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WATER CAPACITY OF DYCK BRIDGES

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We define the notion of capacity, the ability to contain water, for Dyck bridges (also known as Grand-Dyck paths) of semi-length n. The starting point for the analysis is the generating function for Dyck paths according to area and number of steps. We obtain a generating function which tracks the capacity and semi-length of Dyck bridges. The total capacity of all Dyck bridges of semi-length n is investigated as well as the asymptotics thereof as $n \to \infty$.

1. GENERAL INTRODUCTION

A Dyck path is a lattice path, that starts at the origin (0,0) where only up (i.e. $(1,1)$) and down (i.e. $(1,-1)$) steps are allowed. A Dyck path is not allowed below the x-axis and must end on the x-axis. So, a Dyck path with n up steps ends at the point $(2n,0)$; see the definition in [**23**].

Dyck bridges are similar to Dyck paths but are allowed to go below the x-axis. They are also known in the literature as Grand-Dyck paths. See e.g. [**2, 14, 17**].

For further recent work on Dyck paths, see [**1, 3, 8, 9, 10, 12, 13, 18, 22**].

Given an arbitrary Dyck bridge, we mean by its *capacity*, the amount of water the Dyck bridge would retain if the region above it was considered to be a full container with water allowed to escape left and right (but not forwards or backwards) subject to the usual rules of water flow. See [**4, 5, 6, 19, 20**] for the water capacity of Dyck paths, words, compositions and set partitions. The notion of capacity is as the name suggests, a natural two-dimensional model for the extent

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to which various paths may contain water. These can represent the water capacity of dams or underground storage facilities unlike the case of Dyck paths in [**4**] which models only water contained above ground level i.e. above the x-axis. The notion of capacity is general enough to also include permutations as the water containing lattice (see [**7**]). Capacity can also be thought of as a measure of a particular object's degree of "concavity", and as such an interesting concept in itself. This is useful for the theoretical modelling of the holding ability of geomorphic (and human constructed) features and is the motivation of this paper.

We define the area of a Dyck path as the sum of the right hand altitude of each up step because this accords with the usual intuitive notion of area.

As preliminary examples (see Figure 1) the Dyck bridges of length 14 and height 2 (left sketch) or 3 (right sketch), illustrated below have capacities 5 and 8 respectively, as shown by the shading.

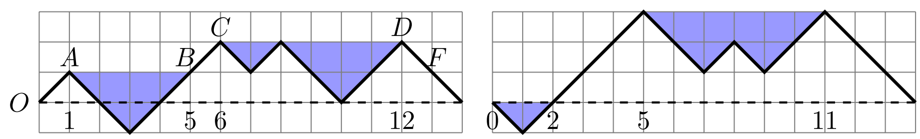

Figure 1: The water capacity of two Dyck bridges of length 14 and heights 2 and 3 respectively. The x axis is shown by the dashed line.

In recent papers by the current authors and others on different lattice path structures such as integer compositions, words over an alphabet $[k]$, bargraphs and integer partitions, amongst others, a standard **localised** technique which is usually termed "adding a slice" has been the main tool used. For numerous examples concerning integer compositions and words over an alphabet, see [**16**] . We characterise this technique as localised because it is based on a recursion where a certain statistic for a lattice path of length n is studied by exploring the effects of adding a single column, thus increasing the length of the path to $n+1$, and altering the generating function by the local effects caused by the additional column.

The study below of (water) capacity involves the natural tendency of the water to distribute itself throughout the entire lattice path structure, depending on the overall shape of the structure. For other references to this approach, see [**4, 6, 7**]. What is involved here is the **global** nature of the lattice path, which in this paper is the particular Dyck bridge. As the reader may discern in what follows, the method employed is the pairing up of leftmost steps of a particular height with rightmost steps of such height and the design of a recursive technique based on reducing such heights step-wise to 0. Such a pairing is inherently global in nature.

2. METHOD TO OBTAIN THE GENERATING FUNCTION FOR THE CAPACITY

We first obtain a generating function for the area contained **under** an arbitrary Dyck path. This is done in Section 3.

Thereafter by a process of sectioning, translating and rotating (explained below), we represent the water contained in a Dyck bridge as areas of disjoint Dyck paths (see for example Figure 2) and employ the formula (1) from Section 3, which was derived in [**21**], to calculate the capacity.

The process of sectioning into maximal pieces is as follows: again consider an arbitrary Dyck bridge of height h. The case $h = 0$ means that the entire bridge is just an inverted Dyck path, whose area represents that capacity of the bridge. Now assume $h \geq 1$. If the height h occurs twice or more, place a vertical line through the x-coordinate of the leftmost such h and likewise at the x-coordinate of the rightmost such h. If there is only one such h, ignore it. Then repeat this procedure twice, once to the left of the leftmost section above and once to the right but now in respect of height $h - 1$, not h. Continue moving both to the left and the right in the same way until height 0 is attained. The sum of the capacity of these sections is the capacity of the Dyck bridge.

So in the first example of Figure 1, the leftmost of the highest multiply occurring points, is C at $x = 6$ and the rightmost is D at $x = 12$. This gives rise to the blue shaded capacity between C and D. On its left there is a section with maximal height 1 lying between the points A and B whilst on the right there is a single maximal height 1 at F which is ignored. The second example in Figure 1 is handled similarly.

Those parts of our Dyck bridge with non-zero capacity are then each vertically translated (until the water surface of that part coincides with the x-axis) and then reflected about $y = 0$. Applying this process to our examples in Figure 1 produces Figure 2 below, i.e., a series of Dyck paths the area of which is precisely the capacity of the original example. Each triangle of width 2 and height 1 is equivalent to 1 unit capacity.

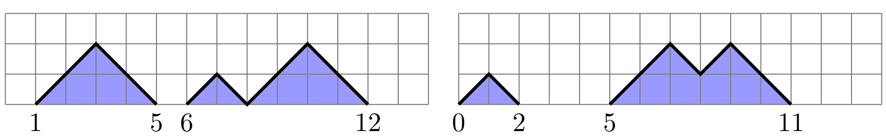

Figure 2: The translated water capacity of the two Dyck bridges in Figure 1

The method of this example is generalized in what follows.

3. GENERATING FUNCTIONS FOR AREAS UNDER DYCK PATHS.

In this section we track the number of steps in any Dyck path by z and the area by q.

From [**15**], the semi-length and area generating function is

$$F(z,q) = \frac{\displaystyle\sum_{n=0}^{\infty} \frac{q^{n^2}(-qz)^n}{\prod_{i=1}^{n}(1-q^i)}}{\displaystyle\sum_{n=0}^{\infty} \frac{q^{n^2}(-z)^n}{\prod_{i=1}^{n}(1-q^i)}}$$

Here, area under a lattice path is taken as the sum of the the starting altitudes of all the up and down steps that encode the path. However we define area to be to the sum of the altitudes of the end points of all steps in the Dyck path which results in

$$G(z,q) := F\left(\frac{z^2}{q}, q^2\right) = \frac{\displaystyle\sum_{n=0}^{\infty} \frac{q^{2n^2+n}\left(-z^2\right)^n}{\prod_{i=1}^{n}(1-q^{2i})}}{\displaystyle\sum_{n=0}^{\infty} \frac{q^{2n^2-n}\left(-z^2\right)^n}{\prod_{i=1}^{n}(1-q^{2i})}} \tag{1}$$

The definition of area used here is equivalent to understanding the area under the Dyck path to made up of triangles of base 2 and height 1. So for example, the Dyck path steps ╱ and ╱╲ would be represented respectively by a term in the generating function of the form $q^{1/2}z$ and qz^2, i.e., area $\frac{1}{2}$ or 1 respectively.

4. GENERATING FUNCTIONS FOR CAPACITY OF DYCK BRIDGES

Now, let us consider all Dyck bridges which attain a maximum height of $h \geq 1$. The Dyck bridge reaches height h at least once. Consider the subpath constituted by the first and last occurrences of the path attaining height h and everything in between. If there is only one occurrence of height h then the "between" part is vacuous.

These subpaths and their containment are in bijection with the Dyck paths and the area under the paths including the vacuous case that was considered earlier. (The bijection is obtained by first translating each subpath vertically down until its initial part is on the x-axis and then reflecting it about $y = 0$). The bijection was illustrated in Figure 2 above.

Hence the generating function $G(z,q)$ defined in Equation (1) is also the generating function for the capacity of this part of the Dyck bridge.

The left up step preceding the inverted Dyck path is not counted in $G(z,q)$ and has up step counted by z. Combining these two parts, we obtain a contribution to the generating function of $zG(z,q)$.

Next, consider the part of our Dyck bridge to the *left* of the above part. If the Dyck bridge passes through level $h-1$ once only on this part, then the generating function for this part is again z.

But if it passes through level $h-1$ more than once on the left of this part, consider this section of the Dyck bridge from the first occurrence of level $h-1$ to the rightmost occurrence of level $h-1$. Similarly, to the right of the rightmost point h, the previous argument also applies yielding altogether the generating function for the left and right side parts to be $z^2G^2(z,q)$.

Reiterating through all heights from h down to 0, we obtain the capacity generating function for all Dyck bridges which attain maximum height r as:

$$C_r(z,q) := z^{2r}G(z,q)^{2r+1}.$$

Finally, summing over all heights, we obtain the capacity generating function

$$C(z,q) := \sum_{r=0}^{\infty} C_r(z,q) = \frac{G(z,q)}{1-z^2G(z,q)^2}. \tag{2}$$

5. GENERATING FUNCTION FOR THE TOTAL CAPACITY OF DYCK BRIDGES

In order to find the total capacity, we begin with a first return decomposition for Dyck paths where we track the area by q and the length by z. The generating function for this is $G(z,q)$, see (1), and the first return decomposition gives rise to the functional equation

$$G(z,q) = 1 + z^2qG(zq,q)G(z,q). \tag{3}$$

We use the notation $\frac{\partial G(z,1)}{\partial q}$ to mean $\frac{\partial G(z,q)}{\partial q}|_{q=1}$. We differentiate (3) with respect to q and set $q=1$ to obtain:

$$\frac{\partial G(z,1)}{\partial q} = z^2\left(G(z,1)\frac{\partial G(z,1)}{\partial q} + G(z,1)\left(G(z,1) + \frac{\partial G(z,1)}{\partial q} + z\frac{\partial G(z,1)}{\partial z}\right)\right)$$

with solution

$$\frac{\partial G(z,1)}{\partial q} = \frac{z^2G(z,1)^2 + z^3G(z,1)\frac{\partial G(z,1)}{\partial z}}{1-2z^2G(z,1)}$$

where

$$G(z,1) = \frac{1-\sqrt{1-4z^2}}{2z^2}$$

and

$$\frac{\partial G(z,1)}{\partial z} = \frac{1-2z^2-\sqrt{1-4z^2}}{z^3\sqrt{1-4z^2}}.$$

This gives

$$\frac{\partial G(z,1)}{\partial q} = \frac{\left(1-\sqrt{1-4z^2}\right)^2}{4z^2\left(1-4z^2\right)}.$$

Now we differentiate (2) with respect to q and set $q=1$ to obtain the total capacity generating function

$$\begin{aligned} T(z) &:= \sum_{r=1}^{\infty} \frac{\partial C_r(z,1)}{\partial q} \\ &= \frac{1+z^2G(z,1)^2}{\left(1-z^2G(z,1)^2\right)^2}\frac{\partial}{\partial q}G(z,1). \end{aligned}$$

This leads to

Theorem 1. *The generating function for the total water capacity of Dyck bridges of length n is given by*

$$T(z) = \frac{\left(1-\sqrt{1-4z^2}\right)^3}{2\left(1-4z^2\right)\left(1-4z^2-\sqrt{1-4z^2}\right)^2}.$$

Remark 2. *The series expansion of $T(z)$ around $z=0$ is:*

$$z^2+9z^4+\mathbf{58z^6}+325z^8+1686z^{10}+8330z^{12}+39796z^{14}+185517z^{16}+848830z^{18}+\cdots.$$

We illustrate the term in bold, for the Dyck bridges of length 6. There are 20 Dyck bridges of length 6; this corresponds to the central binomial coefficient $\binom{2n}{n}$ when n is 3. These 20 paths are illustrated below with the water capacity shaded. The total capacity is indeed 58.

6. ASYMPTOTICS

In this section we find an asymptotic expression for the average water capacity of Dyck bridges of semi-length n as $n \to \infty$.

For semi-length we set $z^2 = x$. Then

$$\frac{\left(-1+\sqrt{1-4x}\right)^3}{2(-1+4x)\left(-1+\sqrt{1-4x}+4x\right)^2} = \frac{1}{2(1-4x)^2} - \frac{1}{2(1-4x)^{3/2}}.$$

For the total capacity of bridges of semi-length n we extract the coefficient of x^n from this which gives

(4)

$$2^{-1+2n}\left(1+n+(-1)^{1+n}\binom{-\frac{3}{2}}{n}\right) = 2^{-1+2n}\left(n-\frac{2\sqrt{n}}{\sqrt{\pi}}+1-\frac{3}{4\sqrt{n}\sqrt{\pi}}\right)+\cdots.$$

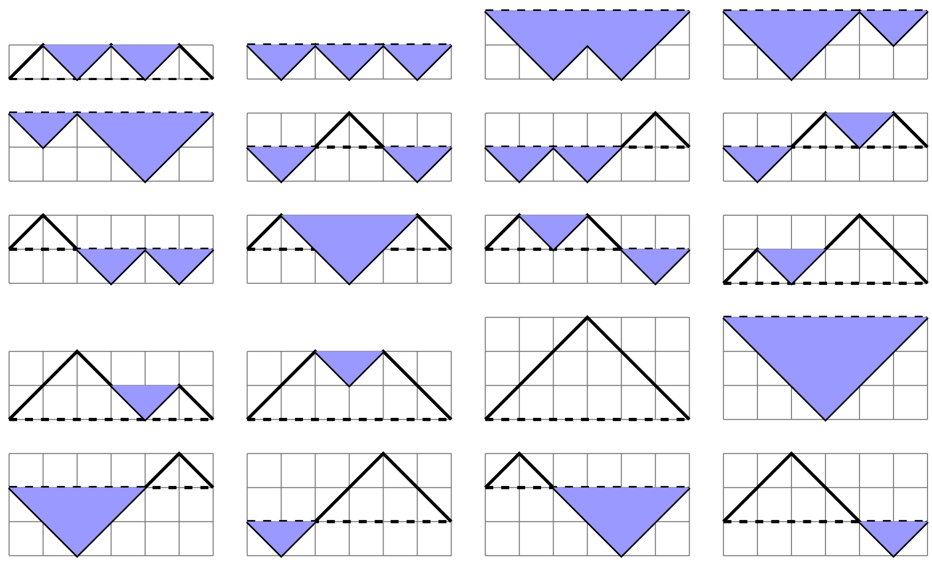

Figure 3: The 20 Dyck bridges of length 6 with their capacity shaded. The dashed line is the x axis.

To obtain the mean value we must divide by the total number of Dyck bridges of semi-length n, i.e., as $n \to \infty$,

$$(5) \qquad \binom{2n}{n} = 2^{2n}\left(\frac{1}{\sqrt{n}\sqrt{\pi}} - \frac{1}{8n^{3/2}\sqrt{\pi}} + \frac{1}{128n^{5/2}\sqrt{\pi}} + \frac{5}{1024n^{7/2}\sqrt{\pi}}\right) + \cdots .$$

Hence, dividing (4) by (5) yields

Theorem 3. *The average water capacity of Dyck bridges of semi-length n, as $n \to \infty$ is*

$$\frac{1}{2}\sqrt{\pi}n^{3/2} - n + \frac{9\sqrt{\pi}\sqrt{n}}{16} - \frac{1}{2} + O(n^{-1/2}).$$

Remark 4. *The constant in front of the main term of order $n^{3/2}$ is $\frac{\sqrt{\pi}}{2} = 0.8862\ldots$ for Dyck bridges, as compared to the constant $\frac{\pi^{5/2}}{3} - 3\sqrt{\pi} = 0.5137\ldots$ for Dyck paths to be found in [**4**].*

7. FURTHER QUESTION

Remark 5. *We leave the following further question for the consideration of the reader. A Dyck walk is defined to be similar to a Dyck bridge with the only difference* *being that the end position of the last step may be at any altitude (i.e., above the x axis) or depth (i.e., below the x axis) and not necessarily on the axis itself. The problem is to obtain a generating function for the total capacity of all Dyck walks.*

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