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SHARP BOUNDS FOR THE ELLIPTIC INTEGRAL OF THE FIRST KIND IN TERMS OF TWO CLASSES OF LOGARITHMIC-TYPE FUNCTIONS

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For $r \in (0,1)$, let $\mathcal{K}(r)$ be the complete elliptic integral of the first kind. In this paper, by introducing two classes of logarithmic-type functions, and proving the monotonicity and absolutely monotonicity properties of certain functions involving $\mathcal{K}(r)$ and the logarithmic-type functions, several new functional inequalities for $\mathcal{K}(r)$ will be derived, which improve some previous known results.

1. INTRODUCTION

For $r \in (0,1)$, Legendre's complete elliptic integrals of the first and second kinds are defined by

$$\mathcal{K} \equiv \mathcal{K}(r) = \int_0^{\pi/2} \frac{dt}{\sqrt{1-r^2\sin^2 t}} = \int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-r^2t^2)}},$$
$$\mathcal{E} \equiv \mathcal{E}(r) = \int_0^{\pi/2} \sqrt{1-r^2\sin^2 t}\,dt = \int_0^1 \sqrt{\frac{1-r^2t^2}{1-t^2}}\,dt,$$

respectively (see [**1, 6**]). It is clear that $\mathcal{K}$ is strictly increasing and $\mathcal{E}$ is strictly decreasing on $(0,1)$, and the limiting values are $\mathcal{K}(0^+) = \mathcal{E}(0^+) = \pi/2$, $\mathcal{E}(1^-) = 1$, $\mathcal{K}(1^-) = +\infty$. Indeed, $\mathcal{K}$ has the following limiting behaviour

$$\lim_{r\to 1^-}\left(\mathcal{K}(r) + \log\sqrt{1-r^2}\right) = \log 4 \qquad (1)$$

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(see [**6**, (3.4)]).

It is well known that the complete elliptic integrals have many important applications in mathematics and physics (see [**9, 12, 18**]). The estimates of the arclength of an ellipse in terms of elementary functions, the computation of electromagnetic and related quantities and so on all depend on $\mathcal{K}$ and $\mathcal{E}$. In the past 30 years, they have occurred frequently in geometric function theory, especially in conformal and quasiconformal mappings (see [**5, 7, 8, 16, 17, 21, 22, 32**]). In particular, several conformal invariants and distortion functions in quasiconformal mappings can be expressed by the complete elliptic integral of the first kind. For example, let $\mathbf{B}^2$ be the unit disk in the complex plane, then the modulus of the plane Grötzsch ring $\mathbf{B}^2 \setminus [0, r]$ $(0 < r < 1)$ is

$$\mu(r) = \frac{\pi}{2}\frac{\mathcal{K}(\sqrt{1-r^2})}{\mathcal{K}(r)}$$

(see [**6**, Chapter 5, 10]).

In face of the above mentioned important applications, the complete elliptic integrals $\mathcal{K}$ and $\mathcal{E}$, especially $\mathcal{K}$, have been studied deeply in many papers for ten years, and a lot of new properties and functional inequalities, including the absolute monotonicity and asymptotic bounds for some certain functions involving $\mathcal{K}$, $\mathcal{E}$ and their generalizations, have been proved in [**2, 3, 4, 10, 11, 14, 15, 20, 23, 24, 25, 26, 27, 28, 29, 30, 31, 35, 37, 38, 39, 40, 41, 43, 44, 45, 46, 47**].

In 2004, Alzer and Qiu [**2**] showed that

$$\frac{\pi}{2}\left(\frac{\mathrm{arth} r}{r}\right)^{\alpha_1} < \mathcal{K}(r) < \frac{\pi}{2}\left(\frac{\mathrm{arth} r}{r}\right)^{\beta_1},$$

$$\frac{\pi}{2}\left[\frac{\alpha_2}{L(1,r')} + \frac{1-\alpha_2}{A(1,r')}\right] < \mathcal{K}(r) < \frac{\pi}{2}\left[\frac{\beta_2}{L(1,r')} + \frac{1-\beta_2}{A(1,r')}\right]$$

hold for all $r \in (0,1)$. Here and in what follows we denote by $r' = \sqrt{1-r^2}$ for $r \in (0,1)$,

$$L(1,r') = \frac{1-r'}{\log(1/r')}, \quad A(1,r') = \frac{1+r'}{2}$$

are the logarithmic and arithmetic means of 1 and r', and $\mathrm{arth}(\cdot)$ stands for the inverse hyperbolic tangent function.

Yang, Tian and Zhu [**43**] established another sharp lower bound for $\mathcal{K}$ in terms of $\mathrm{arth}(\cdot)$:

$$\mathcal{K}(r) > \frac{\pi}{2}\left[1-\alpha_3+\alpha_3\left(\frac{\mathrm{arth} r}{r}\right)^{\beta_3}\right]^{1/\beta_3}$$

for all $r \in (0,1)$ with the best possible constants

$$\alpha_3 = \frac{3}{4}, \quad \beta_3 = \frac{1}{10}.$$

In 2020, Alzer and Richards [**4**] showed that $r \mapsto (8/5 - \log r')/\mathcal{K}(\sqrt{r})$ is strictly concave on $(0,1)$, and therefore obtained the following upper bound for $\mathcal{K}(r)$:

$$\mathcal{K}(r) < \frac{\pi}{2}\frac{16 - 5\log(1-r^2)}{16 + (5\pi - 16)r^2} \tag{2}$$

for all $r \in (0,1)$.

Inspired by (2), we introduce a class of logarithmic-type functions defined on $r \in (0,1)$ with parameter $p \in [0,+\infty)$ as follows

$$A_p(r) = \frac{\pi}{2}\frac{1 - p\log(1-r^2)}{1 + (p\pi - 1)r^2}. \tag{3}$$

By (3), inequality (2) can be rewritten as $\mathcal{K}(r) < A_{5/16}(r)$. Moreover, it is not difficult to verify that, for any fixed $r \in (0,1)$, $p \mapsto A_p(r)$ is strictly decreasing from $[0,+\infty)$ onto $(\log(1/r')/r^2, \pi/(2r'^2)]$ (see Lemma 6(1)), and it was proved in [**6**, Theorem 3.21(7), Exercise 3.43(9)] that $\pi/(2r'^2) > \mathcal{K}(r) > \log(1/r')/r^2$ for all $r \in (0,1)$. Then it is natural to ask that what are the best possible constants $\alpha, \beta \in [0,+\infty)$ such that $A_\alpha(r) < \mathcal{K}(r) < A_\beta(r)$ takes place for each $r \in (0,1)$. The following Theorem 1 is to answer the question.

Theorem 1. *Let $\alpha, \beta \in [0,+\infty)$. Then the inequality*

$$A_\alpha(r) < \mathcal{K}(r) < A_\beta(r)$$

holds for all $r \in (0,1)$ if and only if $\alpha \ge \alpha_0^* = 1/(4\log 2) = 0.360\cdots$ and $\beta \le \beta_0^* = 3/[4(\pi - 1)] = 0.350\cdots$.

Similarly, in this paper, another class of logarithmic-type functions involving inverse hyperbolic tangent function will also be considered. For $q \in [0,+\infty)$, let

$$B_q(r) = \frac{\pi}{2}\frac{1 + 2qr\,\mathrm{arth}r}{1 + (q\pi - 1)r^2}. \tag{4}$$

Then, for any fixed $r \in (0,1)$, the function $q \mapsto B_q(r)$ is also strictly decreasing from $[0,+\infty)$ onto $(\mathrm{arth}r/r, \pi/(2r'^2)]$ (see Lemma 6(2)). Since it was proved in [**6**, Theorem 3.21(7) and Exercises 3.43(30)] that the double inequality

$$\frac{\mathrm{arth}r}{r} < \mathcal{K}(r) < \frac{\pi}{2r'^2}$$

is valid for all $r \in (0,1)$, a similar problem is raised that what are the best possible upper and lower bounds for $\mathcal{K}(r)$ in terms of $B_q(r)$. Our second theorem is to solve this problem.

Theorem 2. *Let $\alpha, \beta \in [0,+\infty)$. Then the inequality*

$$B_\alpha(r) < \mathcal{K}(r) < B_\beta(r)$$

holds for all $r \in (0,1)$ if and only if $\alpha \ge \alpha_1^* = 1/(2\log 2) = 0.721\cdots$ and $\beta \le \beta_1^* = 3/[4(\pi - 2)] = 0.656\cdots$.

In addition, we shall study the difference and ratio between $\mathcal{K}(r)$ and $A_{\beta_0^*}(r)$, and show their monotonicity property or absolutely monotonicity property. Recall that, a function f is called absolutely monotonic on the interval I if

$$f^{(k)}(x) \geq 0 \text{ for } x \in I \text{ and } k = 0, 1, 2, \cdots$$

(see [**33**]). In this paper, we also obtain

Theorem 3. *Let*

$$\begin{aligned} F(r) =& \mathcal{K}(r) - A_{\beta_0^*}(r), \\ G(r) =& \frac{\mathcal{K}(r)}{A_{\beta_0^*}(r)}. \end{aligned}$$

Then

(1) $-F$ *is absolutely monotonic on* $(0,1)$*, and the function* $F(r)/r^4$ *is strictly decreasing on* $(0,1)$ *with the range* $(2(3\log 2 - \pi + 1)/3, \pi(5\pi - 17)/[128(\pi - 1)])$. *Consequently, for all* $r \in (0,1)$*, the double inequality*

$$\begin{aligned} \frac{2(3\log 2 - \pi + 1)}{3} r^4 + \frac{\pi}{2}\left[\frac{4(\pi - 1) - 3\log(1 - r^2)}{4(\pi - 1) + (4 - \pi)r^2}\right] < \mathcal{K}(r) \\ < \frac{\pi(5\pi - 17)}{128(\pi - 1)} r^4 + \frac{\pi}{2}\left[\frac{4(\pi - 1) - 3\log(1 - r^2)}{4(\pi - 1) + (4 - \pi)r^2}\right]; \end{aligned} \tag{5}$$

(2) G *is piecewise monotone on* $(0,1)$*, first decreasing then increasing. Consequently, for all* $r \in (0,1)$*, the inequality*

$$G(r_0)A_{\beta_0^*}(r) \leq \mathcal{K}(r) < A_{\beta_0^*}(r).$$

Here r_0 *is the unique extreme point of* G *on* $(0,1)$.

To end this section, let us recall some basic derivative formulas and series expansions which $\mathcal{K}$ and $\mathcal{E}$ satisfy:

$$\frac{d\mathcal{K}}{dr} = \frac{\mathcal{E} - r'^2\mathcal{K}}{rr'^2}, \quad \frac{d\mathcal{E}}{dr} = \frac{\mathcal{E} - \mathcal{K}}{r}, \quad \frac{d(\mathcal{E} - r'^2\mathcal{K})}{dr} = r\mathcal{K}, \quad \frac{d(\mathcal{K} - \mathcal{E})}{dr} = \frac{r\mathcal{E}}{r'^2},$$

$$\mathcal{K}(r) = \frac{\pi}{2}\sum_{n=0}^{\infty}\left[\frac{\left(\frac{1}{2}\right)_n}{n!}\right]^2 r^{2n}, \quad \mathcal{E}(r) = \frac{\pi}{2}\sum_{n=0}^{\infty}\frac{\left(-\frac{1}{2}\right)_n\left(\frac{1}{2}\right)_n}{(n!)^2} r^{2n}.$$

Here $(a)_0 = 1$ and $(a)_n = \prod_{k=0}^{n-1}(a + k)$ for $n \in \mathbf{N}$.

2. LEMMAS

Lemma 4 ([**6**, Theorem 1.25])**.** Let $a, b \in \mathbf{R}$ with $a < b$, $f, g : [a, b] \to \mathbf{R}$ be continuous on $[a, b]$ and differentiable on (a, b). Let $g'(x) \neq 0$ on (a, b). Then, if f'/g' is increasing (decreasing) on (a, b), then so are the functions

$$\frac{f(x) - f(a)}{g(x) - g(a)} \quad \text{and} \quad \frac{f(x) - f(b)}{g(x) - g(b)}.$$

If f'/g' is strict monotone, then the monotonicity in the conclusion is also strict.

However, sometimes the ratio f'/g' is not monotone on the whole interval. An auxiliary function $H_{f,g}$, called H-function, has been introduced and used frequently to judge the monotonicity property by Yang et al in [**27, 30, 34, 36, 42**] recently. That is, let $a, b \in \mathbf{R}$ with $a < b$, and let f and g be differentiable on (a, b) and $g' \neq 0$ on (a, b). Then $H_{f,g}$ is defined by

$$H_{f,g} := \frac{f'}{g'} g - f. \tag{6}$$

For some basic properties of $H_{f,g}$, the readers can refer to [**34**]. In particular, if f and g are twice differentiable, then we obtain

$$\left(\frac{f}{g}\right)' = \frac{g'}{g^2}\left(\frac{f'}{g'} g - f\right) = \frac{g'}{g^2} H_{f,g}, \tag{7}$$

$$H'_{f,g} = \left(\frac{f'}{g'}\right)' g. \tag{8}$$

Lemma 5 ([**36**, Theorem 2.1],[**19**, Lemma 2.1])**.** *Suppose that the power series $f(x) = \sum_{n=0}^{\infty} a_n x^n$ and $g(x) = \sum_{n=0}^{\infty} b_n x^n$ have the radius of convergence $r > 0$ with $b_n > 0$ for all $n \in \mathbb{N}_0$. Let $h(x) = f(x)/g(x)$, and $H_{f,g}(x)$ be defined as* (6)*, then the following statements are true:*

(1) *If the non-constant sequence $\{a_n/b_n\}_{n=0}^{\infty}$ is increasing (decreasing), then $h(x)$ is strictly increasing (decreasing resp.) on $(0, r)$;*

(2) *If the non-constant sequence $\{a_n/b_n\}$ is increasing (decreasing) for $0 \leq n \leq n_0$ and decreasing (increasing) for $n > n_0$, then the function h is strictly increasing (decreasing) on $(0, r)$ if and only if $H_{f,g}(r^-) \geq (\leq) 0$. Moreover, if $H_{f,g}(r^-) < (>) 0$, then there exists $\eta \in (0, r)$ such that $h(x)$ is strictly increasing (decreasing resp.) on $(0, \eta)$ and strictly decreasing (increasing resp.) on (η, r).*

Lemma 6. *Let $p, q \in [0, +\infty)$, and $A_p(r)$, $B_q(r)$ be defined in* (3) *and* (4)*. Then, for each $r \in (0, 1)$,*

(1) *$p \mapsto A_p(r)$ is strictly decreasing from $[0, \infty)$ onto $(\log(1/r')/r^2, \pi/(2{r'}^2)]$;*

(2) $q \mapsto B_q(r)$ *is strictly decreasing from* $[0,\infty)$ *onto* $(\mathrm{arth}r/r, \pi/(2r'^2)]$.

Proof. By differentiation and simple computations, we obtain

$$\frac{dA_p(r)}{dp} = \frac{\pi}{2}\frac{2r'^2\log(1/r') - \pi r^2}{[1+(p\pi-1)r^2]^2} \tag{9}$$

and

$$\frac{dB_q(r)}{dq} = \frac{\pi r^2}{[1+(q\pi-1)r^2]^2}\left(\frac{r'^2\mathrm{arth}r}{r} - \frac{\pi}{2}\right). \tag{10}$$

It was proved in [**21**, Lemma 3(1)] that $r \mapsto [r'^2\mathrm{arth}r]/r$ is strictly decreasing from $(0,1)$ onto $(0,1)$, and it is easy to check that $r \mapsto (1-r)\log[1/(1-r)] - \pi r$ is strictly decreasing from $(0,1)$ onto $(-\pi, 0)$. Thus $dA_p(r)/dp < 0$ and $dB_q(r)/dq < 0$ for each fixed $r \in (0,1)$ by (9) and (10). So that the monotonicity properties in parts (1) and (2) follow. Obviously,

$$A_0(r) = B_0(r) = \frac{\pi}{2r'^2}, \quad \lim_{p\to+\infty} A_p(r) = \frac{1}{r^2}\log\frac{1}{r'}, \quad \lim_{q\to+\infty} B_q(r) = \frac{\mathrm{arth}r}{r}.$$

□

Lemma 7. *Let*

$$f(r) = \frac{\pi/2 - r'^2\mathcal{K}(r)}{r^2\mathcal{K}(r) + \log r'}, \quad r \in (0,1).$$

Then f *is strictly increasing from* $(0,1)$ *onto* $(3\pi/[4(\pi-1)], \pi/(4\log 2))$.

Proof. Let

$$\begin{aligned}
f_1(r) &= \frac{\pi}{2} - r'^2\mathcal{K}(r), & f_2(r) &= r^2\mathcal{K}(r) + \log r', \\
f_3(r) &= r'^2\left(\frac{\mathcal{K} - \mathcal{E} + r^2\mathcal{K}}{r^2}\right), & f_4(r) &= \mathcal{E} + r'^2\mathcal{K} - 1, \\
f_5(r) &= 2(\mathcal{K}-\mathcal{E}) - r^2(\mathcal{E} - r'^2\mathcal{K}) + 2r^4\mathcal{K} - r^2\mathcal{E}, & f_6(r) &= r^2[2(\mathcal{K}-\mathcal{E}) + r^2\mathcal{K}].
\end{aligned}$$

Then $f(r) = f_1(r)/f_2(r)$, $f_1(0^+) = f_2(0^+) = 0$, $f_3(1^-) = f_4(1^-) = 0$ and

$$\begin{aligned}
\frac{f_1'(r)}{f_2'(r)} &= \frac{2r\mathcal{K} - (\mathcal{E} - r'^2\mathcal{K})/r}{r(\mathcal{E} + r'^2\mathcal{K} - 1)/r'^2} = \frac{f_3(r)}{f_4(r)}, \\
\frac{f_3'(r)}{f_4'(r)} &= \frac{[-2(\mathcal{K}-\mathcal{E}) + r^2(\mathcal{E} - r'^2\mathcal{K}) - 2r^4\mathcal{K} + r^2\mathcal{E}]/r^3}{-[2(\mathcal{K}-\mathcal{E}) + r^2\mathcal{K}]/r} = \frac{f_5(r)}{f_6(r)}.
\end{aligned}$$

Expanding f_5 and f_6 into power series gives

$$f_5(r) = \frac{\pi}{2}\sum_{n=0}^{\infty}\frac{\left[\left(\frac{1}{2}\right)_n\right]^2}{4n!(n+2)!}(16n^2 + 36n + 17)r^{2n+4},$$

$$f_6(r)=\frac{\pi}{2}\sum_{n=0}^{\infty}\frac{\left[\left(\frac{1}{2}\right)_n\right]^2}{n!(n+1)!}(3n+2)r^{2n+4}.$$

Let

$$A_n=\frac{\left[\left(\frac{1}{2}\right)_n\right]^2}{4n!(n+2)!}(16n^2+36n+17),\quad B_n=\frac{\left[\left(\frac{1}{2}\right)_n\right]^2}{n!(n+1)!}(3n+2)$$

and $C_n=A_n/B_n$. Then $C_n=(16n^2+36n+17)/[4(n+2)(3n+2)]$ and

$$C_{n+1}-C_n=\frac{1}{4}\left[\frac{20n^2+46n+21}{(n+2)(n+3)(3n+2)(3n+5)}\right]>0$$

for all $n\in\mathbf{N}$. This together with Lemma 5(1) implies that f_5/f_6 is strictly increasing on $(0,1)$. Applying Lemma 4 twice, we conclude that f is also increasing on $(0,1)$. For the limiting values, by L'Hôpital's rule,

$$\lim_{r\to0^+}f(r)=\lim_{r\to0^+}\frac{f_1'(r)}{f_2'(r)}=\lim_{r\to0^+}\frac{f_3(r)}{f_4(r)}=\frac{3\pi}{4(\pi-1)}.$$

Using (1), we have

$$\begin{aligned}\lim_{r\to1^-}f(r)&=\lim_{r\to1^-}\frac{\frac{\pi}{2}-r'^2\mathcal{K}(r)}{\mathcal{K}(r)+\frac{\log r'}{r^2}}=\lim_{r\to1^-}\frac{\frac{\pi}{2}-r'^2\mathcal{K}(r)}{\mathcal{K}(r)-\log\frac{4}{r'}+\log4+\frac{r'^2\log r'}{r^2}}\\&=\frac{\pi/2}{\log4}=\frac{\pi}{4\log2}.\end{aligned}$$

□

Lemma 8. *Let*

$$g(r)=\frac{\pi/2-r'^2\mathcal{K}(r)}{r[r\mathcal{K}(r)-\operatorname{arth}r]},\quad r\in(0,1).$$

Then g is strictly increasing from $(0,1)$ onto $(3\pi/[4(\pi-2)],\pi/(2\log2))$.

Proof. Let

$$\begin{aligned}g_1(r)&=\frac{\pi/2-r'^2\mathcal{K}(r)}{r},\qquad g_2(r)=r\mathcal{K}(r)-\operatorname{arth}r,\\g_3(r)&=r'^2\left(\frac{2\mathcal{K}-\mathcal{E}-\pi/2}{r^2}\right),\quad g_4(r)=\mathcal{E}-1,\\g_5(r)&=4(\mathcal{K}-\mathcal{E})+r'^2\mathcal{E}+r'^2\mathcal{K}-\pi,\quad g_6(r)=r^2(\mathcal{K}-\mathcal{E}).\end{aligned}$$

Then $g(r)=g_1(r)/g_2(r)$, and $\lim_{r\to0^+}g_1(r)=\lim_{r\to0^+}[3\pi r^2/8+o(r^2)]/r=0$, $g_2(0^+)=0$, $g_3(1^-)=g_4(1^-)=0$ and

$$\frac{g_1'(r)}{g_2'(r)}=\frac{(2\mathcal{K}-\mathcal{E}-\pi/2)/r^2}{(\mathcal{E}-1)/r'^2}=\frac{g_3(r)}{g_4(r)},$$

$$\frac{g_3'(r)}{g_4'(r)} = \frac{-[4(\mathcal{K}-\mathcal{E}) + r'^2\mathcal{E} + r'^2\mathcal{K} - \pi]/r^3}{-(\mathcal{K}-\mathcal{E})/r} = \frac{g_5(r)}{g_6(r)}.$$

Expanding g_5 and g_6 into power series, we have

$$g_5(r) = \frac{\pi}{2}\sum_{n=0}^{\infty}\frac{\left(\frac{1}{2}\right)_n\left(\frac{1}{2}\right)_{n+1}}{4[(n+2)!]^2}(n+1)(16n^2+44n+27)r^{2n+4},$$

$$g_6(r) = \frac{\pi}{2}\sum_{n=0}^{\infty}\frac{\left(\frac{1}{2}\right)_n\left(\frac{1}{2}\right)_{n+1}}{n!(n+1)!}r^{2n+4}.$$

Let

$$A_n^* = \frac{\left(\frac{1}{2}\right)_n\left(\frac{1}{2}\right)_{n+1}}{4[(n+2)!]^2}(n+1)(16n^2+44n+27), \quad B_n^* = \frac{\left(\frac{1}{2}\right)_n\left(\frac{1}{2}\right)_{n+1}}{n!(n+1)!}$$

and $C_n^* = A_n^*/B_n^*$. Then $C_n^* = (16n^2+44n+27)/[4(n+2)^2]$ and

$$C_{n+1}^* - C_n^* = \frac{20n^2+94n+105}{4(n+2)^2(n+3)^2} > 0$$

for all $n \in \mathbf{N}$. This together with Lemma 5(1) showes that g_5/g_6 is strictly increasing on $(0,1)$, so is g by Lemma 4. By L'Hôpital's rule and (1) we get

$$\lim_{r\to 0^+} g(r) = \lim_{r\to 0^+}\frac{\frac{\pi}{2} - r'^2\mathcal{K}(r)}{r^2}\cdot\frac{1}{\mathcal{K}(r) - \frac{\operatorname{arth} r}{r}} = \lim_{r\to 0^+}\frac{\frac{3\pi r^2}{8} + o(r^2)}{r^2}\cdot\frac{1}{\frac{\pi}{2}-1}$$
$$= \frac{3\pi}{4(\pi-2)},$$

$$\lim_{r\to 1^-} g(r) = \lim_{r\to 1^-}\frac{\frac{\pi}{2} - r'^2\mathcal{K}(r)}{\mathcal{K}(r) - \log\frac{4}{r'} + \log 4 - \frac{(1+r)\log(1+r)}{2r} + \frac{(1-r)\log(1-r)}{2r}}$$
$$= \frac{\pi}{2\log 2}.$$

□

3. PROOFS OF THEOREMS 1–3

Proof of Theorem 1. Let $p \in [0, +\infty)$. Then

(11) $$\mathcal{K}(r) - A_p(r) = \frac{[1+(p\pi-1)r^2]\mathcal{K}(r) - \pi[1 - p\log(1-r^2)]/2}{1+(p\pi-1)r^2}$$
$$= \frac{\pi\left(r^2\mathcal{K} + \log r'\right)}{1+(p\pi-1)r^2}\left[p - \frac{\pi/2 - r'^2\mathcal{K}(r)}{\pi\left(r^2\mathcal{K} + \log r'\right)}\right].$$

Therefore, Theorem 1 directly follows from (11) and Lemma 7. □

Proof of Theorem 2. Let $q \in [0, +\infty)$. Then

$$\begin{aligned}\mathcal{K}(r) - B_q(r) = & \frac{[1+(q\pi-1)r^2]\mathcal{K}(r) - \pi/2 - q\pi r \operatorname{arth} r}{1+(q\pi-1)r^2} \\ = & \frac{\pi r^2 \left[\mathcal{K} - (\operatorname{arth} r)/r\right]}{1+(q\pi-1)r^2}\left[q - \frac{\pi/2 - r'^2\mathcal{K}(r)}{\pi r\left(r\mathcal{K} - \operatorname{arth} r\right)}\right].\end{aligned} \tag{12}$$

Therefore, Theorem 2 directly follows from (12) and Lemma 8. □

Proof of Theorem 3. (1) Expanding $\mathcal{K}(r)$ and $A_{\beta_0^*}(r)$ into power series yields

$$\mathcal{K}(r) = \frac{\pi}{2}\sum_{n=0}^{\infty}\mu_n r^{2n}, \quad A_{\beta_0^*}(r) = \frac{\pi}{2}\sum_{n=0}^{\infty}\lambda_n r^{2n},$$

where

$$\mu_n = \left[\frac{\left(\frac{1}{2}\right)_n}{n!}\right]^2, (n \geq 0),$$

and $\lambda_0 = 1$, and λ_n satisfies the following recurrence relation

$$\lambda_{n+1} = \frac{\beta_0^*}{n+1} - \left(\beta_0^*\pi - 1\right)\lambda_n, \quad (n \geq 0). \tag{13}$$

Here (13) can be derived by comparing the coefficients of two sides of the following equation

$$1 - \beta_0^* \log(1-r^2) = [1 + (\beta_0^*\pi - 1)r^2]\sum_{n=0}^{\infty}\lambda_n r^{2n},$$

equivalently,

$$\begin{aligned}1 + \sum_{n=0}^{\infty}\frac{\beta_0^*}{n+1}r^{2n+2} &= \sum_{n=0}^{\infty}\lambda_n r^{2n} + (\beta_0^*\pi - 1)\sum_{n=0}^{\infty}\lambda_n r^{2n+2} \\ &= \lambda_0 + \sum_{n=0}^{\infty}\lambda_{n+1}r^{2n+2} + \sum_{n=0}^{\infty}(\beta_0^*\pi - 1)\lambda_n r^{2n+2}.\end{aligned}$$

Rewrite (13) as

$$\lambda_n = \frac{\beta_0^*}{n} - \left(\beta_0^*\pi - 1\right)\lambda_{n-1}, \quad (n \geq 1),$$

and then substitute into (13), we obtain another recurrence formula involving λ_{n+1} and λ_{n-1}:

$$\lambda_{n+1} = \frac{\beta_0^*[(2-\beta_0^*\pi)n - (\beta_0^*\pi - 1)]}{n(n+1)} + (\beta_0^*\pi - 1)^2\lambda_{n-1} \quad (n \geq 1). \tag{14}$$

In what follows, we shall prove that $\lambda_n \geq \mu_n$ by mathematical induction on n. Obviously, $\lambda_0 = \mu_0 = 1$, $\lambda_1 = \mu_1 = 1/4$ and $\lambda_2 = (\pi+2)/[16(\pi-1)] > \mu_2 = 9/64$. Suppose that $\lambda_k \geq \mu_k$ for $k = 3, \cdots, n$, then from (14) we obtain

$$\begin{aligned}\lambda_{n+1} - \mu_{n+1} =& \frac{\beta_0^*[(2-\beta_0^*\pi)n - (\beta_0^*\pi - 1)]}{n(n+1)} + (\beta_0^*\pi-1)^2\lambda_{n-1} - \left[\frac{\left(\frac{1}{2}\right)_{n+1}}{(n+1)!}\right]^2 \\ \geq& \frac{\beta_0^*[(2-\beta_0^*\pi)n - (\beta_0^*\pi - 1)]}{n(n+1)} + (\beta_0^*\pi-1)^2\mu_{n-1} - \mu_{n-1}\frac{(4n^2-1)^2}{16n^2(n+1)^2} \\ =& \frac{\beta_0^*[(2-\beta_0^*\pi)n - (\beta_0^*\pi - 1)]}{n(n+1)} - \mu_{n-1}\left[\frac{(4n^2-1)^2}{16n^2(n+1)^2} - (\beta_0^*\pi-1)^2\right].\end{aligned}$$

Using the Wallis' inequality $\mu_n \leq 1/[\pi(n+1/4)]$ (see [**13**, Theorem 1]) and the fact that

$$\frac{(4n^2-1)^2}{16n^2(n+1)^2} \geq \left[\frac{(4n^2-1)^2}{16n^2(n+1)^2}\right]_{n=1} = \frac{9}{64} > (\beta_0^*\pi-1)^2$$

for $n \geq 1$, we obtain

$$\begin{aligned}\pi(\lambda_{n+1} - \mu_{n+1}) \geq& \frac{\beta_0^*\pi[(2-\beta_0^*\pi)n - (\beta_0^*\pi - 1)]}{n(n+1)} \\ &- \frac{1}{n-3/4}\left[\frac{(4n^2-1)^2}{16n^2(n+1)^2} - (\beta_0^*\pi-1)^2\right] \\ =& \frac{4(1-4\xi+3\xi^2)n^3 - 4(1+\xi-6\xi^2)n^2 + 12\xi(\xi+1)n - 1}{16n^2(n+1)^2(4n-3)}\end{aligned}$$

for $n \geq 2$, where $\xi = \beta_0^*\pi - 1 = 0.1002\cdots$. Furthermore, inequality

$$\begin{aligned}&4(1-4\xi+3\xi^2)n^3 - 4(1+\xi-6\xi^2)n^2 + 12\xi(\xi+1)n - 1 \\ \geq& 8(1-4\xi+3\xi^2)n^2 - 4(1+\xi-6\xi^2)n^2 + 12\xi(\xi+1)n - 1 \\ =& 4(1-9\xi+12\xi^2)n^2 + 12\xi(1+\xi)n - 1 \\ \geq& (8-60\xi+108\xi^2)n - 1 \geq n - 1 \geq 0\end{aligned}$$

holds for all $n \geq 2$, so that $\lambda_{n+1} \geq \mu_{n+1}$. This proves that $\lambda_k \geq \mu_k$ for $k \in \mathbb{N}$. Until now, $-F$ is absolutely monotonic on $(0,1)$, and $F(r)/r^4$ is decreasing on $(0,1)$. For the limiting values of $F(r)/r^4$ at $r = 0^+$ and $r = 1^-$, it is clear that

$$\lim_{r\to 0^+} \frac{F(r)}{r^4} = \lim_{r\to 0} \frac{\mathcal{K}(r) - A_{\beta_0^*}(r)}{r^4} = \frac{\pi}{2}(\mu_2 - \lambda_2) = \frac{\pi(5\pi-17)}{128(\pi-1)}.$$

By (1) we obtain

$$\begin{aligned}\lim_{r\to 1^-} \frac{F(r)}{r^4} =& \lim_{r\to 1^-} \mathcal{K}(r) - A_{\beta_0^*}(r) = \lim_{r\to 1^-} \mathcal{K}(r) - \log\frac{4}{r'} + \log\frac{4}{r'} - A_{\beta_0^*}(r) \\ =& \lim_{r\to 1^-} \frac{[1+(\beta_0^*\pi-1)r^2]\log 4 - \pi/2 - (1-\pi\beta_0^*)(1-r^2)\log r'}{1+(\beta_0^*\pi-1)r^2}\end{aligned}$$

$$=\frac{\beta_0^*\pi\log 4-\pi/2}{\beta_0^*\pi}=\frac{2(3\log 2-\pi+1)}{3}.$$

Therefore, inequality (5) follows, and this completes the proof of part (1).

(2) Let

$$G_1(r)=[1+(\beta_0^*\pi-1)r^2]\mathcal{K}(r),\quad G_2(r)=1-\beta_0^*\log(1-r^2).$$

Then $G(r)=2G_1(r)/[\pi G_2(r)]$. If we prove that G_1'/G_2' is piecewise monotone on $(0,1)$, first decreasing and then increasing, then by the formula (8)

$$\frac{dH_{G_1,G_2}(r)}{dr}=\left(\frac{G_1'}{G_2'}\right)'G_2,$$

we can conclude that there exists $r_0\in(0,1)$ such that $H_{G_1,G_2}(r)$ is strictly decreasing on $(0,r_0)$, and strictly increasing on $(r_0,1)$; if we further show that $H_{G_1,G_2}(0^+)=0$ and $H_{G_1,G_2}(1^-)>0$, then $H_{G_1,G_2}(r)$ is first negative then positive on $(0,1)$. Then by the formula (7)

$$G'(r)=\left(\frac{2G_1(r)}{\pi G_2(r)}\right)'=\frac{2G_2'(r)}{\pi G_2(r)^2}H_{G_1,G_2}(r),$$

and the fact that $G_2'(r)>0$ for all $r\in(0,1)$, one has that $G'(r)$ is also first negative then positive on $(0,1)$. So that G is piecewise monotone on $(0,1)$. Moreover, by (1),

$$\lim_{r\to1^-}G(r)=\lim_{r\to1^-}\frac{2}{\pi}\frac{[1+(\beta_0^*\pi-1)r^2][\mathcal{K}(r)/\log(1/r')])}{1/\log(1/r')+2\beta_0^*}=1.$$

The asserted inequality holds true. Now we prove part (2) by two steps.

Step 1: We show that G_1'/G_2' is piecewise monotone on $(0,1)$, first decreasing and then increasing. Differentiating G_1 and G_2, and then expanding them into power series give

$$\begin{aligned}(15)\qquad G_1'(r)=&2(\beta_0^*\pi-1)r\mathcal{K}(r)+[1+(\beta_0^*\pi-1)r^2]\frac{\mathcal{E}(r)-r'^2\mathcal{K}(r)}{rr'^2}\\ =&\pi\sum_{n=0}^{\infty}\frac{\left[\left(\frac12\right)_n\right]^2}{n!(n+1)!}\left[\left(n+\frac12\right)^2+(\beta_0^*\pi-1)(n+1)^2\right]r^{2n+1},\\ (16)\qquad G_2'(r)=&\frac{2\beta_0^*r}{1-r^2}=2\beta_0^*\sum_{n=0}^{\infty}r^{2n+1},\\ \frac{G_1'(r)}{G_2'(r)}=&\frac{\pi}{2\beta_0^*}\frac{\sum_{n=0}^{\infty}D_nr^{2n+1}}{\sum_{n=0}^{\infty}r^{2n+1}},\end{aligned}$$

where

$$D_n=\frac{\left[\left(\frac12\right)_n\right]^2}{n!(n+1)!}\left[\left(n+\frac12\right)^2+(\beta_0^*\pi-1)(n+1)^2\right].$$

By simple computation we get

$$\frac{D_{n+1}}{D_n}-1=\frac{(7\pi-16)n^2+(12\pi-36)n+5\pi-17}{4(n+1)(n+2)[3\pi n^2+(2\pi+4)n+13]}.$$

This implies, for $n\in\mathbf{N}\cup\{0\}$,

$$D_0>D_1<D_2<D_3<\cdots,$$

that is, the sequence $\{D_n\}$ is strictly decreasing for $0\le n\le 1$ and strictly increasing for $n\ge 1$. According to Lemma 5(2), it suffices to prove that $H_{G_1',G_2'}(1^-)>0$. Indeed,

$$\begin{aligned}G_1''(r)=&2(\beta_0^*\pi-1)\mathcal{K}(r)+4(\beta_0^*\pi-1)\frac{\mathcal{E}-r'^2\mathcal{K}}{r'^2}\\&+[1+(\beta_0^*\pi-1)r^2]\frac{r'^2(\mathcal{K}-\mathcal{E})+2r^2(\mathcal{E}-r'^2\mathcal{K})}{r^2r'^4},\\G_2''(r)=&\frac{2\beta_0^*(1+r^2)}{(1-r^2)^2}\end{aligned}$$

and thereby

$$\begin{aligned}&\lim_{r\to1^-}H_{G_1',G_2'}(r)\\=&\lim_{r\to1^-}\frac{G_1''(r)}{G_2''(r)}G_2'(r)-G_1'(r)\\=&\lim_{r\to1^-}\frac{1}{1+r^2}\Big\{2(\beta_0^*\pi-1)rr'^2\mathcal{K}(r)+4(\beta_0^*\pi-1)r(\mathcal{E}-r'^2\mathcal{K})\\&\qquad+[1+(\beta_0^*\pi-1)r^2]\frac{r'^2(\mathcal{K}-\mathcal{E})+2r^2(\mathcal{E}-r'^2\mathcal{K})}{r^2r'^4}\Big\}\\&\quad-\Big\{2(\beta_0^*\pi-1)r\mathcal{K}(r)+[1+(\beta_0^*\pi-1)r^2]\frac{\mathcal{E}(r)-r'^2\mathcal{K}(r)}{rr'^2}\Big\}\\=&\lim_{r\to1^-}\frac{r\mathcal{K}(r)}{1+r^2}\Big\{2(\beta_0^*\pi-1)r'^2+4(\beta_0^*\pi-1)\frac{\mathcal{E}-r'^2\mathcal{K}}{\mathcal{K}(r)}\\&\quad+[1+(\beta_0^*\pi-1)r^2]\frac{\mathcal{K}-\mathcal{E}}{r^2\mathcal{K}}-2(\beta_0^*\pi-1)(1+r^2)\\&\quad-[1+(\beta_0^*\pi-1)r^2]\frac{\mathcal{E}(r)-r'^2\mathcal{K}(r)}{r^2\mathcal{K}}\Big\}\\=&\lim_{r\to1^-}[\beta_0^*\pi-4(\beta_0^*\pi-1)]\frac{r\mathcal{K}(r)}{1+r^2}=\lim_{r\to1^-}\frac{7\pi-16}{4(\pi-1)}\left[\frac{r\mathcal{K}(r)}{1+r^2}\right]=+\infty.\end{aligned}$$

Step 2: We show that $H_{G_1,G_2}(0^+)=0$ and $H_{G_1,G_2}(1^-)>0$. Use of (15), (16) and (1) gives

$$\lim_{r\to0^+}H_{G_1,G_2}(r)=\lim_{r\to0^+}\frac{G_1'(r)}{G_2'(r)}\lim_{r\to0^+}G_2(r)-\lim_{r\to0^+}G_1(r)$$

$$
\begin{aligned}
&= \frac{\pi}{2\beta_0^*}\left[\frac{1}{4}+(\beta_0^*\pi-1)\right]-\frac{\pi}{2}=0,\\
\lim_{r\to 1^-} H_{G_1,G_2}(r) &= \lim_{r\to 1^-}\left[\frac{\beta_0^*\pi-1}{\beta_0^*}r'^2\mathcal{K}(r)+\frac{1+(\beta_0^*\pi-1)r^2}{2\beta_0^*}\frac{\mathcal{E}(r)-r'^2\mathcal{K}(r)}{r^2}\right]\\
&\quad\times\left[1+\beta_0^*\log\frac{1}{1-r^2}\right]-\left[1+(\beta_0^*\pi-1)r^2\right]\mathcal{K}(r)\\
&= \lim_{r\to 1^-}\frac{\pi}{2}\left(1+\beta_0^*\log\frac{1}{1-r^2}\right)-\left[1+(\beta_0^*\pi-1)r^2\right]\log\frac{4}{r'}\\
&= \lim_{r\to 1^-}\frac{\pi}{2}+\beta_0^*\pi\log\frac{1}{r'}-\beta_0^*\pi\log\frac{4}{r'}+(\beta_0^*\pi-1)(1-r^2)\log\frac{4}{r'}\\
&= \frac{\pi}{2}-\beta_0^*\pi\log 4=\frac{\pi(\pi-1-3\log 2)}{2\pi-2}=1.0853\cdots>0.
\end{aligned}
$$

□

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