geq 1$*, the functions*

$$\mathcal{G}_{2,n}(x) = \frac{1}{x^{2n}}\left[\sum_{k=0}^{n-1} b_k x^{2k} - \frac{x \arcsin x}{(1 - \frac{41}{45}x^2)^{45/82}(\operatorname{arctanh} x)^2}\right],$$

$$\mathcal{G}^*_{2,n}(x) = \frac{1}{x^{2n}}\left[\frac{(1 - \frac{41}{45}x^2)^{45/82}(\operatorname{arctanh} x)^2}{x \arcsin x} - \sum_{k=0}^{n-1} b^*_k x^{2k}\right]$$

are absolutely monotonic on $(0, 1)$. *Consequently, the double inequality*

$$\frac{(1 - \frac{41}{45}x^2)^{-45/82} x \arcsin x}{\sum_{k=0}^{n} b_k x^{2k}} < (\operatorname{arctanh} x)^2 < \frac{(1 - \frac{41}{45}x^2)^{-45/82} x \arcsin x}{\sum_{k=0}^{n-1} b_k x^{2k} - \left(\sum_{k=0}^{n-1} b_k\right) x^{2n}} \tag{46}$$

holds for $x \in (0, 1)$ *and the inequality*

$$(\operatorname{arctanh} x)^2 > \frac{\left(\sum_{k=0}^{n} b^*_k x^{2k}\right) x \arcsin x}{\left(1 - \frac{41}{45}x^2\right)^{45/82}} \tag{47}$$

holds for $x \in (0, 1)$.

Taking $n = 3$ into (46) and (47), we derive the following corollary.

Corollary 3. *The inequalities*

$$\frac{x \arcsin x}{\left(1 - \frac{214}{42525}x^6\right)(1 - \frac{41}{45}x^2)^{45/82}} < (\operatorname{arctanh} x)^2 < \frac{x \arcsin x}{(1 - x^6)(1 - \frac{41}{45}x^2)^{45/82}}, \tag{48}$$

$$(\operatorname{arctanh} x)^2 < \frac{\left(1 - \frac{214}{42525}x^6\right) x \arcsin x}{(1 - \frac{41}{45}x^2)^{45/82}} \tag{49}$$

hold for $x \in (0, 1)$.

Remark 7. Clearly, the left side of (48) completely improves (9) (i.e. Chen-Malešević's conjecture). While the right side of (48) and (49) also provide two new upper bounds for $(\operatorname{arctanh} x)^2$.

At the end, we introduce a very interesting situation that the absolute monotonicity of $F_q(x)$ as q tends to infinite. Since

$$\lim_{q\to\infty}\left(1-\frac{x^2}{2q}\right)^q = e^{-x^2/2},$$

we have

$$F_\infty(x) = \lim_{q\to\infty} F_q(x) = \frac{x\arcsin x}{e^{-x^2/2}(\operatorname{arctanh} x)^2}.$$

Then Theorem 1(ii) reveals that

Proposition 7. *The function $-F'_\infty(x)$ is absolutely monotonic on $(0,1)$. Consequently, the double inequality*

$$1-\frac{41}{180}x^4-\frac{139}{180}x^6 < \frac{xe^{x^2/2}\arcsin x}{(\operatorname{arctanh} x)^2} < 1-\frac{41}{180}x^4-\frac{271}{1890}x^6$$

holds for $x\in(0,1)$.

5. CONCLUSIONS

In this paper, by using the recurrence method (Proposition 1) and absolute monotone rule (Proposition 2) we proved that the function

$$x\mapsto F_q(x) = \frac{x\arcsin x}{(1-x^2/(2q))^q\,(\operatorname{arctanh} x)^2}$$

is absolutely monotonic on $(0,1)$ if and only if $q=1/2$ and $-F'_q$ is absolutely monotonic on $(0,1)$ if and only if $q\geq 45/82$. This gives answers to Conjectures 1 and 2. As consequences of Theorem 1, some known and new inequalities follow.

It should be emphasize that the absolute monotonicity of $F_q(x)$ is very challenging, and fortunately, we have Proposition 1 to help. This proposition offers a recurrence relation such that certain complicated problems involving hypergeometric function can be easily treated; see for example, [**20, 21, 26, 27, 32**].

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