Applicable Analysis and Discrete Mathematics
available online at http://pefmath.etf.rs

Appl. Anal. Discrete Math. **19** (2025), 046–066.
https://doi.org/10.2298/AADM230402001A

# ON A PROBLEM OF MEZŐ AND ITS GENERALIZATIONS TO THREE CLASSES OF RATIONAL ZETA SERIES

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We evaluate in closed form three special classes of alternating zeta series with one and two additional parameters. Two classes are expressed as linear combinations of polylogarithms while for the third class we prove an expression involving the incomplete gamma function and the exponential integral. We also present some related series that can be deduced from the main results as well as some series with Fibonacci and Lucas numbers as coefficients.

## 1. MOTIVATION

This paper has two sources of motivation. The first source is a problem proposal by Mező from 2015 that appeared in the American Mathematical Monthly [**13**]. It asks to prove the identity

$$\frac{1}{2\pi}\operatorname{Li}_2(e^{-2\pi}) = \ln(2\pi) - 1 - \frac{5\pi}{12} - \sum_{k=1}^{\infty}\frac{(-1)^k\zeta(2k)}{k(2k+1)}, \tag{1}$$

where $\zeta(s) = \sum\limits_{k=1}^{\infty}\frac{1}{k^s}$, $\Re(s) > 1$, is the Riemann zeta function and $\operatorname{Li}_2(z) = \sum\limits_{k=1}^{\infty}\frac{z^k}{k^2}$, $|z| < 1$, is the dilogarithm. During the course of solving this problem, we found an interesting generalization for which we could provide two different proofs. Searching

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2020 Mathematics Subject Classification. 11M99, 11B39, 33B15.
Keywords and Phrases. Zeta Riemann function, Rational zeta series, Polylogarithm, Clausen function, Fibonacci numbers.

deeper in this direction we got familiar with the recent papers by Orr [**12, 14**] which became the second source of motivation for writing this paper. Orr has derived beautiful results for two families of rational zeta series which he expressed using the Clausen functions $\mathrm{Cl}_n(x)$. One such evaluation involving $\zeta(2n)$ is [**14**, Eq. (2.4)]

$$-2\sum_{n=0}^{\infty}\frac{\zeta(2n)z^{2n}}{2n+p}=\sum_{k=0}^{p}\frac{p!(-1)^{\lfloor (k+3)/2\rfloor}}{(p-k)!(2\pi z)^k}\,\mathrm{Cl}_{k+1}(2\pi z)+\delta_{\lfloor p/2\rfloor,p/2}\frac{p!(-1)^{p/2}}{(2\pi z)^p}\zeta(p+1),$$

where $\delta_{j,k}$ is the Kronecker delta function. Orr [**14**, Eq. (3.5)] also evaluates series of the form

$$\sum_{n=1}^{\infty}\frac{\zeta(2n)z^{2n}}{(2n)(2n+1)\cdots(2n+m-1)(2n+m+p)}.$$

Here, $\mathrm{Cl}_n(x)$ are the Clausen functions defined by

$$\mathrm{Cl}_1(x)=-\ln\left(2\sin\left(\frac{x}{2}\right)\right),\qquad |x|<2\pi,$$

and, for $n\geq 2$, by [**11**, Formulas (7.9) and (7.10)]

$$\mathrm{Cl}_n(x)=\begin{cases}\Im\big(\mathrm{Li}_n(e^{ix})\big)=\displaystyle\sum_{k=1}^{\infty}\frac{\sin(kx)}{k^n}, & \text{if } n \text{ is even};\\ \Re\big(\mathrm{Li}_n(e^{ix})\big)=\displaystyle\sum_{k=1}^{\infty}\frac{\cos(kx)}{k^n}, & \text{if } n \text{ is odd}.\end{cases}\tag{2}$$

See also [**16**] for new information about the Clausen functions.

In this article, for a positive integer $n$, we first consider series of the form

$$P(n,z)=\sum_{k=1}^{\infty}\frac{(-1)^k\zeta(2k)z^{2k}}{k(2k+n)},\qquad 0<|z|\leq 1.$$

In addition, for positive integers $m$ and $n$ with $m\neq n$, we also treat the class of rational series

$$P(m,n,z)=\sum_{k=1}^{\infty}\frac{(-1)^k\zeta(2k)z^{2k}}{k(2k+m)(2k+n)},\qquad 0<|z|\leq 1,$$

and its degenerated counterpart for $m\geq 1$

$$Q(m,z)=\sum_{k=1}^{\infty}\frac{(-1)^k\zeta(2k)z^{2k}}{k(2k+m)^2},\qquad 0<z\leq 1.$$

We express $P(n,z)$ and $P(m,n,z)$ as linear combinations of polylogarithms $\mathrm{Li}_s(x)$. For $Q(m,z)$ we prove an identity involving the incomplete gamma function $\Gamma(a,x)$ and the exponential integral $\mathrm{Ein}(x)$. We also present some related series that can be deduced from the main results as well as some series with Fibonacci and Lucas numbers as coefficients. The paper concludes with a discussion of further possible generalizations of the present results to series with an arbitrary power $k^p, p \geq 1$, in the denominator of $P(n,z), P(m,n,z)$ and $Q(m,z)$, respectively.

## 2. A GENERALIZATION OF THE ZETA IDENTITY OF MEZŐ

First, we prove the following generalization of (1), for which we offer two proofs.

**Theorem 1.** *Let $n$ be a positive integer. For all $z$ such that $0 < |z| \leq 1$, we have the identity*

$$P(n,z) = \sum_{k=1}^{\infty} \frac{(-1)^k \zeta(2k) z^{2k}}{k(2k+n)} = -\frac{1}{n^2} - \frac{\pi z}{n+1} + \frac{\ln(2\pi z)}{n} + \frac{(n-1)!}{(2\pi z)^n}\zeta(n+1) - (n-1)! \sum_{j=1}^{n} \frac{\operatorname{Li}_{j+1}\big(e^{-2\pi z}\big)}{(n-j)!(2\pi z)^j}, \tag{3}$$

*where $\operatorname{Li}_s(z) = \sum\limits_{k=1}^{\infty} \frac{z^k}{k^s}$ is the polylogarithm of order $s$ evaluated at $z$.*

*First proof.* Let

$$S_1(z) = \sum_{k=1}^{\infty} \frac{(-1)^k \zeta(2k)}{k} z^{2k}. \tag{4}$$

For a variable $y$, we have

$$y^{n-1} S_1(zy) = \sum_{k=1}^{\infty} \frac{(-1)^k \zeta(2k)}{k} z^{2k} y^{2k+n-1}$$

and hence

$$P(n,z) = \int_0^1 y^{n-1} S_1(zy)dy. \tag{5}$$

The evaluation of $S_1(z)$ is a classical result and equals

$$\begin{aligned} S_1(z) &= -\ln\Big(\frac{\sinh(\pi z)}{\pi z}\Big) = \pi z + \ln\Big(\frac{2\pi z}{e^{2\pi z}-1}\Big) \\ &= -\pi z + \ln(2\pi z) - \ln\big(1 - e^{-2\pi z}\big); \end{aligned}$$

so that

$$S_1(zy) = -\pi z y + \ln(2\pi z) + \ln y - \ln\big(1 - e^{-2\pi z y}\big). \tag{6}$$

Thus, from (5) and (6) we have

$$P(n,z) = -\pi z\int_0^1 y^n dy + \ln(2\pi z)\int_0^1 y^{n-1}dy + \int_0^1 y^{n-1}\ln y\,dy$$
$$-\int_0^1 y^{n-1}\ln\big(1-e^{-2\pi yz}\big)dy$$
$$= -\frac{\pi z}{n+1} + \frac{\ln(2\pi z)}{n} - \frac{1}{n^2} - \int_0^1 y^{n-1}\ln\big(1-e^{-2\pi yz}\big)dy. \tag{7}$$

It now remains to evaluate the remaining integral in (7). Let

$$I(z) = -\int_0^1 y^{n-1}\ln\big(1-e^{-2\pi yz}\big)dy.$$

Using the identity

$$\ln\Big(1-\frac{1}{x}\Big) = -\sum_{m=1}^{\infty}\frac{1}{mx^m}, \qquad |x|>1,$$

we have

$$I(z) = \int_0^1 y^{n-1}\sum_{m=1}^{\infty}\frac{e^{-2\pi zym}}{m}dy = \sum_{m=1}^{\infty}\frac{1}{m}\int_0^1 y^{n-1}e^{-2\pi zym}dy,$$

where the interchange of integration and summation is justified by uniform convergence.

Next, using the standard integral [**8**, Entry 3.351]

$$\int_0^1 x^w e^{-\mu x}dx = \frac{w!}{\mu^{w+1}} - e^{-\mu}\frac{w!}{\mu^{w+1}}\sum_{j=0}^{w}\frac{\mu^j}{j!}, \tag{8}$$

we have

$$I'(z) = \int_0^1 y^{n-1}e^{-2\pi zym}dy = \frac{(n-1)!}{(2\pi zm)^n} - \frac{(n-1)!e^{-2\pi zm}}{(2\pi zm)^n}\sum_{j=0}^{n-1}\frac{(2\pi zm)^j}{j!}$$

and hence

$$I(z) = \frac{(n-1)!}{(2\pi z)^n}\sum_{m=1}^{\infty}\frac{1}{m^{n+1}} - \frac{(n-1)!}{(2\pi z)^n}\sum_{j=0}^{n-1}\frac{(2\pi z)^j}{j!}\sum_{m=1}^{\infty}\frac{e^{-2\pi mz}}{m^{n-j+1}}. \tag{9}$$

Plugging (9) into (7) gives the identity stated in the theorem.

*Second proof.* We start with the obvious observation that $\frac{1}{k(2k+n)} = \frac{1}{n}\Big(\frac{1}{k} - \frac{2}{2k+n}\Big)$, and hence $nP(n,z) = S_1(z) - 2S_2(z)$, where $S_1(z)$ is defined and evaluated as in the first proof and $S_2(z)$ equals

$$S_2(z) = \sum_{k=1}^{\infty}\frac{(-1)^k\zeta(2k)}{2k+n}z^{2k}.$$

We get

$$S_2(z) = \sum_{k=1}^{\infty}\sum_{s=1}^{\infty} \frac{(-1)^k}{2k+n} s^{-2k} z^{2k} = \sum_{k=1}^{\infty}\sum_{s=1}^{\infty} (-1)^k \Big(\frac{z^2}{s^2}\Big)^k \int_0^1 x^{2k+n-1} dx$$
$$= \int_0^1 x^{n-1} \sum_{s=1}^{\infty} \left( \sum_{k=0}^{\infty} \Big( -\frac{z^2x^2}{s^2}\Big)^k - 1 \right) dx$$
$$= -\int_0^1 x^{n+1} z^2 \sum_{s=1}^{\infty} \frac{dx}{s^2 + z^2x^2}.$$

In view of the known identity [**8**, Entry 1.421]

$$\coth(\pi x) = \frac{1}{\pi x} + \frac{2x}{\pi} \sum_{k=1}^{\infty} \frac{1}{k^2 + x^2}$$

we get

$$\sum_{s=1}^{\infty} \frac{1}{s^2 + z^2 x^2} = \frac{1}{2z^2x^2} \big(\pi z x \coth(\pi z x) - 1\big),$$

and hence

$$S_2(z) = \frac{1}{2n} - \frac{\pi z}{2} \int_0^1 x^n \coth(\pi z x) dx$$
$$= \frac{1}{2n} - \frac{\pi z}{2} \int_0^1 x^n \Big(1 + \frac{2e^{-2\pi z x}}{1 - e^{-2\pi z x}}\Big) dx$$
$$= \frac{1}{2n} - \frac{\pi z}{2} \Big( \frac{1}{n+1} + 2 \sum_{m=1}^{\infty} \int_0^1 x^n e^{-2\pi z m x} dx \Big).$$

Again we can use (8) to simplify. The result is

$$P(n,z) = -\frac{\pi z}{n} + \frac{\ln(2\pi z)}{n} - \frac{\ln(1 - e^{-2\pi z})}{n} - \frac{1}{n^2} + \frac{\pi z}{n(n+1)}$$
$$+ \frac{(n-1)!}{(2\pi z)^n} \Big( \zeta(n+1) - \sum_{j=0}^{n} \frac{(2\pi z)^j}{j!} \operatorname{Li}_{n+1-j}(e^{-2\pi z}) \Big).$$

As $\operatorname{Li}_1(z) = -\ln(1-z)$ the proof is completed. □

**Remark 2.** *Identity* (1) *is deduced from the evaluation of* $P(1,1)$*.*

**Example 3.** *Taking particular values of the* $n$ *and* $z$ *in* (3) *leads to the following series:*

$$\sum_{k=1}^{\infty} \frac{(-1)^{k-1}\zeta(2k)}{k(2k+1)} = 1 + \frac{5\pi}{12} - \ln(2\pi) + \frac{\operatorname{Li}_2(e^{-2\pi})}{2\pi},$$
$$\sum_{k=1}^{\infty} \frac{(-1)^{k-1}\zeta(2k)}{2^k k(2k+1)} = 1 + \frac{\pi}{3\sqrt{2}} - \ln(\sqrt{2}\pi) + \frac{\operatorname{Li}_2(e^{-\sqrt{2}\pi})}{\sqrt{2}\pi},$$

$$\sum_{k=1}^{\infty} \frac{(-1)^{k-1}\zeta(2k)}{4^k k(2k+1)} = 1 + \frac{\pi}{12} - \ln\pi + \frac{\mathrm{Li}_2(e^{-\pi})}{\pi},$$
$$\sum_{k=1}^{\infty} \frac{(-1)^{k-1}\zeta(2k)}{k(k+1)} = \frac{1}{2} + \frac{2\pi}{3} - \ln(2\pi) - \frac{\zeta(3) - \mathrm{Li}_3(e^{-2\pi}) - 2\pi\,\mathrm{Li}_2(e^{-2\pi})}{2\pi^2}.$$

**Theorem 4.** *Let $n$ be a positive integer and let $z$ be a real number such that $0 < |z| \le 1$. Then*

$$\begin{aligned}\sum_{k=1}^{\infty}\frac{\zeta(2k)z^{2k}}{k(k+n)} &= -\frac{1}{2n^2} + \frac{\ln(2\pi z)}{n} + \frac{(-1)^n 2(2n-1)!}{(2\pi z)^{2n}}\,\zeta(2n+1) \\ &\quad - 2(2n-1)!\sum_{j=1}^{n}(-1)^j\,\frac{(2n+1-2j)\,\mathrm{Cl}_{2j+1}(2\pi z) + 2\pi z\,\mathrm{Cl}_{2j}(2\pi z)}{(2n+1-2j)!(2\pi z)^{2j}},\end{aligned} \tag{10}$$

$$\begin{aligned}\sum_{k=1}^{\infty}\frac{\zeta(2k)z^{2k}}{k(2k-1+2n)} &= -\frac{1}{(2n-1)^2} + \frac{\ln\big(\pi z\csc(\pi z)\big)}{2n-1} \\ &\quad - (2n-2)!\sum_{j=1}^{n}(-1)^j\,\frac{(2n+1-2j)\,\mathrm{Cl}_{2j}(2\pi z) - 2\pi z\,\mathrm{Cl}_{2j-1}(2\pi z)}{(2n+1-2j)!(2\pi z)^{2j-1}},\end{aligned} \tag{11}$$

*where $\mathrm{Cl}_j(x)$ is the Clausen function defined in* (2).

*Proof.* Evaluate $P(2n, -iz)$ and $P(2n-1, -iz)$, where $i$ denotes the imaginary unit, and simplify. □

The following numerical relations of the Clausen functions and polylogarithm are known [**11**, Sections 4.3, 4.5, 7.2, 7.3, 7.5].

**Lemma 5.** *We have*

$$\mathrm{Cl}_2(n\pi) = 0, \quad n \in \mathbb{Z},$$
$$\mathrm{Cl}_2\left(\frac{\pi}{2}\right) = G, \qquad \mathrm{Cl}_2\left(\frac{3\pi}{2}\right) = -G,$$
$$\mathrm{Cl}_2\left(\frac{\pi}{3}\right) = \frac{3}{2}\,\mathrm{Cl}_2\left(\frac{2\pi}{3}\right), \qquad \mathrm{Cl}_2\left(\frac{\pi}{6}\right) + \mathrm{Cl}_2\left(\frac{5\pi}{6}\right) = \frac{4G}{3},$$
$$\mathrm{Cl}_{2n+1}(\pi) = (2^{-2n}-1)\zeta(2n+1), \qquad \mathrm{Cl}_{2n+1}(2\pi) = \zeta(2n+1),$$
$$\mathrm{Cl}_{2n+1}\left(\frac{\pi}{2}\right) = 2^{-(2n+1)}(2^{-2n}-1)\zeta(2n+1),$$
$$\mathrm{Cl}_{2n+1}\left(\frac{\pi}{3}\right) = \frac{1}{2}(2^{-2n}-1)(3^{-2n}-1)\zeta(2n+1),$$
$$\mathrm{Cl}_{2n+1}\left(\frac{2\pi}{3}\right) = \frac{1}{2}(3^{-2n}-1)\zeta(2n+1),$$
$$\mathrm{Li}_n(1) = \zeta(n), \qquad \mathrm{Li}_n(-1) = (2^{1-n}-1)\zeta(n),$$

*where $G = \sum_{n=1}^{\infty}\frac{(-1)^n}{(2n+1)^2}$ is Catalan's constant.*

**Corollary 6.** *If $z$ is a real number such that $0 < |z| \le 1$, then*

$$\sum_{k=1}^{\infty} \frac{\zeta(2k) z^{2k}}{k(k+1)} = -\frac{1}{2} + \ln(2\pi z) - \frac{1}{2\pi^2 z^2}\Big(\zeta(3) - \mathrm{Cl}_3(2\pi z) + \frac{\mathrm{Cl}_2(2\pi z)}{\pi z}\Big),$$
$$\sum_{k=1}^{\infty} \frac{\zeta(2k) z^{2k}}{k(2k+1)} = -1 + \ln(2\pi z) + \frac{\mathrm{Cl}_2(2\pi z)}{2\pi z}.$$

*Proof.* Set $n = 1$ in (10) and (11), respectively. □

**Corollary 7.** *If $n$ is a positive integer, then*

$$\sum_{k=1}^{\infty} \frac{\zeta(2k)}{k(k+n)} = -\frac{1}{2n^2} + \frac{\ln(2\pi)}{n} - 2(2n-1)! \sum_{j=1}^{n-1} \frac{(-1)^j \zeta(2j+1)}{(2\pi)^{2j}(2n-2j)!}, \tag{12}$$

$$\sum_{k=1}^{\infty} \frac{\zeta(2k)}{k(2k+2n-1)} = -\frac{1}{(2n-1)^2} + \frac{\ln(2\pi)}{2n-1} - (2n-2)! \sum_{j=1}^{n-1} \frac{(-1)^j \zeta(2j+1)}{(2\pi)^{2j}(2n-2j-1)!}. \tag{13}$$

*Proof.* Evaluate $2P(2n, i)$ and $P(2n-1, i)$ using $\ln(2\pi i) = \ln(2\pi) + i\pi/2$ and $\mathrm{Li}_n(e^{-2\pi i}) = \zeta(n)$. □

**Example 8.** *From* (12) *and* (13) *one gets the following series:*

$$\sum_{k=1}^{\infty} \frac{\zeta(2k)}{k(k+1)} = -\frac{1}{2} + \ln(2\pi),$$
$$\sum_{k=1}^{\infty} \frac{\zeta(2k)}{k(k+2)} = -\frac{1}{8} + \frac{\ln(2\pi)}{2} + \frac{3\,\zeta(3)}{2\pi^2},$$
$$\sum_{k=1}^{\infty} \frac{\zeta(2k)}{k(2k+1)} = -1 + \ln(2\pi), \tag{14}$$
$$\sum_{k=1}^{\infty} \frac{\zeta(2k)}{k(2k+3)} = -\frac{1}{9} + \frac{\ln(2\pi)}{3} + \frac{\zeta(3)}{2\pi^2}.$$

Identity (14) was also reported by Yun-Fei [**19**, Identity (2.54)].

**Corollary 9.** *If $n$ is a positive integer, then*

$$\sum_{k=1}^{\infty} \frac{\zeta(2k)}{4^k k(k+n)} = \frac{\ln \pi}{n} - \frac{1}{2n^2} + \frac{(-1)^n (2n)!}{\pi^{2n} n}\zeta(2n+1) + 2(2n-1)! \sum_{j=1}^{n-1} \frac{(-1)^j (2^{2j}-1)}{(2\pi)^{2j}(2n-2j)!}\zeta(2j+1), \tag{15}$$

$$\sum_{k=1}^{\infty}\frac{\zeta(2k)}{4^k k(2k+2n-1)} = \frac{\ln\pi}{2n-1} - \frac{1}{(2n-1)^2} - (2n-2)!\sum_{j=1}^{n-1}\frac{(-1)^j(2^{2j}-1)}{(2\pi)^{2j}(2n+1-2j)!}\zeta(2j+1). \tag{16}$$

*Proof.* Evaluate $2P(2n, i/2)$ and $P(2n-1, i/2)$. □

**Example 10.** *At $n = 1$ and $n = 2$, from (15), (16) we obtain*

$$\sum_{k=1}^{\infty}\frac{\zeta(2k)}{4^k k(k+1)} = -\frac{1}{2} + \ln\pi - \frac{7\zeta(3)}{2\pi^2}, \tag{17}$$

$$\sum_{k=1}^{\infty}\frac{\zeta(2k)}{4^k k(k+2)} = -\frac{1}{8} + \frac{\ln\pi}{2} - \frac{9\,\zeta(3)}{2\pi^2} + \frac{93\,\zeta(5)}{4\pi^4},$$

$$\sum_{k=1}^{\infty}\frac{\zeta(2k)}{4^k k(2k+1)} = \ln\pi - 1, \tag{18}$$

$$\sum_{k=1}^{\infty}\frac{\zeta(2k)}{4^k k(2k+3)} = -\frac{1}{9} + \frac{\ln\pi}{3} - \frac{3\zeta(3)}{2\pi^2}. \tag{19}$$

Identities (17), (18) and (19) were derived by Zhang and Williams [**20**, p. 1585]; see also [**4**, Formulas (2.16), (2.17)] and [**17**].

**Theorem 11.** *For all $z$ such that $0 < |z| \le 1$,*

$$\sum_{k=1}^{\infty}\frac{\zeta(2k)z^{2k}}{k(2k+1)} = -1 + \ln(2\pi|z|) + \frac{\mathrm{Cl}_2(2\pi z)}{2\pi z}, \tag{20}$$

$$\sum_{k=1}^{\infty}\frac{\zeta(2k)z^{2k}}{k(k+1)} = -\frac{1}{2} + \ln(2\pi|z|) - \frac{\zeta(3)}{2\pi^2 z^2} + \frac{\mathrm{Cl}_3(2\pi z)}{2\pi^2 z^2} + \frac{\mathrm{Cl}_2(2\pi z)}{\pi z}. \tag{21}$$

*Proof.* Evaluate $P(1, -iz)$ and $2P(2, -iz)$, respectively. □

Identity (20) is equivalent to, but much simpler and more useful than that from [**15**, Formula (566)].

**Example 12.** *From (20) and (21) we have*

$$\sum_{k=1}^{\infty}\frac{\zeta(2k)}{16^k k(2k+1)} = \ln\left(\frac{\pi}{2}\right) - 1 + \frac{2G}{\pi}, \tag{22}$$

$$\sum_{k=1}^{\infty}\left(\frac{9}{16}\right)^k\frac{\zeta(2k)}{k(2k+1)} = \ln\left(\frac{3\pi}{2}\right) - 1 - \frac{2G}{3\pi}, \tag{23}$$

$$\sum_{k=1}^{\infty}\frac{\zeta(2k)}{16^k k(k+1)} = \ln\left(\frac{\pi}{2}\right) - \frac{1}{2} - \frac{35\,\zeta(3)}{4\pi^2} + \frac{4G}{\pi}, \tag{24}$$

*as well as*

$$\sum_{k=1}^{\infty}\frac{\zeta(2k)}{9^k k(2k+1)}=\ln\Big(\frac{2\pi}{3}\Big)-1-\frac{\sqrt{3}}{9}\pi+\frac{\sqrt{3}}{6\pi}\,\psi'\Big(\frac{1}{3}\Big),$$
$$\sum_{k=1}^{\infty}\frac{\zeta(2k)}{36^k k(2k+1)}=\ln\Big(\frac{\pi}{3}\Big)-1-\frac{\pi}{\sqrt{3}}+\frac{\sqrt{3}}{2\pi}\,\psi'\Big(\frac{1}{3}\Big),$$
$$\sum_{k=1}^{\infty}\frac{\zeta(2k)}{64^k k(2k+1)}=\ln\Big(\frac{\pi}{4}\Big)-1-\frac{\sqrt{2}+1}{4}\pi-\frac{(2\sqrt{2}-1)G}{\pi}+\frac{\sqrt{2}}{8\pi}\,\psi'\Big(\frac{1}{8}\Big),$$

*where* $\psi(z)=\frac{\Gamma'(z)}{\Gamma(z)}$ *is the digamma function with* $\psi'(z)=\sum\limits_{n=0}^{\infty}\frac{1}{(z+n)^2}$.

Note that in the last three examples we have used evaluations

$$\mathrm{Cl}_2\Big(\frac{2\pi}{3}\Big)=\frac{\sqrt{3}}{9}\Big(\psi'\Big(\frac{1}{3}\Big)-\frac{2\pi^2}{3}\Big),$$
$$\mathrm{Cl}_2\Big(\frac{\pi}{3}\Big)=\frac{\sqrt{3}}{6}\Big(\psi'\Big(\frac{1}{3}\Big)-\frac{2\pi^2}{3}\Big),$$
$$\mathrm{Cl}_2\Big(\frac{\pi}{4}\Big)=\frac{1}{32}\Big(\sqrt{2}\psi'\Big(\frac{1}{8}\Big)-2(\sqrt{2}+1)\pi^2-8(2\sqrt{2}-1)G\Big),$$

which were derived by Grosjean in [**9**].

Formulas (22)–(24) are known results. For example, (22) appears in [**10**, Entry (54.5.6)]. See also [**15**, Formulas (698), (699)] and [**2**, Formulas (5.21), (5.22)].

**Theorem 13.** *For all* $z$ *such that* $0<|\ln z|\le 2\pi$,

$$\sum_{k=1}^{\infty}\frac{(\ln z)^{2k+1}B_{2k}}{k(2k+1)!}=\frac{\pi^2}{3}-\frac{1}{2}\ln^2 z+2\ln z\big(1-\ln(-\ln z)\big)-2\,\mathrm{Li}_2(z),\tag{25}$$

*where* $B_j$ *denotes the Bernoulli numbers.*

*Proof.* Evaluate $P(1,-\frac{\ln z}{2\pi})$ and use $\zeta(2n)=\frac{(-1)^{n+1}(2\pi)^{2n}}{2(2n)!}B_{2n},\ n\ge 1$. □

It is instructive to compare (25) with [**11**, Identity (1.76)]:

$$\mathrm{Li}_2(e^{-z})=\frac{\pi^2}{6}+z\ln z-z-\frac{z^2}{4}+\frac{B_1z^3}{2\cdot 3\cdot 2!}-\frac{B_2z^5}{4\cdot 5\cdot 4!}+\cdots.$$

**Corollary 14.** *If* $z$ *is a real number such that* $0<|z|\le 2\pi$, *then*

$$\sum_{k=1}^{\infty}\frac{B_{2k}z^{2k+1}}{k(2k+1)!}=\frac{4z+z^2}{2}-2z\ln z-\frac{\pi^2}{3}+2\,\mathrm{Li}_2(e^{-z}).$$

*Proof.* Evaluate (25) at $z=e^{-z}$ with $z>0$. □

In particular,

$$\sum_{k=1}^{\infty} \frac{B_{2k}}{k(2k+1)!} = \frac{5}{2} - \frac{\pi^2}{3} + 2\,\mathrm{Li}_2(e^{-1}).$$

**Corollary 15.** *For all $z$ such that $0 < |z| \le 2\pi$,*

$$\sum_{k=1}^{\infty} \frac{B_{2k} z^{2k}}{k(2k)!} = 2\ln\left(\frac{2}{z}\sinh\left(\frac{z}{2}\right)\right). \tag{26}$$

*Proof.* Differentiate (25) with respect to $z$ and write $e^{-z}$ for $z$. □

Differentiating (26) with respect to $z$, in the next corollary we obtain the generating function of even-indexed Bernoulli numbers.

**Corollary 16.** *For all real $z$ such that $0 < |z| \le 2\pi$,*

$$\sum_{k=0}^{\infty} \frac{B_{2k} z^{2k}}{(2k)!} = \frac{z}{2} + \frac{z}{e^z - 1}.$$

**Corollary 17.** *For all $z$ such that $0 < |z| \le 1$,*

$$\sum_{k=1}^{\infty} \frac{(-1)^{k-1}\zeta(2k) z^{2k}}{2k+1} = -\frac{1}{2} + \frac{\pi z}{4} - \frac{\pi}{24z} + \frac{\ln\bigl(-2\sinh(\pi z)\bigr)}{2} + \frac{\mathrm{Li}_2(e^{2\pi z})}{4\pi z}. \tag{27}$$

*Proof.* Differentiate $P(1, z)$ with respect to $z$. □

**Corollary 18.** *For all real $z$ such that $0 < |z| < 1$,*

$$\sum_{k=1}^{\infty} \frac{\zeta(2k) z^{2k}}{2k+1} = \frac{1}{2} - \frac{\ln\bigl(2\sin(\pi z)\bigr)}{2} - \frac{\mathrm{Cl}_2(2\pi z)}{4\pi z}. \tag{28}$$

*Proof.* Write $iz$ for $z$ in (27) and take the imaginary part. □

Note that Formula (28) is given in a slightly different way in [**15**, Formula (490)]. Also, one can find it in [**10**, Formula (54.5.4)] and [**18**, Formula (2.18)].

**Example 19.** *Taking $z = 1/2$, $z = 1/4$ and $z = 3/4$ in (28) yield the following identities:*

$$\sum_{k=1}^{\infty} \frac{\zeta(2k)}{4^k(2k+1)} = \frac{1}{2} - \frac{\ln 2}{2}, \tag{29}$$

$$\sum_{k=1}^{\infty} \frac{\zeta(2k)}{16^k(2k+1)} = \frac{1}{2} - \frac{\ln 2}{4} - \frac{G}{\pi}, \tag{30}$$

$$\sum_{k=1}^{\infty} \left(\frac{9}{16}\right)^k \frac{\zeta(2k)}{2k+1} = \frac{1}{2} - \frac{\ln 2}{4} + \frac{G}{3\pi}. \tag{31}$$

Identity (29) is found in [**15**, p. 313, Formula (493)] and [**14**]. One can find (30) and (31) in [**15**, Formulas (670) and (671)].

**Theorem 20.** *For all $z$ such that $0 < |z| \le 1$,*

$$\sum_{k=1}^{\infty} \frac{(-1)^{k-1}\zeta(2k)z^{2k}}{(2k+1)(k+1)} = -\frac{1}{2} - \frac{\pi z}{6} - \frac{\pi}{12z} - \frac{\zeta(3)}{2\pi^2 z^2} + \frac{\mathrm{Li}_3(e^{2\pi z})}{2\pi^2 z^2} - \frac{\mathrm{Li}_2(e^{2\pi z})}{2\pi z}. \tag{32}$$

*Proof.* Multiply through (27) by $z$ and integrate with respect to $z$. □

**Corollary 21.** *For all real $z$ such that $0 < |z| \le 1$,*

$$\sum_{k=1}^{\infty} \frac{\zeta(2k)z^{2k}}{(k+1)(2k+1)} = \frac{1}{2} - \frac{\zeta(3)}{2\pi^2 z^2} + \frac{\mathrm{Cl}_3(2\pi z)}{2\pi^2 z^2} + \frac{\mathrm{Cl}_2(2\pi z)}{2\pi z}. \tag{33}$$

*Proof.* Write $iz$ for $z$ in (32) and take real parts. □

**Example 22.** *For certain $z$, from* (33) *we have*

$$\sum_{k=1}^{\infty} \frac{\zeta(2k)}{(k+1)(2k+1)} = \frac{1}{2},$$

$$\sum_{k=1}^{\infty} \frac{\zeta(2k)}{4^k(k+1)(2k+1)} = \frac{1}{2} - \frac{7\zeta(3)}{2\pi^2}, \tag{34}$$

$$\sum_{k=1}^{\infty} \frac{\zeta(2k)}{16^k(k+1)(2k+1)} = \frac{1}{2} - \frac{35\zeta(3)}{4\pi^2} + \frac{2G}{\pi}, \tag{35}$$

$$\sum_{k=1}^{\infty} \Big(\frac{9}{16}\Big)^k \frac{\zeta(2k)}{(k+1)(2k+1)} = \frac{1}{2} - \frac{35\zeta(3)}{36\pi^2} - \frac{2G}{3\pi},$$

$$\sum_{k=1}^{\infty} \frac{\zeta(2k)}{36^k(k+1)(2k+1)} = \frac{1}{2} - \frac{\pi}{\sqrt{3}} - \frac{12\zeta(3)}{\pi^2} + \frac{\sqrt{3}}{2\pi}\,\psi'\Big(\frac{1}{3}\Big).$$

The series representation (34) is found in one of Euler's papers and has been rediscovered by many mathematicians (see [**5**] and [**20**], among others). The identity (35) is also known ([**3**, Formula (5.10)], [**12**, Formula (16)]).

## 3. CONNECTION WITH SECOND-ORDER SEQUENCES

In this section, we study some series involving the Riemann zeta function and Fibonacci (Lucas) numbers. The results are closely related to our studies in [**1, 6, 7**]. As usual, let $F_n$ and $L_n$ denote the $n$-th Fibonacci and Lucas numbers, both satisfying the recurrence $w_n = w_{n-1} + w_{n-2}$ for $n \ge 2$, but with the initial values $F_0 = 0$, $F_1 = 1$ and $L_0 = 2$, $L_1 = 1$, respectively.

The Binet formulas are

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \quad L_n = \alpha^n + \beta^n, \quad n \geq 0, \tag{36}$$

where $\alpha = (1+\sqrt{5})/2$ and $\beta = -1/\alpha = (1-\sqrt{5})/2$.

**Theorem 23.** *If $n$ is a positive integer and $z$ is any real number such that $0 < |z| \leq 1$, then*

$$\begin{aligned}\sqrt{5}\sum_{k=1}^{\infty}\frac{(-1)^{k-1}F_{2k}\zeta(2k)z^{2k}}{k(2k+n)} &= \frac{\pi z}{n+1}\\ &\quad - \frac{2\ln\alpha}{n} + \frac{2(n-1)!\sinh(n\ln\alpha)}{(2\pi z)^n}\zeta(n+1)\\ &\quad - (n-1)!\sum_{j=1}^{n}\frac{\alpha^j \operatorname{Li}_{j+1}\big(e^{2\pi\beta z}\big) - (-\beta)^j \operatorname{Li}_{j+1}\big(e^{-2\pi\alpha z}\big)}{(n-j)!(2\pi z)^j},\end{aligned} \tag{37}$$

$$\begin{aligned}\sum_{k=1}^{\infty}\frac{(-1)^{k-1}L_{2k}\zeta(2k)z^{2k}}{k(2k+n)} &= \frac{\sqrt{5}\pi z}{n+1} + \frac{2}{n^2}\\ &\quad - \frac{2\ln(2\pi z)}{n} - \frac{2(n-1)!\cosh(n\ln\alpha)}{(2\pi z)^n}\zeta(n+1)\\ &\quad + (n-1)!\sum_{j=1}^{n}\frac{\alpha^j \operatorname{Li}_{j+1}\big(e^{2\pi\beta z}\big) + (-\beta)^j \operatorname{Li}_{j+1}\big(e^{-2\pi\alpha z}\big)}{(n-j)!(2\pi z)^j}.\end{aligned} \tag{38}$$

*Proof.* Evaluate $P(n, \alpha z)$ and $P(n, -\beta z)$ and combine these equations according to (3) and the Binet formulas (36). □

**Theorem 24.** *If $z$ is any real number such that $0 < |z| \leq 1/\alpha$, then*

$$\sum_{k=1}^{\infty}\frac{F_{2k}\zeta(2k)z^{2k}}{k(2k+1)} = \frac{2\ln\alpha}{\sqrt{5}} - \frac{\beta\operatorname{Cl}_2(2\pi\alpha z) - \alpha\operatorname{Cl}_2(2\pi\beta z)}{2\sqrt{5}\pi z}, \tag{39}$$

$$\sum_{k=1}^{\infty}\frac{L_{2k}\zeta(2k)z^{2k}}{k(2k+1)} = -2 + 2\ln(2\pi z) - \frac{\beta\operatorname{Cl}_2(2\pi\alpha z) + \alpha\operatorname{Cl}_2(2\pi\beta z)}{2\pi z}. \tag{40}$$

*Proof.* Evaluate $P(1, i\alpha z)$ and $P(1, -i\beta z)$ and combine these equations according to (20) and the Binet formulas (36). □

**Example 25.** *When $n = 1$ and $z = 1/2$, then from* (37)–(40) *we get*

$$\sqrt{5}\sum_{k=1}^{\infty}\frac{(-1)^{k-1}F_{2k}\zeta(2k)}{4^k k(2k+1)} = \frac{\pi}{4} - 2\ln\alpha + \frac{\pi\sinh(\ln\alpha)}{3} - \frac{\alpha\operatorname{Li}_2(e^{\pi\beta}) + \beta\operatorname{Li}_2(e^{-\pi\alpha})}{\pi},$$

$$\sum_{k=1}^{\infty}\frac{(-1)^{k-1}L_{2k}\zeta(2k)}{4^k k(2k+1)}=\frac{8+\pi\sqrt{5}}{4}-2\ln\pi-\frac{\pi\cosh(\ln\alpha)}{3}+\frac{\alpha\operatorname{Li}_2(e^{\pi\beta})-\beta\operatorname{Li}_2(e^{-\pi\alpha})}{\pi},$$

$$\sum_{k=1}^{\infty}\frac{F_{2k}\zeta(2k)}{4^k k(2k+1)}=\frac{2\ln\alpha}{\sqrt{5}}-\frac{\beta\operatorname{Cl}_2(\pi\alpha)-\alpha\operatorname{Cl}_2(\pi\beta)}{\sqrt{5}\pi},$$

$$\sum_{k=1}^{\infty}\frac{L_{2k}\zeta(2k)}{4^k k(2k+1)}=2\ln\pi-2-\frac{\beta\operatorname{Cl}_2(\pi\alpha)+\alpha\operatorname{Cl}_2(\pi\beta)}{\pi}.$$

**Theorem 26.** *If $n$ is a positive integer and $z$ is any real number such that $0<|z|\le 1/\alpha$, then*

$$\begin{aligned}\frac{\sqrt{5}}{2}\sum_{k=1}^{\infty}\frac{F_{2k}\zeta(2k)z^{2k}}{k(k+n)}&=\frac{\ln\alpha}{n}-\frac{(-1)^n\sqrt{5}(2n-1)!F_{2n}\zeta(2n+1)}{(2\pi z)^{2n}}\\&\quad-(2n-1)!\sum_{j=1}^{n}(-1)^j\frac{\beta^{2j}\operatorname{Cl}_{2j+1}(2\pi\alpha z)-\alpha^{2j}\operatorname{Cl}_{2j+1}(2\pi\beta z)}{(2n-2j)!(2\pi z)^{2j}}\\&\quad+(2n-1)!\sum_{j=1}^{n}(-1)^j\frac{\beta^{2j-1}\operatorname{Cl}_{2j}(2\pi\alpha z)-\alpha^{2j-1}\operatorname{Cl}_{2j}(2\pi\beta z)}{(2n+1-2j)!(2\pi z)^{2j-1}},\end{aligned}$$

$$\begin{aligned}\frac{1}{2}\sum_{k=1}^{\infty}\frac{L_{2k}\zeta(2k)z^{2k}}{k(k+n)}&=-\frac{1}{2n^2}+\frac{\ln(2\pi z)}{n}-\frac{(-1)^n(2n-1)!L_{2n}\zeta(2n+1)}{(2\pi z)^{2n}}\\&\quad-(2n-1)!\sum_{j=1}^{n}\frac{(-1)^j\big(\beta^{2j}\operatorname{Cl}_{2j+1}(2\pi\alpha z)+\alpha^{2j}\operatorname{Cl}_{2j+1}(2\pi\beta z)\big)}{(2n-2j)!(2\pi z)^{2j}}\\&\quad+(2n-1)!\sum_{j=1}^{n}\frac{(-1)^j\big(\beta^{2j-1}\operatorname{Cl}_{2j}(2\pi\alpha z)+\alpha^{2j-1}\operatorname{Cl}_{2j}(2\pi\beta z)\big)}{(2n+1-2j)!(2\pi z)^{2j-1}}.\end{aligned}$$

*Proof.* Both results follow immediately from (10). We omit details. □

**Theorem 27.** *If $n$ is a positive integer and $z$ is any real number such that $0<|z|\le 1/\alpha$, then*

$$\begin{aligned}\sqrt{5}\sum_{k=1}^{\infty}\frac{F_{2k}\zeta(2k)z^{2k}}{k(2k-1+2n)}&=\frac{1}{2n-1}\ln\left(-\alpha^2\,\frac{\sin(\pi\beta z)}{\sin(\pi\alpha z)}\right)\\&\quad-(2n-2)!\sum_{j=1}^{n}(-1)^j\frac{(2n+1-2j)\operatorname{Cl}_{2j}(2\pi\alpha z)-2\pi\alpha z\operatorname{Cl}_{2j-1}(2\pi\alpha z)}{(2n+1-2j)!(2\pi\alpha z)^{2j-1}}\\&\quad+(2n-2)!\sum_{j=1}^{n}(-1)^j\frac{(2n+1-2j)\operatorname{Cl}_{2j}(2\pi\beta z)-2\pi\beta z\operatorname{Cl}_{2j-1}(2\pi\beta z)}{(2n+1-2j)!(2\pi\beta z)^{2j-1}},\end{aligned}$$

$$\sum_{k=1}^{\infty}\frac{L_{2k}\zeta(2k)z^{2k}}{k(2k-1+2n)}=-\frac{2}{(2n-1)^2}+\frac{1}{2n-1}\ln\left(\frac{\pi^2z^2}{\sin(\pi\alpha z)\sin(\pi\beta z)}\right)$$
$$-(2n-2)!\sum_{j=1}^{n}(-1)^j\frac{(2n+1-2j)\operatorname{Cl}_{2j}(2\pi\alpha z)-2\pi\alpha z\operatorname{Cl}_{2j-1}(2\pi\alpha z)}{(2n+1-2j)!(2\pi\alpha z)^{2j-1}}$$
$$-(2n-2)!\sum_{j=1}^{n}(-1)^j\frac{(2n+1-2j)\operatorname{Cl}_{2j}(2\pi\beta z)-2\pi\beta z\operatorname{Cl}_{2j-1}(2\pi\beta z)}{(2n+1-2j)!(2\pi\beta z)^{2j-1}}.$$

*Proof.* Both formulas follow immediately from (11). □

## 4. TWO OTHER INTERESTING SERIES

Our analysis allows the evaluation of two other interesting series which are similar to the series considered by Orr [**14**]. For positive integers $m$ and $n$ with $m\neq n$, let us consider the two series

$$P(m,n,z)=\sum_{k=1}^{\infty}\frac{(-1)^k\zeta(2k)z^{2k}}{k(2k+m)(2k+n)},\qquad 0<|z|\le 1,$$

and its degenerated counterpart for $m\ge 1$

$$Q(m,z)=\sum_{k=1}^{\infty}\frac{(-1)^k\zeta(2k)z^{2k}}{k(2k+m)^2},\qquad 0<z\le 1,$$

Then the following result holds.

**Theorem 28.** *For non-equal positive integers $m$, $n$ and all $z$ such that $0<|z|\le 1$, we have*

$$\sum_{k=1}^{\infty}\frac{(-1)^{k-1}\zeta(2k)z^{2k}}{k(2k+m)(2k+n)}=\frac{m+n}{(mn)^2}+\frac{\pi z}{(m+1)(n+1)}-\frac{\ln(2\pi z)}{mn}$$
$$+\frac{1}{m-n}\left(\frac{(m-1)!}{(2\pi z)^m}\zeta(m+1)-\frac{(n-1)!}{(2\pi z)^n}\zeta(n+1)\right)\tag{41}$$
$$-\frac{1}{m-n}\left((m-1)!\sum_{j=1}^{m}\frac{\operatorname{Li}_{j+1}\left(e^{-2\pi z}\right)}{(m-j)!(2\pi z)^j}-(n-1)!\sum_{j=1}^{n}\frac{\operatorname{Li}_{j+1}\left(e^{-2\pi z}\right)}{(n-j)!(2\pi z)^j}\right)$$

*and, for all* $0 < z \le 1$,

$$\sum_{k=1}^{\infty} \frac{(-1)^{k-1}\zeta(2k) z^{2k}}{k(2k+m)^2} = \frac{2}{m^3} + \frac{\pi z}{(m+1)^2} - \frac{\ln(2\pi z)}{m^2} + \frac{(m-1)!}{(2\pi z)^m}\zeta(m+1) H_{m-1} + \frac{m!}{m^2}\sum_{j=1}^{m} \frac{\mathrm{Li}_{j+1}\left(e^{-2\pi z}\right)}{(m-j)!(2\pi z)^j} - \frac{(m-1)!}{(2\pi z)^m}\sum_{k=1}^{\infty} \frac{\mathrm{Ein}(2\pi z k) + \sum_{j=1}^{m} \frac{\Gamma(j, 2\pi z k)}{j!}}{k^{m+1}}, \tag{42}$$

*where*

$$H_n = \sum_{s=1}^{n} \frac{1}{s}, \quad H_0 = 0,$$

*are the harmonic numbers,* $\Gamma(a,x)$ *is the incomplete gamma function* $\Gamma(a,x) = \int_x^\infty t^{a-1}\, e^{-t}\, dt$, *and* $\mathrm{Ein}(x)$ *being the exponential integral* $\mathrm{Ein}(x) = \int_0^x \frac{1-e^{-t}}{t}\, dt$.

*Proof.* From the partial fraction decomposition

$$\frac{1}{k(2k+m)(2k+n)} = \frac{2}{m(m-n)(2k+m)} - \frac{2}{n(m-n)(2k+n)} + \frac{1}{mnk}$$

we immediately see that we can write

$$P(m,n,z) = \frac{1}{mn} S_1(z) + \frac{2}{m-n}\left(\frac{1}{m} S_{2,m}(z) - \frac{1}{n} S_{2,n}(z)\right),$$

where $S_1(z)$ is defined in (4) and $S_{2,v}(z)$ equals

$$S_{2,v}(z) = \sum_{k=1}^{\infty} \frac{(-1)^k \zeta(2k) z^{2k}}{2k+v},$$

which was also evaluated in Section 2. To complete the proof of (41) we simplify making use of the elementary identity

$$\frac{n(n+1) - m(m+1)}{(m+1)(n+1)(m-n)} + 1 = \frac{mn}{(m+1)(n+1)}.$$

For (42) we start with the partial fraction decomposition

$$\frac{1}{k(2k+m)^2} = \frac{1}{m^2 k} - \frac{2}{m^2(2k+m)} - \frac{2}{m(2k+m)^2}.$$

This yields

$$Q(m,z) = \frac{1}{m^2} S_1(z) - \frac{2}{m^2} S_{2,m}(z) - \frac{2}{m} S_{3,m}(z) \tag{43}$$

with $S_1(z)$ and $S_{2,m}(z)$ as above and

$$S_{3,m}(z) = \sum_{k=1}^{\infty} \frac{(-1)^k \zeta(2k) z^{2k}}{(2k+m)^2}. \tag{44}$$

We have

$$\begin{aligned}
S_{3,m}(z) &= \int_0^1 y^{m-1} S_{2,m}(zy)\, dy \\
&= \frac{1}{2m^2} - \frac{\pi z}{2(m+1)^2} - \frac{m!}{2^{m+1}(\pi z)^m}\zeta(m+1)\int_0^1 \frac{dy}{y} \\
&\quad + \pi z \sum_{j=0}^{m} \frac{m!}{j!}\frac{1}{(2\pi z)^{m+1-j}} \int_0^1 y^{j-1}\operatorname{Li}_{m+1-j}(e^{-2\pi z y})dy \\
&= \frac{1}{2m^2} - \frac{\pi z}{2(m+1)^2} + \frac{m!}{2^{m+1}(\pi z)^m}\int_0^1 \frac{\operatorname{Li}_{m+1}(e^{-2\pi z y}) - \zeta(m+1)}{y} dy \\
&\quad + \frac{\pi z}{(2\pi z)^{m+1}} \sum_{j=1}^{m} \frac{m!}{j!}(2\pi z)^j \int_0^1 y^{j-1}\operatorname{Li}_{m+1-j}(e^{-2\pi z y})dy.
\end{aligned}$$

Next,

$$\begin{aligned}
\int_0^1 \frac{1}{y}\left(\operatorname{Li}_{m+1}(e^{-2\pi z y}) - \zeta(m+1)\right) dy &= \sum_{k=1}^{\infty} \frac{1}{k^{m+1}} \int_0^1 \frac{e^{-2\pi z y k} - 1}{y} dy \\
&= -\sum_{k=1}^{\infty} \frac{\operatorname{Ein}(2\pi z k)}{k^{m+1}}.
\end{aligned}$$

Also,

$$\begin{aligned}
\int_0^1 y^{j-1}\operatorname{Li}_{m+1-j}(e^{-2\pi z y})dy &= \sum_{k=1}^{\infty} \frac{1}{k^{m+1-j}} \int_0^1 y^{j-1} e^{-2\pi z y k} dy \\
&= \sum_{k=1}^{\infty} \frac{1}{k^{m+1-j}} \frac{(j-1)! - \Gamma(j, 2\pi z k)}{(2\pi z k)^j},
\end{aligned}$$

as

$$\int_0^1 x^{j-1} e^{-2\pi a x}\, dx = \frac{(j-1)! - \Gamma(j, 2\pi a)}{(2\pi a)^j}, \qquad \Re(a) > 0.$$

The expression for $S_{3,m}(z)$ becomes

$$\begin{aligned} S_{3,m}(z) &= \frac{1}{2m^2} - \frac{\pi z}{2(m+1)^2} + \frac{m!}{2^{m+1}(\pi z)^m}\left(\sum_{k=1}^{\infty}\frac{\gamma - \operatorname{Ein}(2\pi zk)}{k^{m+1}} - \gamma\zeta(m+1)\right) \\ &\quad + \frac{\pi z}{(2\pi z)^{m+1}}\sum_{j=1}^{m}\frac{m!}{j!}(2\pi z)^j\sum_{k=1}^{\infty}\frac{1}{k^{m+1-j}}\frac{(j-1)! - \Gamma(j,2\pi zk)}{(2\pi zk)^j} \\ &= \frac{1}{2m^2} - \frac{\pi z}{2(m+1)^2} + \frac{m!}{2^{m+1}(\pi z)^m}\zeta(m+1)(H_m - \gamma) \\ &\quad + \frac{m!}{2^{m+1}(\pi z)^m}\sum_{k=1}^{\infty}\frac{\gamma - \operatorname{Ein}(2\pi zk)}{k^{m+1}} - \frac{m!}{2^{m+1}(\pi z)^m}\sum_{j=1}^{m}\frac{1}{j!}\sum_{k=1}^{\infty}\frac{\Gamma(j,2\pi zk)}{k^{m+1}}, \end{aligned}$$

where $\gamma = \lim_{n\to\infty}(H_n - \ln n)$ is the Euler–Mascheroni constant.

The expression for $Q(m,z)$ follows from simplifications according to (43). □

**Example 29.** *Series identity* (41) *yields*

$$\begin{aligned} \sum_{k=1}^{\infty}\frac{(-1)^{k-1}\zeta(2k)}{k(2k+1)(2k+2)} &= \frac{3}{4} + \frac{\pi}{12} - \frac{\ln(2\pi)}{2} + \frac{\zeta(3)}{(2\pi)^2} - \frac{\operatorname{Li}_3(e^{-2\pi})}{(2\pi)^2}, \\ \sum_{k=1}^{\infty}\frac{(-1)^{k-1}\zeta(2k)}{4^k k(2k+1)(2k+2)} &= \frac{3}{4} - \frac{\pi}{12} - \frac{\ln\pi}{2} + \frac{\zeta(3)}{\pi^2} - \frac{\operatorname{Li}_3(e^{-\pi})}{\pi^2}, \\ \sum_{k=1}^{\infty}\frac{(-1)^{k-1}\zeta(2k)}{4^k k(2k+1)(2k+3)} &= \frac{4}{9} - \frac{7\pi}{720} - \frac{\ln\pi}{3} - \frac{\operatorname{Li}_3(e^{-\pi})}{\pi^2} - \frac{\operatorname{Li}_4(e^{-\pi})}{\pi^3}. \end{aligned}$$

**Corollary 30.** *If $m$ and $n$ are non-equal positive integers having the same parity, then*

$$\begin{aligned} \sum_{k=1}^{\infty}\frac{\zeta(2k)}{k(2k+m)(2k+n)} &= -\frac{m+n}{(mn)^2} + \frac{\ln(2\pi)}{mn} \\ &\quad - \frac{1+(-1)^n}{2(m-n)}\left(i^m(m-1)!\frac{\zeta(m+1)}{(2\pi)^m} - i^n(n-1)!\frac{\zeta(n+1)}{(2\pi)^n}\right) \\ &\quad + \frac{1}{m-n}\left((m-1)!\sum_{j=1}^{\lfloor m/2\rfloor}\frac{(-1)^j\zeta(2j+1)}{(m-2j)!(2\pi)^{2j}} - (n-1)!\sum_{j=1}^{\lfloor n/2\rfloor}\frac{(-1)^j\zeta(2j+1)}{(n-2j)!(2\pi)^{2j}}\right). \end{aligned}$$

*Proof.* Evaluate (41) at $z = i$, where $i = \sqrt{-1}$. □

**Example 31.** *We have*

$$\sum_{k=1}^{\infty}\frac{\zeta(2k)}{k(2k+1)(2k+3)}=-\frac{4}{9}+\frac{\ln(2\pi)}{3}-\frac{\zeta(3)}{4\pi^2},$$
$$\sum_{k=1}^{\infty}\frac{\zeta(2k)}{k(k+1)(k+2)}=-\frac{3}{8}+\frac{\ln(2\pi)}{2}-\frac{3\zeta(3)}{2\pi^2},$$
$$\sum_{k=1}^{\infty}\frac{\zeta(2k)}{k(2k+3)(2k+5)}=-\frac{8}{225}+\frac{\ln(2\pi)}{15}-\frac{\zeta(3)}{4\pi^2}+\frac{3\zeta(5)}{4\pi^4}.$$

**Corollary 32.** *If $m$ is a positive even integer and $n$ is a positive odd integer, then*

$$\sum_{k=1}^{\infty}\frac{\zeta(2k)}{k(2k+m)(2k+n)}=-\frac{m+n}{(mn)^2}+\frac{\ln(2\pi)}{mn}-i^m\frac{(m-1)!}{m-n}\frac{\zeta(m+1)}{(2\pi)^m}$$
$$+\frac{1}{m-n}\left((m-1)!\sum_{j=1}^{m/2}\frac{(-1)^j\zeta(2j+1)}{(m-2j)!(2\pi)^{2j}}-(n-1)!\sum_{j=1}^{(n-1)/2}\frac{(-1)^j\zeta(2j+1)}{(n-2j)!(2\pi)^{2j}}\right).$$

**Example 33.** *We have*

$$\sum_{k=1}^{\infty}\frac{\zeta(2k)}{k(k+2)(2k+3)}=-\frac{7}{72}+\frac{1}{6}\ln(2\pi)-\frac{\zeta(3)}{2\pi^2},$$
$$\sum_{k=1}^{\infty}\frac{\zeta(2k)}{k(k+1)(2k+5)}=-\frac{7}{50}+\frac{1}{5}\ln(2\pi)-\frac{2\zeta(3)}{3\pi^2}+\frac{\zeta(5)}{\pi^4}.$$

## 5. CONCLUDING REMARKS

We conclude with the following observations. Another natural generalization of the function $P(n,z)$ is one of the form

$$P(n,z,p)=\sum_{k=1}^{\infty}\frac{(-1)^k\zeta(2k)z^{2k}}{k^p(2k+n)},\quad 0<|z|\le 1,\ p\ge 1. \tag{45}$$

The recursion

$$\frac{1}{k^p(2k+n)}=\frac{1}{nk^p}-\frac{2}{n}\frac{1}{k^{p-1}(2k+n)}$$

can be solved by standard methods to get

$$\frac{1}{k^p(2k+n)} = \sum_{j=0}^{p-1} \frac{(-2)^j}{n^{j+1}k^{p-j}} + \Big(\frac{2}{n}\Big)^p \frac{(-1)^p}{2k+n}.$$

Then

$$P(n,z,p) = \sum_{j=0}^{p-1} \frac{(-2)^j}{n^{j+1}} \sum_{k=1}^{\infty} \frac{(-1)^k \zeta(2k) z^{2k}}{k^{p-j}} + \Big(-\frac{2}{n}\Big)^p S_{2,n}(z). \tag{46}$$

Hence, to extend Theorem 1 to $P(n,z,p)$ it suffices to find a closed form for

$$\sum_{k=1}^{\infty} \frac{(-1)^k \zeta(2k) z^{2k}}{k^p} = \sum_{t=1}^{\infty} \mathrm{Li}_p\Big(-\Big(\frac{z}{t}\Big)^2\Big), \quad p \geq 2, \tag{47}$$

as

$$\sum_{t=1}^{\infty} \mathrm{Li}_1\big(-\big(\tfrac{z}{t}\big)^2\big) = -\ln\Big(\frac{\sinh(\pi z)}{\pi z}\Big) = S_1(z).$$

Similarly, from the partial fraction decomposition

$$\frac{1}{k(2k+m)(2k+n)} = \frac{1}{mnk} + \frac{2}{m(m-n)(2k+m)} - \frac{2}{n(m-n)(2k+n)},$$

we also get

$$\begin{aligned}\frac{1}{k^p(2k+m)(2k+n)} &= \frac{1}{mnk^p} + \frac{2}{m(m-n)} \left( \sum_{j=0}^{p-2} \frac{(-2)^j}{m^{j+1}k^{p-1-j}} + \Big(\frac{2}{m}\Big)^{p-1} \frac{(-1)^{p-1}}{2k+m} \right) \\ &\quad - \frac{2}{n(m-n)} \left( \sum_{j=0}^{p-2} \frac{(-2)^j}{n^{j+1}k^{p-1-j}} + \Big(\frac{2}{n}\Big)^{p-1} \frac{(-1)^{p-1}}{2k+n} \right).\end{aligned}$$

This shows that a closed form evaluation of (47) would also allow us to evaluate the generalization of $P(m,n,z)$ to

$$P(m,n,z,p) = \sum_{k=1}^{\infty} \frac{(-1)^k \zeta(2k) z^{2k}}{k^p(2k+m)(2k+n)}, \quad 0 < |z| \leq 1, \;\; m \neq n, \;\; p \geq 1,$$

via

$$\begin{aligned}P(m,n,z,p) &= \frac{1}{mn} \sum_{k=1}^{\infty} \frac{(-1)^k \zeta(2k) z^{2k}}{k^p} \\ &\quad + \frac{2}{m(m-n)} \left( \sum_{j=0}^{p-2} \frac{(-2)^j}{m^{j+1}} \sum_{k=1}^{\infty} \frac{(-1)^k \zeta(2k) z^{2k}}{k^{p-1-j}} + \Big(-\frac{2}{m}\Big)^{p-1} S_{2,m}(z) \right) \\ &\quad - \frac{2}{n(m-n)} \left( \sum_{j=0}^{p-2} \frac{(-2)^j}{n^{j+1}} \sum_{k=1}^{\infty} \frac{(-1)^k \zeta(2k) z^{2k}}{k^{p-1-j}} + \Big(-\frac{2}{n}\Big)^{p-1} S_{2,n}(z) \right).\end{aligned}$$

Finally, considering the generalization of $Q(m,z)$ to $Q(m,z,p)$ defined by

$$Q(m,z,p)=\sum_{k=1}^{\infty}\frac{(-1)^k\zeta(2k)z^{2k}}{k^p(2k+m)^2},\qquad 0<|z|\le 1,\ p\ge 1,$$

we start with the partial fraction decomposition

$$\frac{1}{k(2k+m)^2}=\frac{1}{m^2k}-\frac{2}{m^2(2k+m)}-\frac{2}{m(2k+m)^2}.$$

The recursion is solved as

$$\frac{1}{k^p(2k+m)^2}=\sum_{j=0}^{p-1}\frac{(-2)^j}{m^{j+2}k^{p-j}}-2\sum_{j=0}^{p-1}\frac{(-2)^j}{m^{j+2}k^{p-1-j}(2k+m)}+\left(\frac{2}{m}\right)^p\frac{(-1)^p}{(2k+m)^2}$$

giving the identity

$$\begin{aligned}Q(m,z,p)=&\sum_{j=0}^{p-1}\frac{(-2)^j}{m^{j+2}}\sum_{k=1}^{\infty}\frac{(-1)^k\zeta(2k)z^{2k}}{k^{p-j}}\\&-2\sum_{j=0}^{p-1}\frac{(-2)^j}{m^{j+2}}P(m,z,p-1-j)+\left(-\frac{2}{m}\right)^pS_{3,m}(z),\end{aligned}$$

where $P(n,z,p)$ is defined in (45) and $S_{3,m}(z)$ is defined in (44).

**Acknowledgements.** The authors sincerely thank the anonymous referee for their thorough examination, insightful review, and constructive feedback.

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(Received 02. 04. 2023.)
(Revised 31. 12. 2024.)

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