Applicable Analysis and Discrete Mathematics
available online at http://pefmath.etf.rs

Appl. Anal. Discrete Math. **19** (2025), 076–087.
https://doi.org/10.2298/AADM231013011L

SEVERAL NEW INEQUALITIES OF TANC FUNCTION WITH APPLICATIONS TO SEIFFERT MEAN

*Wen-Hui Li and Bai-Ni Guo**

This paper is dedicated to Professor Dr. Feng Qi for his retirement in 2025

In the paper, the authors discover necessary and sufficient conditions such that an inequality for the ratio between the differences of powers of the tanc function and the cosine function is sound. As a result, the authors derive some new inequalities involving the tangent function, the inverse tangent function, and the second Seiffert mean.

1. INTRODUCTION

For $x \in \mathbb{R}$, the functions

$$\operatorname{sinc} x = \begin{cases} \dfrac{\sin x}{x}, & x \neq 0; \\ 1, & x = 0, \end{cases} \qquad \operatorname{sinhc} x = \begin{cases} \dfrac{\sinh x}{x}, & x \neq 0; \\ 1, & x = 0, \end{cases}$$

$$\operatorname{tanc} x = \begin{cases} \dfrac{\tan x}{x}, & x \neq 0; \\ 1, & x = 0, \end{cases} \qquad \operatorname{tanhc} x = \begin{cases} \dfrac{\tanh x}{x}, & x \neq 0; \\ 1, & x = 0 \end{cases}$$

are called the sinc function, the hyperbolic sinc function, the tanc function, and the hyperbolic tanc function, respectively. The function $\operatorname{sinc} x$ is also called the sine cardinal or sampling function, as well as the function $\operatorname{sinhc} x$ is also called

*Corresponding author. Bai-Ni Guo
2020 Mathematics Subject Classification. Primary 26E60; Secondary 26D05, 33B10
Keywords and Phrases. Seiffert mean; necessary and sufficient condition; inequality; difference; ratio; tanc function; cosine function; tangent function; inverse tangent function

hyperbolic sine cardinal, see the papers [**13, 41**]. The sinc function $\operatorname{sinc} x$ arises frequently in signal processing, the theory of the Fourier transforms, and other areas in mathematics, physics, and engineering.

Inequalities involving the sinc function, the hyperbolic sinc function, the tanc function, and the hyperbolic tanc function, such as Wilker's inequality, Huygens' inequality, Jordan's inequality, Caus–Huygens' inequality, Becker–Stark's inequality, and so on, arouse great enthusiasm of researchers, see the literatures [**2, 4, 5, 6, 7, 8, 9, 10, 11, 12, 15, 16, 17, 18, 19, 20, 22, 23, 25, 26, 29, 30, 32, 33, 34, 35, 36, 37, 38, 39, 47, 48, 49, 50, 51**] and closely-related references therein.

For $s, t > 0$ with $s \neq t$, the Neuman–Sándor mean $M(s,t)$, the first Seiffert means $P(s,t)$, the second Seiffert means $T(s,t)$ are, respectively, defined in [**31, 42, 43**] by

$$M(s,t) = \frac{s-t}{2\operatorname{arsinh}[(s-t)/(s+t)]}, \quad T(s,t) = \frac{s-t}{2\arctan[(s-t)/(s+t)]},$$

and

$$P(s,t) = \frac{s-t}{4\arctan(\sqrt{s/t}\,) - \pi} = \frac{s-t}{2\arcsin[(s-t)/(s+t)]}.$$

For $s, t > 0$, let

$$A(s,t) = \frac{s+t}{2}, \quad H(s,t) = \frac{2st}{s+t}, \quad C(s,t) = \frac{s^2+t^2}{s+t}$$

be the arithmetic, harmonic, and contra-harmonic mean.

In the paper [**27**], Li, Miao, and Guo established three double inequalities for bounding the Neuman–Sándor mean $M(s,t)$ in terms of the arithmetic mean $A(s,t)$ and the contra-harmonic mean $C(s,t)$ and deduced that the double inequalities

$$1 - \beta_1\left(1 - \frac{1}{\cosh^2 x}\right) < \frac{1}{\operatorname{sinhc} x} < 1 - \alpha_1\left(1 - \frac{1}{\cosh^2 x}\right),$$
$$1 - \beta_2\left(1 - \frac{1}{\cosh^4 x}\right) < \left(\frac{1}{\operatorname{sinhc} x}\right)^2 < 1 - \alpha_2\left(1 - \frac{1}{\cosh^4 x}\right),$$

and

$$1 + \alpha_3\left(\cosh^4 x - 1\right) < \operatorname{sinhc}^2 x < 1 + \beta_3\left(\cosh^4 x - 1\right)$$

hold for $x \in (0, \ln(1+\sqrt{2}\,))$ if and only if

$$\alpha_1 \le \frac{1}{6}, \quad \beta_1 \ge 2\left[1 - \ln\left(1+\sqrt{2}\,\right)\right] = 0.237253\ldots,$$
$$\alpha_2 \le \frac{1}{6}, \quad \beta_2 \ge \frac{4}{3}\left[1 - \ln^2\left(1+\sqrt{2}\,\right)\right] = 0.297574\ldots,$$

and

$$\alpha_3 \le \frac{1 - \ln^2\left(1+\sqrt{2}\,\right)}{3\ln^2\left(1+\sqrt{2}\,\right)} = 0.095767\ldots, \quad \beta_3 \ge \frac{1}{6}.$$

In the article [**28**], motivated by the results obtained in the paper [**27**], Li, Shen, and Guo obtained the following two theorems.

Theorem 1. *When $r \ge \frac{8}{25}$, the double inequality*

$$\alpha < \frac{1 - \operatorname{sinhc}^r x}{1 - \cosh^{2r} x} < \beta, \quad x \in \mathbb{R} \tag{1}$$

holds if and only if $\alpha \le 0$ and $\beta \ge \frac{1}{6}$.

When $r < 0$, the right-hand side inequality in (1) holds if and only if $\beta \le \frac{1}{6}$.

Theorem 2. *When $r \ge \frac{1}{2}$, the double inequality*

$$\beta < \frac{1 - \operatorname{sinc}^r x}{1 - \cos^{2r} x} < \alpha, \quad x \in \left(0, \frac{\pi}{2}\right) \tag{2}$$

holds if and only if $\alpha \ge 1 - \left(\frac{2}{\pi}\right)^r$ and $\beta \le \frac{1}{6}$.

When $0 < r \le \frac{8}{25}$, the double inequality (2) holds if and only if $\alpha \ge \frac{1}{6}$ and $\beta \le 1 - \left(\frac{2}{\pi}\right)^r$.

When $r < 0$, the right-hand side inequality in (2) holds if and only if $\beta \ge \frac{1}{6}$.

In this paper, motivated by the double inequalities in (1) and (2), we will find out the largest scalar α and the smallest number β such that the double inequality

$$\beta < \frac{1 - \operatorname{tanc}^r x}{1 - \cos^{2r} x} < \alpha, \quad x \in \left(0, \frac{\pi}{2}\right) \tag{3}$$

holds for $r \in \mathbb{R}$. Hereafter, substituting the double inequalities (3) into the second Seiffert means $T(s,t)$, we derive generalizations of some inequalities for the second Seiffert means $T(s,t)$.

2. LEMMAS

To attain our main purposes, we need the following lemmas.

Lemma 1 ([**3**, Theorem 1.25])**.** *For $a, b \in (-\infty, \infty)$ with $a < b$, let $f, g : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$, differentiable on (a,b), and $g'(x) \ne 0$ on (a,b). If the derivative ratio $\frac{f'(x)}{g'(x)}$ is increasing on (a,b), then the functions*

$$\frac{f(x) - f(a)}{g(x) - g(a)} \quad \text{and} \quad \frac{f(x) - f(b)}{g(x) - g(b)}$$

are also increasing on (a,b).

Lemma 2 ([**44, 46**])**.** *Suppose that the Maclaurin power series $f(x) = \sum_{n=0}^{\infty} a_n x^n$ and $g(x) = \sum_{n=0}^{\infty} b_n x^n$ converge on $(-r,r)$ for $r > 0$ and $b_n > 0$ for all $n \ge 0$. If the sequence $\frac{a_n}{b_n}$ is increasing in $n \ge 0$, then the function $h(x) = \frac{f(x)}{g(x)}$ is increasing in $x \in (0,r)$.*

Lemma 3 ([**1, 14, 21, 24, 40, 45**]). *The power series expansions*

$$\frac{x}{\sin x} = 1 + \sum_{n=1}^{\infty} \frac{2^{2n}-2}{(2n)!}|B_{2n}|x^{2n}, \quad 0 < |x| < \pi,$$

$$\cot x = \frac{1}{x} - \sum_{n=1}^{\infty} \frac{2^{2n}}{(2n)!}|B_{2n}|x^{2n-1}, \quad 0 < |x| < \pi,$$

$$\frac{1}{\sin^2 x} = \frac{1}{x^2} + \sum_{n=1}^{\infty} \frac{2^{2n}(2n-1)}{(2n)!}|B_{2n}|x^{2n-2}, \quad 0 < |x| < \pi$$

hold, where B_{2n} *be the even-indexed Bernoulli numbers which are generated by*

$$\frac{z}{\mathrm{e}^z - 1} = \sum_{n=0}^{\infty} B_n \frac{z^n}{n!} = 1 - \frac{z}{2} + \sum_{n=1}^{\infty} B_{2n} \frac{z^{2n}}{(2n)!}, \quad |z| < 2\pi.$$

3. NEW INEQUALITIES FOR THE TANC FUNCTION

Now we are in a position to state and prove several new inequalities for the tanc function.

Theorem 3. *When* $r > 0$*, the right-hand side of the inequality* (3) *holds if and only if* $\alpha \geq -\frac{1}{3}$*. When* $r \leq -\frac{1}{10}$*, the double inequality* (3) *holds if and only if* $\alpha \leq -\frac{1}{3}$ *and* $\beta \geq 0$.

Proof. Let

$$f_1(x) = \operatorname{tanc}^r x - 1, \quad f_2(x) = \cos^{2r} x - 1, \quad F(x) = \frac{f_1(x)}{f_2(x)}.$$

Then direct differentiation gives

$$\frac{f_1'(x)}{f_2'(x)} = \frac{1}{2}\left(\frac{\sin x}{x\cos^3 x}\right)^{r-1} \frac{\sin x \cos x - x}{x^2 \sin x \cos^3 x}$$

and

$$\begin{aligned}\left[\frac{f_1'(x)}{f_2'(x)}\right]' &= \frac{r-1}{2}\left(\frac{\sin x}{x\cos^3 x}\right)^{r-2} \frac{x - \sin x\cos x + 2x\sin^2 x}{x^2\cos^4 x}\,\frac{\sin x\cos x - x}{x^2\sin x\cos^3 x}\\ &\quad + \frac{1}{2}\left(\frac{\sin x}{x\cos^3 x}\right)^{r-1} \frac{\begin{pmatrix}2x\sin^3 x\cos x + x\sin x\cos x + x^2\\ -2\sin^2 x\cos^2 x - 4x^2\sin^2 x\end{pmatrix}}{x^3\sin^2 x\cos^4 x}\\ &= \frac{1}{2}\left(\frac{\sin x}{x\cos^3 x}\right)^{r-2} \frac{\begin{bmatrix}x^2 + [(r-1)(x - \sin x\cos x\\ +2x\sin^2 x)(\sin x\cos x - x)\\ +2x\sin^3 x\cos x + x\sin x\cos x\\ -2\sin^2 x\cos^2 x - 4x^2\sin^2 x\end{bmatrix}}{x^4\sin x\cos^7 x}\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2}\left(\frac{\sin x}{x\cos^3 x}\right)^{r-2} \frac{\begin{pmatrix}2x\sin x\cos x - \sin^2 x\cos^2 x - x^2\\ +2x\sin^3 x\cos x - 2x^2\sin^2 x\end{pmatrix}}{x^4\sin x\cos^7 x}\\
&\quad\times\left[r - \frac{x\sin x\cos x + 2x^2\sin^2 x + \sin^2 x\cos^2 x - 2x^2}{\begin{pmatrix}2x\sin x\cos x - \sin^2 x\cos^2 x - x^2\\ +2x\sin^3 x\cos x - 2x^2\sin^2 x\end{pmatrix}}\right]\\
&= \frac{1}{2}\left(\frac{\sin x}{x\cos^3 x}\right)^{r-2} \frac{\begin{pmatrix}4x\sin x\cos x - \sin^2 x\cos^2 x - 3x^2\\ -2x\sin x\cos^3 x + 2x^2\cos^2 x\end{pmatrix}}{x^4\sin x\cos^7 x}\\
&\quad\times\left[r - \frac{x\sin x\cos x + \sin^2 x\cos^2 x - 2x^2\cos^2 x}{\begin{pmatrix}4x\sin x\cos x - \sin^2 x\cos^2 x - 3x^2\\ -2x\sin x\cos^3 x + 2x^2\cos^2 x\end{pmatrix}}\right]\\
&= \frac{1}{2}\left(\frac{\sin x}{x\cos^3 x}\right)^{r-2} \frac{\sin x}{x^4\cos^5 x}\left(-1 - 2x\cot x + \frac{8x}{\sin 2x} + \frac{2x^2}{\sin^2 x}\right.\\
&\quad\left. - \frac{12x^2}{\sin^2 2x}\right)\left(r - \frac{1 + \frac{2x}{\sin 2x} - \frac{2x^2}{\sin^2 x}}{-1 - 2x\cot x + \frac{8x}{\sin 2x} + \frac{2x^2}{\sin^2 x} - \frac{12x^2}{\sin^2 2x}}\right)\\
&= \frac{1}{2}\left(\frac{\sin x}{x\cos^3 x}\right)^{r-2} \frac{B(x)\sin x}{x^4\cos^5 x}\left[r - \frac{A(x)}{B(x)}\right],
\end{aligned}
$$

where

$$A(x) = 1 + \frac{2x}{\sin 2x} - \frac{2x^2}{\sin^2 x}$$

and

$$B(x) = -1 - 2x\cot x + \frac{8x}{\sin 2x} + \frac{2x^2}{\sin^2 x} - \frac{12x^2}{\sin^2 2x}.$$

Making use of the series expansions in Lemma 3 results in

$$
\begin{aligned}
A(x) &= \sum_{n=1}^{\infty}\frac{2^{2n}-2}{(2n)!}|B_{2n}|(2x)^{2n} - \sum_{n=1}^{\infty}\frac{2^{2n+1}(2n-1)}{(2n)!}|B_{2n}|x^{2n}\\
&= \sum_{n=2}^{\infty}\frac{2^{2n}(2^{2n}-4n)}{(2n)!}|B_{2n}|x^{2n}\\
&= \sum_{n=2}^{\infty}a_n x^{2n}
\end{aligned}
$$

and

$$B(x) = 2x\sum_{n=1}^{\infty}\frac{2^{2n}}{(2n)!}|B_{2n}|x^{2n-1} + 4\sum_{n=1}^{\infty}\frac{2^{2n}-2}{(2n)!}|B_{2n}|(2x)^{2n}$$

$$
\begin{aligned}
&+2x^2\sum_{n=1}^{\infty}\frac{2^{2n}(2n-1)}{(2n)!}|B_{2n}|x^{2n-2}-12x^2\sum_{n=1}^{\infty}\frac{2^{2n}(2n-1)}{(2n)!}|B_{2n}|(2x)^{2n-2}\\
&=\sum_{n=2}^{\infty}\frac{\left[2^{2n}(7-6n)+4n-8\right]2^{2n}}{(2n)!}|B_{2n}|x^{2n}\\
&=-\sum_{n=2}^{\infty}b_nx^{2n},
\end{aligned}
$$

where

$$
a_n=\frac{2^{2n}(2^{2n}-4n)}{(2n)!}|B_{2n}|>0 \quad \text{and} \quad b_n=\frac{\left[2^{2n}(6n-7)-4n+8\right]2^{2n}}{(2n)!}|B_{2n}|>0.
$$

Let

$$
\xi_n=\frac{a_n}{b_n}=\frac{2^{2n}-4n}{2^{2n}(6n-7)-4n+8}, \quad n\geq 2.
$$

By induction, it is not difficult to verify that the sequence ξ_n is decreasing in $n\geq 2$. Hence, in light of Lemma 2, we conclude that the function $\frac{A(x)}{B(x)}$ is increasing on $\left(0,\frac{\pi}{2}\right)$ and onto $\left(-\frac{1}{10},0\right)$.

It is clear that $B(x)<0$. By Lemma 1, we obtain the following conclusions:

1. when $r>0$, it is obvious that $r-\frac{A(x)}{B(x)}>0$, that is, $\left[\frac{f_1'(x)}{f_2'(x)}\right]'<0$, which is equivalent to that the derivative ratio $\frac{f_1'(x)}{f_2'(x)}$ is decreasing on $\left(0,\frac{\pi}{2}\right)$, therefore the function $F(x)=\frac{f_1(x)}{f_2(x)}=\frac{f_1(x)-f_1(0)}{f_2(x)-f_2(0)}$ is decreasing on $\left(0,\frac{\pi}{2}\right)$;

2. when $r\leq-\frac{1}{10}$, it is easy to deduce that $r-\frac{A(x)}{B(x)}<0$, accordingly the derivative $\left[\frac{f_1'(x)}{f_2'(x)}\right]'>0$, that is, the derivative ratio $\frac{f_1'(x)}{f_2'(x)}$ is increasing on $\left(0,\frac{\pi}{2}\right)$, consequently the function $F(x)=\frac{f_1(x)}{f_2(x)}=\frac{f_1(x)-f_1(0)}{f_2(x)-f_2(0)}$ is increasing on $\left(0,\frac{\pi}{2}\right)$.

Moreover, it is easy to compute $\lim_{x\to 0^+}F(x)=-\frac{1}{3}$. The proof of Theorem 3 is thus complete. □

By Theorem 3, we derive the following corollaries.

Corollary 1. *For $x\in\left(0,\frac{\pi}{2}\right)$, the inequality*

$$
\frac{1-\operatorname{tanc}x}{1-\cos^2 x}<\alpha
$$

holds if and only if $\alpha\geq-\frac{1}{3}$.

Corollary 2. *For $r \ge \frac{1}{10}$ and $0 < x < \frac{\pi}{2}$, the inequality*

$$\alpha < \frac{1 - \left(\frac{1}{\operatorname{tanc} x}\right)^r}{1 - \left(\frac{1}{\cos x}\right)^{2r}}$$

holds if and only if $\alpha \le -\frac{1}{3}$.

Corollary 3. *For $0 < x < \frac{\pi}{2}$, the inequality*

$$\frac{4}{3} - \frac{1}{3}\left(\frac{1}{\cos x}\right)^2 < \frac{1}{\operatorname{tanc} x} < 1 < \frac{4}{3} - \frac{1}{3}\cos^2 x < \operatorname{tanc} x$$

holds.

Corollary 4. *For $0 < x < 1$, the inequality*

$$\frac{4}{3} - \frac{1}{3}(1 + x^2) < \frac{\arctan x}{x} < 1 < \frac{4}{3} - \frac{1}{3(1 + x^2)} < \frac{x}{\arctan x}$$

holds.

4. APPLICATIONS TO THE SECOND SEIFFERT MEAN

Applying Theorem 3, we can obtain the following inequalities for bounding the second Seiffert mean $T(s, t)$.

Theorem 4. *Let $s, t > 0$ with $s \ne t$.*

1. *When $r > 0$, the double inequality*

$$\alpha\left[\frac{A^r(s,t)}{C^r(s,t)} - 1\right] < \frac{T^r(s,t)}{A^r(s,t)} - 1 < \beta\left[\frac{A^r(s,t)}{C^r(s,t)} - 1\right] \tag{4}$$

holds if and only if $\alpha \ge -\frac{1}{3}$ and $\beta \le -\frac{2^r(4^r - \pi^r)}{\pi^r(2^r - 1)}$.

2. *When $r \le -\frac{1}{10}$, the double inequality (4) holds if and only if $\alpha \le -\frac{1}{3}$ and $\beta \ge -\frac{2^r(4^r - \pi^r)}{\pi^r(2^r - 1)}$.*

Proof. Without loss of generality, we assume that $s > t > 0$. Let $u = \frac{s-t}{s+t}$. Then $u \in (0, 1)$ and

$$\frac{T^r(s,t)C^r(s,t) - A^r(s,t)C^r(s,t)}{A^{2r}(s,t) - A^r(s,t)C^r(s,t)} = \frac{T^r(s,t) - A^r(s,t)}{\frac{A^{2r}(s,t)}{C^r(s,t)} - A^r(s,t)} = \frac{\frac{u^r}{\arctan^r u} - 1}{\left(\frac{1}{1+u^2}\right)^r - 1}.$$

Let $u = \tan\theta$. Then $\theta \in \left(0, \frac{\pi}{4}\right)$ and

$$\frac{T^r(s,t)C^r(s,t) - A^r(s,t)C^r(s,t)}{A^{2r}(s,t) - A^r(s,t)C^r(s,t)} = \frac{\operatorname{tanc}^r\theta - 1}{\cos^{2r}\theta - 1} = F(\theta).$$

By virtue of Theorem 3, we obtain that, when $r > 0$, the function $F(\theta)$ is decreasing on the interval $(0, \frac{\pi}{4})$, whereas $F(\theta)$ is increasing on $(0, \frac{\pi}{4})$ for $r < -\frac{1}{10}$.

According to L'Hospital's rule, we have

$$\lim_{\theta\to 0^+} F(\theta) = -\frac{1}{3} \quad \text{and} \quad \lim_{\theta\to(\pi/4)^-} F(\theta) = -\frac{2^r(4^r - \pi^r)}{\pi^r(2^r - 1)}.$$

The proof of Theorem 4 is thus complete. □

As consequences of Theorem 4, the following corollaries can be derived.

Corollary 5. *For $s, t > 0$ with $s \neq t$.*

1. *The double inequality*

$$\alpha_1\left[\frac{C^2(s,t)}{A^2(s,t)} - 1\right] < \frac{A^2(s,t)}{T^2(s,t)} - 1 < \beta_1\left[\frac{C^2(s,t)}{A^2(s,t)} - 1\right]$$

holds if and only if

$$\alpha_1 \leq -\frac{1}{3} \quad \textit{and} \quad \beta_1 \geq \frac{\pi^2}{48} - \frac{1}{3} = -0.127717\dots.$$

2. *The double inequality*

$$\alpha_4\left[\frac{A^2(s,t)}{C^2(s,t)} - 1\right] < \frac{T^2(s,t)}{A^2(s,t)} - 1 < \beta_4\left[\frac{A^2(s,t)}{C^2(s,t)} - 1\right]$$

holds if and only if

$$\alpha_4 \geq -\frac{1}{3} \quad \textit{and} \quad \beta_4 \leq \frac{4}{3} - \frac{64}{\pi^2} = -0.828185\dots.$$

3. *The double inequality*

$$\alpha_2\left[\frac{C(s,t)}{A(s,t)} - 1\right] < \frac{A(s,t)}{T(s,t)} - 1 < \beta_2\left[\frac{C(s,t)}{A(s,t)} - 1\right]$$

holds if and only if

$$\alpha_2 \leq -\frac{1}{3} \quad \textit{and} \quad \beta_2 \geq \frac{\pi}{4} - 1 = -0.214602\dots.$$

4. *The double inequality*

$$\alpha_3\left[\frac{A(s,t)}{C(s,t)} - 1\right] < \frac{T(s,t)}{A(s,t)} - 1 < \beta_3\left[\frac{A(s,t)}{C(s,t)} - 1\right]$$

holds if and only if

$$\alpha_3 \geq -\frac{1}{3} \quad \textit{and} \quad \beta_3 \leq 2 - \frac{8}{\pi} = -0.546479\dots.$$

Acknowledgements. The authors appreciate the anonymous referees for their valuable comments and careful adding of two references on the original version of this paper.

REFERENCES

1. M. Abramowitz and I. A. Stegun (Eds), *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, National Bureau of Standards, Applied Mathematics Series **55**, Reprint of the 1972 edition, Dover Publications, Inc., New York, 1992.
2. R. P. Agarwal, E. Karapinar, M. Kostić, J. Cao, and W.-S. Du, *A brief overview and survey of the scientific work by Feng Qi*, Axioms **11** (2022), no. 8, Art. No. 385, 27 pages; available online `https://doi.org/10.3390/axioms11080385`.
3. G. D. Anderson, M. K. Vamanamurthy, and M. Vuorinen, *Conformal Invariants, Inequalities, and Quasiconformal Maps*, John Wiley & Sons, New York, 1997.
4. Y. J. Bagul, M. Kostić, C. Chesneau, and R. M. Dhaigude, *On the generalized Becker–Stark type inequalities*, Acta Univ. Sapientiae Math. **13** (2021), no. 1, 88–104; available online at `https://doi.org/10.2478/ausm-2021-0005`.
5. C.-P. Chen, *Sharp Wilker- and Huygens-type inequalities for inverse trigonometric and inverse hyperbolic functions*, Integral Transforms Spec. Funct. **23** (2012), no. 12, 865–873; available online at `https://doi.org/10.1080/10652469.2011.644851`.
6. C.-P. Chen and W.-S. Cheung, *Wilker- and Huygens-type inequalities and solution to Oppenheim's problem*, Integral Transforms Spec. Funct. **23** (2012), no. 5, 325–336; available online at `https://doi.org/10.1080/10652469.2011.586637`.
7. C.-P. Chen and F. Qi, *Inequalities of some trigonometric functions*, Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. **15** (2004), 71–78; available online at `http://dx.doi.org/10.2298/PETF0415071C`.
8. C.-P. Chen, J.-W. Zhao, and F. Qi, *Three inequalities involving hyperbolically trigonometric functions*, Octogon Math. Mag. **12** (2004), no. 2, 592–596.
9. X.-D. Chen, J. Ma, J. Jin, and Y. Wang, *A two-point-Padé-approximant-based method for bounding some trigonometric functions*, J. Inequal. Appl. **2018**, Art. 140, 15 pages; available online at `https://doi.org/10.1186/s13660-018-1726-7`.
10. X.-D. Chen, J. Shi, Y. Wang, and P. Xiang, *A new method for sharpening the bounds of several special functions*, Results Math. **72** (2017), 695–702; available online at `https://doi.org/10.1007/s00025-017-0700-x`.
11. R. M. Dhaigude and Y. J. Bagul, *A note on the Becker-Stark type inequalities*, Probl. Anal. Issues Anal. **11(29)** (2022), no. 1, 58–66; available online at `https://doi.org/10.15393/j3.art.2022.10770`.
12. W.-S. Du, R. P. Agarwal, E. Karapinar, M. Kostić, and J. Cao, *Preface to the Special Issue "A Themed Issue on Mathematical Inequalities, Analytic Combinatorics and Related Topics in Honor of Professor Feng Qi"*, Axioms **12** (2023), no. 9, Article 846, 5 pages; available online at `https://doi.org/10.3390/axioms12090846`.
13. W. B. Gearhart and H. S. Shultz, *The function* $\frac{\sin x}{x}$, College Math. J. **21** (1990), no. 2, 90–99; available online at `https://doi.org/10.1080/07468342.1990.11973290`.

14. I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series, and Products*, Translated from the Russian, Translation edited and with a preface by Daniel Zwillinger and Victor Moll, Eighth edition, Revised from the seventh edition, Elsevier/Academic Press, Amsterdam, 2015; available online at https://doi.org/10.1016/B978-0-12-384933-5.00013-8.

15. B.-N. Guo, W. Li, and F. Qi, *Proofs of Wilker's inequalities involving trigonometric functions*, The 7th International Conference on Nonlinear Functional Analysis and Applications, Chinju, South Korea, August 6-10, 2001; Inequality Theory and Applications, Volume **3**, Yeol Je Cho, Jong Kyu Kim, and Sever S. Dragomir (Eds), Nova Science Publishers, Hauppauge, NY, ISBN 1-59033-866-9, 2003, pp. 109–112.

16. B.-N. Guo, D. Lim, and F. Qi, *Maclaurin's series expansions for positive integer powers of inverse (hyperbolic) sine and tangent functions, closed-form formula of specific partial Bell polynomials, and series representation of generalized logsine function*, Appl. Anal. Discrete Math. **16** (2022), no. 2, 427–466; available online at https://doi.org/10.2298/AADM210401017G.

17. B.-N. Guo and F. Qi, *Alternative proofs for inequalities of some trigonometric functions*, Internat. J. Math. Ed. Sci. Tech. **39** (2008), no. 3, 384–389; available online at http://dx.doi.org/10.1080/00207390701639516.

18. B.-N. Guo, B.-M. Qiao, F. Qi, and W. Li, *On new proofs of Wilker's inequalities involving trigonometric functions*, Math. Inequal. Appl. **6** (2003), no. 1, 19–22; available online at https://doi.org/10.7153/mia-06-02.

19. Y. Hua and F. Qi, *Sharp inequalities between the hyperbolic cosine function and the sine and cosine functions*, Pakistan J. Statist. **29** (2013), no. 3, 315–321.

20. Z.-H. Huo, D.-W. Niu, J. Cao, and F. Qi, *A generalization of Jordan's inequality and an application*, Hacet. J. Math. Stat. **40** (2011), no. 1, 53–61.

21. A. Jeffrey, *Handbook of Mathematical Formulas and Integrals*, Third edition, Elsevier Academic Press, San Diego, CA, 2004.

22. W.-D. Jiang, Q.-M. Luo, and F. Qi, *Refinements and sharpening of some Huygens and Wilker type inequalities*, Turkish J. Anal. Number Theory **2** (2014), no. 4, 134–139; available online at https://doi.org/10.12691/tjant-2-4-6.

23. W.-D. Jiang, M.-K. Wang, Y.-M. Chu, Y.-P. Jiang, and F. Qi, *Convexity of the generalized sine function and the generalized hyperbolic sine function*, J. Approx. Theory **174** (2013), 1–9; available online at http://dx.doi.org/10.1016/j.jat.2013.06.005.

24. J.-L. Li, *An identity related to Jordan's inequality*, Int. J. Math. Math. Sci. **2006**, Art. ID 76782, 6 pages; available online at https://doi.org/10.1155/IJMMS/2006/76782.

25. W.-H. Li and B.-N. Guo, *Several inequalities for bounding sums of two (hyperbolic) sine cardinal functions*, Filomat **38** (2024), no. 11, 3937–3943; available online at https://doi.org/10.2298/FIL2411937L.

26. W.-H. Li and B.-N. Guo, *Several sharp inequalities involving (hyperbolic) tangent, tanc, cosine, and their reciprocals*, Math. Slovaca (2024), accepted on 22 July 2024; available online at https://www.researchgate.net/publication/382448610.

27. W.-H. Li, P. Miao, and B.-N. Guo, *Bounds for the Neuman–Sándor mean in terms of the arithmetic and contra-harmonic means*, Axioms **11** (2022), no. 5, Art. 236, 12 pages; available online at https://doi.org/10.3390/axioms11050236.

28. W.-H. Li, Q.-X. Shen, and B.-N. Guo, *Several double inequalities for integer powers of the sinc and sinhc functions with applications to the Neuman–Sándor mean and the first Seiffert mean*, Axioms **11** (2022), no. 7, Art. 304, 12 pages; available online at https://doi.org/10.3390/axioms11070304.

29. X.-L. Liu, H.-X. Long, and F. Qi, *A series expansion of a logarithmic expression and a decreasing property of the ratio of two logarithmic expressions containing sine*, Mathematics **11** (2023), no. 14, Art. 3107, 12 pages; available online at https://doi.org/10.3390/math11143107.

30. C. Mortici, *The natural approach of Wilker–Cusa–Huygens inequalities*, Math. Inequal. Appl. **14** (2011), no. 3, 535–541; available online at https://doi.org/10.7153/mia-14-46.

31. E. Neuman and J. Sándor, *On the Schwab-Borchardt mean*, Math. Pannon. **14** (2003), no. 2, 253–266.

32. D.-W. Niu, J. Cao, and F. Qi, *Generalizations of Jordan's inequality and concerned relations*, Politehn. Univ. Bucharest Sci. Bull. Ser. A Appl. Math. Phys. **72** (2010), no. 3, 85–98.

33. D.-W. Niu, Z.-H. Huo, J. Cao, and F. Qi, *A general refinement of Jordan's inequality and a refinement of L. Yang's inequality*, Integral Transforms Spec. Funct. **19** (2008), no. 3, 157–164; available online at https://doi.org/10.1080/10652460701635886.

34. D.-W. Niu, Y.-J. Zhang, and F. Qi, *A double inequality for the harmonic number in terms of the hyperbolic cosine*, Turkish J. Anal. Number Theory **2** (2014), no. 6, 223–225; available online at https://doi.org/10.12691/tjant-2-6-6.

35. F. Qi and B.-N. Guo, *Alternative proofs for summation formulas of some trigonometric series*, Glob. J. Math. Anal. **5** (2017), no. 2, 44–46; available online at https://doi.org/10.14419/gjma.v5i2.7471.

36. F. Qi and Q.-D. Hao, *Refinements and sharpenings of Jordan's and Kober's inequality*, Mathematics and Informatics Quarterly **8** (1998), no. 3, 116–120.

37. F. Qi and M. Mahmoud, *Bounding the gamma function in terms of the trigonometric and exponential functions*, Acta Sci. Math. (Szeged) **83** (2017), no. 1-2, 125–141; available online at https://doi.org/10.14232/actasm-016-813-x.

38. F. Qi, D.-W. Niu, and B.-N. Guo, *Refinements, generalizations, and applications of Jordan's inequality and related problems*, J. Inequal. Appl. **2009**, Art. ID 271923, 52 pages; available online at https://doi.org/10.1155/2009/271923.

39. F. Qi, X.-T. Shi, F.-F. Liu, and Z.-H. Yang, *A double inequality for an integral mean in terms of the exponential and logarithmic means*, Period. Math. Hungar. **75** (2017), no. 2, 180–189; available online at http://dx.doi.org/10.1007/s10998-016-0181-9.

40. F. Qi and P. Taylor, *Series expansions for powers of sinc function and closed-form expressions for specific partial Bell polynomials*, Appl. Anal. Discrete Math. **18** (2024), no. 1, 92–115; available online at https://doi.org/10.2298/AADM230902020Q.

41. J. Sánchez-Reyes, *The hyperbolic sine cardinal and the catenary*, College Math. J. **43** (2012), no. 4, 285–290; available online at https://doi.org/10.4169/college.math.j.43.4.285.

42. H.-J. Seiffert, *Aufgabe β 16*, Wurzel **29** (1995), 221–222.

43. H.-J. Seiffert, *Problem 887*, Nieuw Arch. Wiskd. **11** (1993), no. 2, 176–176.

44. S. Simić and M. Vuorinen, *Landen inequalities for zero-balanced hypergeometric functions*, Abstr. Appl. Anal. **2012**, Art. ID 932061, 11 pages; available online at https://doi.org/10.1155/2012/932061.

45. S.-H. Wu and H. M. Srivastava, *A further refinement of Wilker's inequality*, Integral Transforms Spec. Funct. **19** (2008), no. 9-10, 757–765; available online at https://doi.org/10.1080/10652460802340931.

46. Z.-H. Yang, Y.-M. Chu, and M.-K. Wang, *Monotonicity criterion for the quotient of power series with applications*, J. Math. Anal. Appl. **428** (2015), no. 1, 587–604; available online at https://doi.org/10.1016/j.jmaa.2015.03.043.

47. L. Yin, L.-G. Huang, and F. Qi, *Some inequalities for the generalized trigonometric and hyperbolic functions*, Turkish J. Anal. Number Theory **2** (2014), no. 3, 96–101; available online at https://doi.org/10.12691/tjant-2-3-8.

48. L. Yin, X.-L. Lin, and F. Qi, *Monotonicity, convexity and inequalities related to complete (p,q,r)-elliptic integrals and generalized trigonometric functions*, Publ. Math. Debrecen **97** (2020), no. 1-2, 181–199; available online at https://doi.org/10.5486/PMD.2020.8793.

49. L. Zhu, *A source of inequalities for circular functions*, Comput. Math. Appl. **58** (2009), no. 10, 1998–2004; available online at https://doi.org/10.1016/j.camwa.2009.07.076

50. L. Zhu, *High precision Wilker-type inequality of fractional powers*, Mathematics **9** (2021), no. 13, Art. 1476, 24 pages; https://doi.org/10.3390/math9131476.

51. L. Zhu, *New inequalities of Wilker's type for circular functions*, AIMS Math. **5** (2020), no. 5, 4874–4888; available online at https://doi.org/10.3934/math.2020311.

(Received 13.10.2023.)
(Revised 08.04.2025.)

Wen-Hui Li
School of Economics
Henan Kaifeng College of Science Technology and Communication
Kaifeng 475001, Henan, China
E-mail: *wen.hui.li@foxmail.com*
URL: https://orcid.org/0000-0002-1848-8855

Bai-Ni Guo
School of Mathematics and Informatics
Henan Polytechnic University
Jiaozuo, Henan, 454010, China
Independent researcher, University Village
Dallas, TX 75252, USA
E-mail: *bai.ni.guo@gmail.com*
URL: https://orcid.org/0000-0001-6156-2590