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# RECOGNIZING SIGNED LINE GRAPHS WITH A SINGLE ROOT

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A signed line graph of a simply signed graph extends the notion of a generalized line graph defined in the framework of ordinary graphs. In this paper, the Krausz theorem on covering characterization of line graphs and the Whitney theorem on isomorphism are extended to the context of signed line graphs.

## 1. INTRODUCTION

Line graphs of ordinary graphs have been studied for nearly a century, during which many of their properties have been uncovered. One of them states that the spectrum of the standard $\{0, 1\}$-adjacency matrix of every line graph is bounded below by $-2$. There are graphs that are not line graphs but have the same spectral property, and all graphs with this property are characterized by Cameron et al. [**5**] when they proved that the least eigenvalue of a graph is greater than or equal to $-2$ if and only if it can be constructed from the vectors in the root system $D_n$ or $E_8$. The graphs corresponding to $D_n$ are the generalized line graphs, a class that includes all line graphs, while those corresponding to $E_8$ are the so-called exceptional graphs, namely graphs that are not generalized line graphs but nevertheless satisfy the required property.

The same characterization has been extended to the framework of signed graphs by Greaves et al. [**7**]; see [**11, 12**] for earlier results. It relies on the same root systems, this time corresponding to signed line graphs of simply signed graphs and the exceptional signed graphs, respectively. Simply signed graphs generalize

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the concept of signed graphs by allowing parallel edges between the same pair of vertices, provided that one edge is positive and the other is negative.

A classical result of Krausz [**8**] establishes a structural characterization of line graphs, that is, it determines precisely when a graph arises as the line graph of some other graph. Another classical result, due to Whitney [**13**], describes all pairs of non-isomorphic graphs that share the same line graph. These results are extended to generalized line graph by Cvetković, Doob and Simić [**6**]. Recently, they are extended by Cavaleri, D'Angeli and Donno [**4**] to signed line graphs of signed graphs that do not allow parallel edges.

In this paper, we prove the Krausz-type and Whitney-type theorems for signed line graphs of simply signed graphs. In doing so, we complete the characterization of all non-exceptional signed graphs whose spectrum is bounded below by $-2$, and we determine which of these graphs arise from a unique simply signed graph. Compared with the results of [**4**], we see that the Krausz-type theorem provides a natural extension of the corresponding classical result, whereas the Whitney-type theorem identifies a significantly larger set of signed line graphs that arise from different simply signed graphs.

In the remainder of this section we give necessary notions, terminology and mention some known results. The main results are formulated and proved in Sections 2 and 3, respectively.

The *line graph* of a graph $G$ is the graph whose vertices are the edges of $G$, with two vertices adjacent whenever the corresponding edges have a common vertex. Henceforth, we call $G$ a *root graph* (that corresponds to its line graph). In 1943, Krausz gave the following characterization.

**Theorem 1** (Krausz [**8**]). *A graph is a line graph if and only if its edges can be partitioned into cliques in such a way that*

*(i) each vertex is in at most two cliques,*

*(ii) two cliques have at most one common vertex.*

This edge partition is called a cover of the corresponding graph, and the theorem itself is also known as the Krausz's covering characterization. Other characterizations are given by Beineke [**1**] (in terms of forbidden subgraphs) and van Rooij and Wilf [**9**] (in terms of prescribed induced triangles). Eleven years earlier, Whitney has offered the following result known as the Whitney isomorphism theorem.

**Theorem 2** (Whitney [**13**]). *Every connected line graph distinct from the triangle $K_3$ has a unique root (up to isolated vertices).*

A set of isolated vertices may be added to any graph without affecting the line graph. In this paper, we will deal with graphs that admit the existence of at most two parallel edges between a pair of vertices. A pair of such edges is called a *digon*, and such a graph is called a *simply graph*. This term is transferred from the domain of signed graphs, see below. The graph whose vertices are the edges of a given simply graph $G$, with two vertices adjacent whenever the corresponding edges have exactly one common vertex is denoted by $\Lambda(G)$. This is not a definition of a generalized line graph, since a generalized line graph allows only certain vertices of the root graph to be connected by parallel edges. We do not provide a formal definition here, as it is not relevant to this paper.

We proceed with signed graphs. A *signed graph* $\dot{G} = (G, \sigma)$ consists of the *underlying graph* $G = (V, E)$ with the *signature function* $\sigma$ that maps the edge set $E$ into $\{1, -1\}$. The edges mapped to 1 are *positive*, those mapped to $-1$ are *negative*, and together they comprise the edge set of $\dot{G}$. The adjacency matrix is defined according to the signature. A graph is interpreted as a signed graph in which all edges are positive; it is recognized in the text by the absence of a dot symbol. The number of vertices is called the *order* of $\dot{G}$.

Signed graphs $\dot{G}$ and $\dot{H}$ are *switching equivalent* if $\dot{H}$ is obtained by selecting a vertex subset of $\dot{G}$ and reversing the sign of every edge with exactly one end in the selected subset. Switching equivalent signed graphs share the same spectrum since the corresponding adjacency matrices are similar. A signed graph is *balanced* if it switches to its underlying graph. In the context of signed graphs, isomorphism is usually combined with switching equivalence to the more general concept of switching isomorphism of signed graphs. Accordingly, two signed graphs are *switching isomorphic* if one of them is isomorphic to a signed graph that is switching equivalent to the other one. This is designated by the symbol $\backsimeq$, similar to $\cong$ which traditionally stands between isomorphic (unsigned) graphs. When we say that a signed graph is unique for some property, we will always mean 'up to switching isomorphism'.

For a signed graph $\dot{G}$, we introduce the vertex-edge orientation $\eta\colon V(G) \times E(G) \longrightarrow \{0, 1, -1\}$ formed by obeying the following rules:

(O1) $\eta(i, jk) = 0$ if $i \notin \{j, k\}$,

(O2) $\eta(i, ij) \in \{1, -1\}$,

(O3) $\eta(i, ij)\eta(j, ij) = -\sigma(ij)$.

The vertex-edge incidence matrix $B_\eta$ is the matrix whose $(i, e)$ entry is $\eta(i, e)$. The $\{0, 1, -1\}$-adjacency matrix of a *signed line graph* $\mathcal{L}(\dot{G})$ is

$$A(\mathcal{L}(\dot{G})) = B_\eta^\intercal B_\eta - 2I,$$

where $I$ is the identity matrix. A signed line graph depends on the orientation $\eta$, but different orientations generate switching equivalent singed line graphs. Also, switching equivalent signed graphs generate switching equivalent signed line graphs [**3**]. It is worth mentioning that the concept of signed line graphs extends but not generalizes the concept of line graphs, as in general a signed line graph of an ordinary graph (considered as a signed graph with the all positive signature) may differ from its line graph [**3, 14**].

A signed digon consists of two edges connecting the same pair of vertices. A digon is positive if its edges have the same sign, and negative if they differ in sign. It follows that the existence of a positive digon in $\dot{G}$ implies the existence of parallel edges in its signed line graph. On the contrary, a negative digon produces non-adjacent vertices. A signed graph which admits parallel edges if and only if they form negative digons is called by Zaslavsky [**14**] a *simply signed graph*. Accordingly, $\mathcal{L}(\dot{G})$ has no multiple edges if and only if $\dot{G}$ is a simply signed graph.

Every signed graph naturally belongs to the class of simply signed graphs. Moreover, all non-exceptional signed graphs are signed line graphs of simply signed roots. We follow the approach of [**2, 3, 10, 14**], according to which a 'signed line graph' is understood to have a root that is a simply signed graph, that is, a graph that may contain negative digons. As pointed out, Theorems 1 and 2 are extended to generalized line graphs and line graphs of signed graphs without negative digons. These results are not quoted here, and the reader is referred to the papers mentioned at the beginning of this section. In what follows, we do the same for simply signed roots.

More notions are introduced in the next section. We conclude this section with the following conventions. An edge of a root graph (resp. a simply signed root graph) and the corresponding vertex of its line graph (resp. signed line graph) will be considered interchangeably. Typically, two edges are said to be adjacent, or neighbours, if they share a common vertex. In this paper, the same terminology will also apply to edges that share two common vertices.

## 2. NECESSARY AND SUFFICIENT CONDITIONS FOR A SIGNED GRAPH TO BE A SIGNED LINE GRAPH

A *generalized cocktail party graph* (a *GCP*, for short) is obtained from a complete graph of even order by removing a set of independent edges. If $n$ is the order of the complete graph, then every vertex in a GCP is of degree $n-1$ or $n-2$. A vertex of the former degree is called 1-*type*, while the others are called 2-*type*. The previous notions are extended to the domain of signed graphs with the same terminology.

Two vertices $u, v$ of a graph $G$ are called *twins* if they are non-adjacent and share the same neighbourhood in $G$. This notion is extended to signed graphs in such a way that *twins* in a signed graph are non-adjacent, and have the same set of positive neighbours (the neighbours joined by a positive edge) and the same set of negative neighbours. We will need the notion of *antitwin* vertices: They are non-adjacent, share the same neighbourhood and every positive (resp. negative) neighbour of one of them is a negative (positive) neighbour of the other one.

Observe that the two vertices of a signed line graph that arise from a negative digon and lie in the intersection of two GCPs are twins in one GCP and antitwins in the other.

**Theorem 3.** *A graph $G$ is isomorphic to $\Lambda(H)$ for some simply graph $H$ if and only if the edges of $G$ can be partitioned into GCPs in such a way that* *(a) each vertex is in at most two GCPs,*

*(b) two GCPs have at most two common vertices,*

*(c) if two GCPs have exactly one common vertex (resp. two common vertices), then it is (they are) of* 1*-type (*2*-type) in each.*

*Proof.* We may suppose that $G$ is connected, since otherwise every component is considered separately.

Let $G \cong \Lambda(H)$ for some $H$. For each vertex $u$ adjacent to at least two vertices in $H$, the edges incident with $u$ belong to a GCP in $G$. In this way we have obtained an edge partition of $G$ into GCPs.

Let $uv$ be the edge of $H$ corresponding to a vertex $w$ of $G$. If $u, v$ are the unique neighbours to each other, then $w$ does not belong to any GCP of $G$. If exactly one of $u, v$ is the unique neighbour of the other one, $w$ is in exactly one GCP. Otherwise, $w$ is in exactly two GCPs. Hence, (a) follows.

Part (b) follows since two vertices of $H$ are incident to at most two common edges. Moreover, if two GCPs of $G$ share exactly one common vertex, say $w$, then the corresponding edge of $H$ is not in a digon, which means that $w$ is adjacent to every vertex of both GCPs, i.e., $w$ is of 1-type. Similarly, if two GCPs share exactly two common vertices then these vertices are of 2-type, as the corresponding edges form a digon in $H$. This completes (c).

Suppose now that the edges of $G$ are partitioned into GCPs satisfying the hypothesis (a), (b) and (c).

We construct a simply graph $H$ in the following way. If a vertex of $G$ does not belong to any GCP then $G$ consists of two isolated vertices, and we fix $H$ to either two disjoint edges or a single digon. Otherwise, proceed with the following steps: (i) each GCP is a single vertex of $H$; (ii) GCPs with exactly one common vertex are joined by a single edge, and GCPs with two common vertices are joined by two parallel edges; (iii) each vertex that belongs to exactly one GCP and has no twin gives rise to an additional vertex of $H$ joined to this GCP by a single edge, while twin vertices belonging to exactly one GCP give an additional vertex joined to this GCP by two parallel edges. □

In spirit of Theorem 1, the edge partition into GCPs can be interpreted as a cover of $G$, which is proper whenever the GCPs satisfy the assumptions of the previous theorem.

By introducing a signature on the edge set of $\Lambda(H)$, we obtain a signed graph. This signed graph may or may not be a signed line graph, as illustrated in the following example. The forthcoming Theorem 5 addresses this question.

**Example 4.** The complete graph $K_4$ is isomorphic to $\Lambda(K_{1,4})$. By introducing the all-positive signature on $K_4$, we obtain a signed graph (in fact, again $K_4$, now interpreted as a signed graph). This signed graph appears as the signed line graph of $K_{1,4}$.

However, the signed graph obtained by negating exactly one edge of $K_4$ is not a signed line graph. This is an easy exercise, and also follows from the next theorem.

Here is the main result of this section, a Krausz-type theorem for signed line graphs. The proof relies on the forthcoming Algorithm 1.

**Theorem 5.** *A signed graph $\dot{G} = (G, \sigma)$ is a signed line graph if and only if $G$ satisfies the three assumptions of Theorem 3 and*

*(i) every GCP is the underlying graph of a balanced subgraph of $\dot{G}$,*

*(ii) if two GCPs have two common vertices, then these vertices are twins in one and antitwins in the other one.*

*Proof.* As before, we may assume that $\dot{G}$ is connected. Suppose that $\dot{G}$ is the signed line graph of a simply signed graph $\dot{H}$. Then $G$ satisfies the assumptions of Theorem 3, and it remains to prove parts (i) and (ii) of this statement.

The vertices of each GCP correspond to the edges incident to a single vertex, say $w$, of $\dot{H}$. The corresponding GCP switches to the all-positive signed graph obtained by choosing the vertex-edge orientation on $\dot{H}$ satisfying $\eta(u, e) = 1$, for each edge incident to $u$. Hence, we have arrived at (i).

If $u, v$ are vertices that belong to two GCPs, then the corresponding edges of $\dot{H}$, denoted again by $u$ and $v$, form a negative digon. Let $a, b$ be the ends of $u, v$ in $\dot{H}$. For every vertex-edge orientation, we have $\eta(a, u)\eta(a, v)\eta(b, u)\eta(b, v) = -1$ which, together with the defining rule (O3), leads to (ii).

Suppose now that the three assumptions of Theorem 3 and the two assumptions of this statement hold. An explicit construction of a simply signed root graph is given in Algorithm 1. In what follows, we prove the correctness of the algorithm, using the notation introduced therein.

For Step 1, we need to prove that if $\eta(u, v_{j_1})$ and $\eta(u, v_{j_2})$ are fixed, then the inequality $\eta(u, v_{j_1}) \neq \eta(u, v_{j_2})\sigma(v_{j_1}v_{j_2})$ does not occur. This follows from the assumption (i) of this theorem. Namely, since the GCP is balanced, there is a switching in which all its edges are positive, and then we have $\eta(u, v_i) = 1$ and $\sigma(v_i, v_j) = 1$, for every $i, j$. Obviously, the previous inequality does not occur in this setting. Applying the inverse switching, we arrive at the same conclusion for the initial GCP. We also need to prove that this step reaches its end, i.e., that at some point every orientation $\eta(u, v_i)$ is determined. This follows since every GCP is connected.

Step 2 is clear. In Step 3, a random choice between two options determines which of the two parallel edges is positive and which is negative. These choices are equivalent up to switching.

For Step 4, it is clear that each edge has received both orientations, so we proceed to create the simply signed root graph. Actually, this can be performed only if every pair of parallel edges has received orientations fixing them to a positive and a negative edge. This is ensured by (ii). □

**Algorithm 1** From a signed line graph with given GCPs to a simply signed root graph

**Require:** A connected signed line graph $\dot{G} = (G, \sigma)$ whose edges are partitioned into GCPs satisfying the assumptions of Theorem 5

**Ensure:** A simply signed graph $\dot{H}$ such that $\mathcal{L}(\dot{H}) \backsimeq \dot{G}$

**Step 1** Each GCP is a single vertex $u$ of $\dot{H}$. The vertices $v_1, v_2, \dots, v_k$ of this GCP are the edges incident to $u$ in $\dot{H}$. Set $\eta(u, v_1) = 1$ and $\eta(u, v_j) = \eta(u, v_i)\sigma(v_i v_j)$ for $v_j \sim v_i$, until every orientation $\eta(u, v_i)$ is fixed. Do this for each GCP.

**Step 2** For every vertex $v$ that belongs to exactly one GCP, insert an additional vertex $u$ incident to the edge $v$ in $\dot{H}$. Do this for every such a vertex. If $v$ has no twin in $G$, then chose $\eta(u, v)$ randomly. If $v_1$ is a twin of $v$ and $u_1$ is the added vertex incident to $v_1$, then $v$ and $v_1$ are incident to a common vertex, say $w$, in $\dot{H}$ with already fixed orientations $\eta(w, v)$ and $\eta(w, v_1)$. Identify $u$ and $u_1$, choose $\eta(u, v)$ randomly and set $\eta(u, v_1) = -\eta(u, v)\eta(w, v_1)\eta(w, v)$.

**Step 3** For a pair of CGPs sharing at least one common vertex, identify the corresponding edges formed in Step 1. Do this for each pair.

**Step 4** After the previous step, each edge has received two orientations, one for each of its ends. Such a bi-orientation uniquely determines the sign of this edge. Replace the orientations with edge signs and return the resulting simply signed graph.

The time complexity of Algorithm 1 is $O(|V(\dot{G})|)$, i.e., $O(|E(\dot{H})|)$, since for every vertex $u \in \dot{H}$ the amount of work done is of the order $O(\deg(u))$. Here is an illustration.

**Example 6.** Consider the signed graph $\dot{G}$ of Figure 1. We first recognize the four GCPs: For $1 \leq i \leq 4$, the $i$th GCP contains the vertices denoted by $i\ell$ where $\ell$ is an additional symbol defined below. Concerning these GCPs, we deduce that all assumptions of Theorem 5 are satisfied, and therefore $\dot{G}$ is a signed line graph of a simply signed graph, say $\dot{H}$.

In what follows, we apply Steps 1–4 to construct $\dot{H}$.

The first GCP contains the vertices $1a, 1b, 1c$ and $1d$. Following Step 1, we introduce the vertex denoted by 1 together with edges $a, b, c, d$. In the next iteration, we assign the orientation $\eta(1, \ell)$ for $\ell \in \{a, b, c, d\}$. The same procedure is performed for the remaining three GCPs. This is illustrated in the second row of Figure 1. For the sake of simplicity, only positive orientation is indicated by an arrow pointing to the vertex, whereas the absence of an arrow indicates a negative orientation.

In Step 2, we locate exactly three vertices belonging to only one GCP: $1a, 2e$ and $2f$. Additional vertices are added to the corresponding edges. These are the white vertices in the second row of the figure. The corresponding orientations are assigned according to the algorithm.

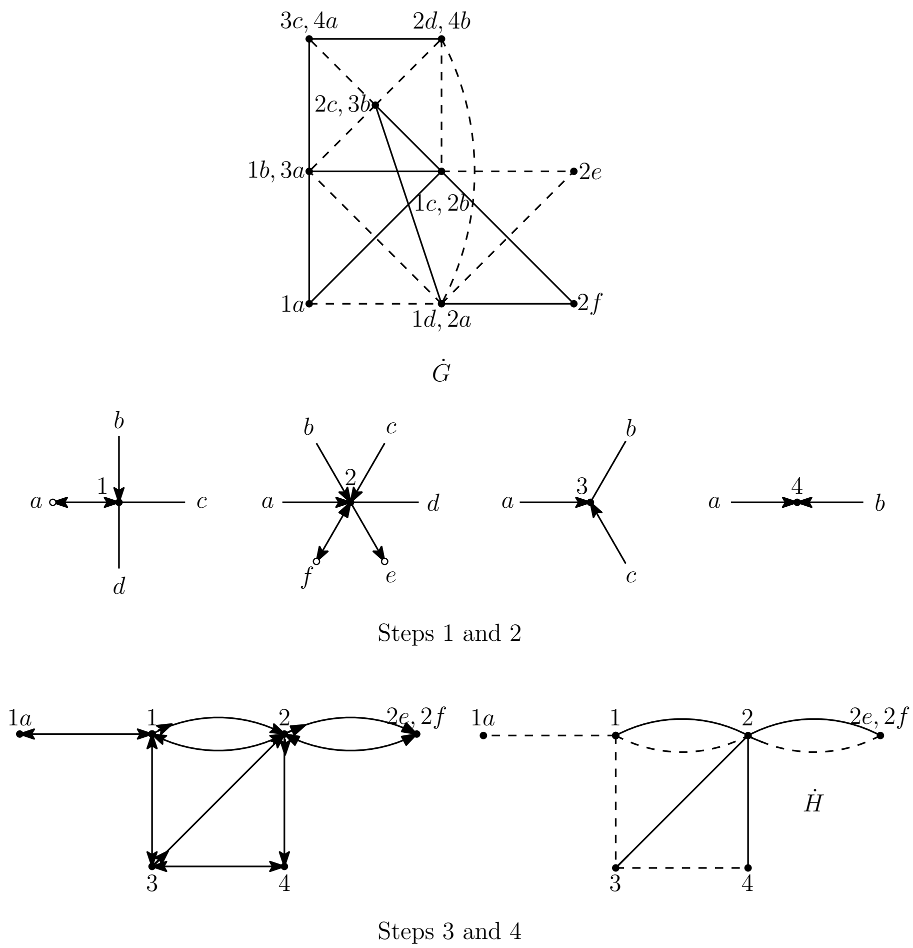



Figure 1: An example for Theorem 5

In Step 3, we identify the edges that arise from the vertices belonging to two GCPs. For example, the vertex denoted by $1b$ and $3a$ in $\dot{G}$ belongs to the first and the third GCP, and so we identify the edges $1b$ and $3a$ given in the second row. Proceeding in the same way, we obtain the vertex-edge oriented graph illustrated in the third row (in this case, all orientations are indicated by arrows). The second graph of the same row is the final result obtained in the last step.

We conclude this section with a consequence, significant for the next section.

**Corollary 7.** *Up to the switching equivalence, Algorithm 1 results in one and only one simply signed root.*

*Proof.* This result follows from the algorithm itself, as we had some instances in which an orientation has been chosen randomly, but in each of them a different choice leads to a switching equivalent output. □

## 3. SIGNED LINE GRAPHS WITH A UNIQUE SIMPLY SIGNED ROOT

As in the previous section, we first treat the underlying graph. We say that two twin vertices in the underlying graph of a signed line graph are *partners* if they correspond to edges forming a negative digon in a simply signed root.

**Theorem 8.** *Let $G$ be the underlying graph of a connected signed line graph. If $G$ has at least* 13 *vertices, then two twin vertices in a GCP are partners.*

*Proof.* Since $G$ is the underlying graph of a signed line graph, there is a simply graph $H$ such that $G \cong \Lambda(H)$. Let $u, v$ be twins in $G$ that are not partners. Considered as the edges of $H$ they are non-adjacent, and the set of edges that are incident to exactly one end of $u$ coincides with the set of edges that are incident to exactly one end of $v$. Moreover, we claim that, apart from possible edges that are parallel with either $u$ or $v$, there are no other edges in $H$. Indeed, since $G$ is connected, so is $H$ (up to isolated vertices), and the existence of an edge that does not belong to the previous set necessarily implies that either this edge is parallel with some of $u$ or $v$ or there exists an edge sharing one and only one end with exactly one of $u$ or $v$. Since the latter is impossible, we have arrived at the desired assertion.

Counting the edges in $H$, we deduce that in the worst case scenario $H$ has 12 edges: $u, v$, two edges parallel with them, and 8 edges belonging to digons lying between $u$ and $v$. Therefore, the desired statement holds if $\dot{H}$ has more than 12 edges, i.e., $G$ has more than 12 vertices. □

We proceed with the following lemma.

**Lemma 9.** *If $G$ is the underlying graph of a connected signed line graph with at least* 13 *vertices, then there exists exactly one partition of edges of $G$ into GCPs that satisfies the three assumptions of Theorem 3.*

*Proof.* Due to Theorem 8, twins necessarily feature as 2-type vertices in a GCP. Consequently, a vertex that does not have a twin is of 1-type. For two twin vertices, the 1-type vertices in that GCP are those vertices mutually adjacent to both twins, and multiple twin pairs belonging to the same GCP are easily recognized. Hence, any GCP that is not a clique has been uniquely constructed. Thus, the case with twin vertices is resolved, and question of uniqueness is reduced to the case without twins, but this means that $G$ is a line graph, and this situation has been settled in [**13**]. □

The main result of this section, a Whitney-type theorem, reads as follows.

**Theorem 10.** *Let $\dot{G} = (G, \sigma)$ and $\dot{H} = (H, \sigma)$ be two connected simply signed graphs without isolated vertices. The implication*

$$\mathcal{L}(\dot{G}) \backsimeq \mathcal{L}(\dot{H}) \implies \dot{G} \backsimeq \dot{H}$$

*holds whenever at least one of $G, H$ does not appear in Figure 2.*

*Proof.* Throughout the proof we suppose that $\dot{G}$ and $\dot{H}$ are specified in the statement of the theorem. If $\mathcal{L}(\dot{G})$ has at least 13 vertices, by Lemma 9 there is exactly one edge partition of $\Lambda(G)$ into GCPs satisfying the three assumptions of Theorem 3. The same holds for $\Lambda(H)$, since it is isomorphic. Corollary 7 ensures that simply signed root graphs are switching isomorphic.

Assume that $\mathcal{L}(\dot{G})$ has fewer than 13 vertices. If the twins of its underlying graph necessarily arise from digons of $G$ and $H$, then we may replace every pair of parallel edges in each of them with a single edge, and proceed with signed line graphs of signed graphs without negative digons. We know from [**4**] that, in this case, the implication of the theorem fails to hold if and only if $G$ is a triangle and $H$ is a star $K_{1,3}$ (see row (1) of Figure 2). In return to parallel edges, we find an other possibility: $G$ is a triangle with parallel edges and $H$ consists of three pairs of parallel edges sharing the same end (see row (7) of the figure).

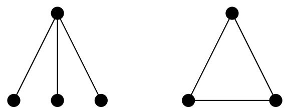
If $\Lambda(H)$ contains twins that do not correspond to a digon of $H$, then $H$ is a simply graph encountered in the proof of Theorem 8, i.e., it has 4 vertices and two non-adjacent edges which correspond to the specified twins. There are the three possibilities: both edges are single, one single the other in a digon, and each in a digon. By considering them, eliminating isomorphic cases and selecting those that satisfy $\Lambda(G) \cong \Lambda(H)$, we obtain the simply graphs of Figure 2. Precisely, the first simply graph in rows (2)–(16) contains two non-adjacent single edges. In the remainder of its row, all that satisfy the previous condition are listed. The last three rows contain the remaining possibilities for the first simply graph.

We have verified by hand that, for every simple graph in the figure, the same row contains at least one simple graph whose associated simple signed graphs generate switching isomorphic signed line graphs. Clearly, these simple signed graphs themselves are not switching isomorphic, since their underlying graphs are non-isomorphic. □

We conclude the paper with a few remarks on the simple graphs in Figure 2. Observe that the second graph in row (2) can be obtained from the first by a single edge shift. Once the initial graph in each row is fixed, the remaining graphs in that row arise through successive applications of this shifting operation.

In row (5), each of the three simple graphs generates a signed quadrangle. For the first graph, this quadrangle preserves the signature of the root graph; for the second, it is positive for every possible signature; and for the third, it is negative for every possible signature. Thus, it may occur that two graphs in the same row do not yield switching isomorphic signed line graphs, as is the case here where the second and third graphs do not.

(1)



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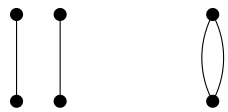

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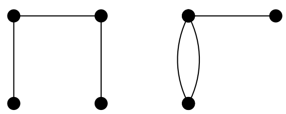

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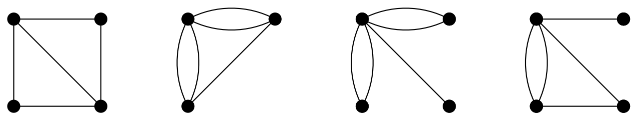

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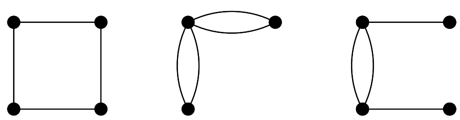

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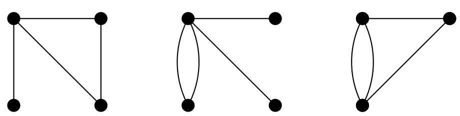

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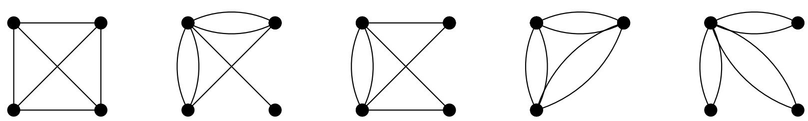

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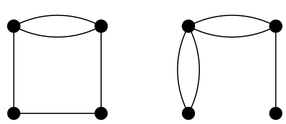

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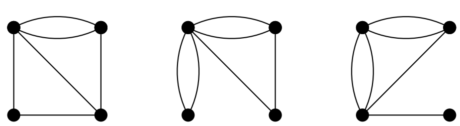

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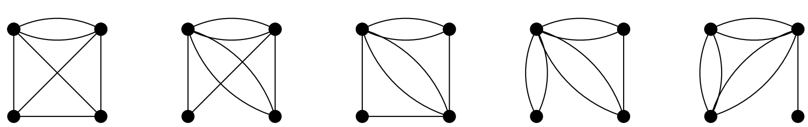

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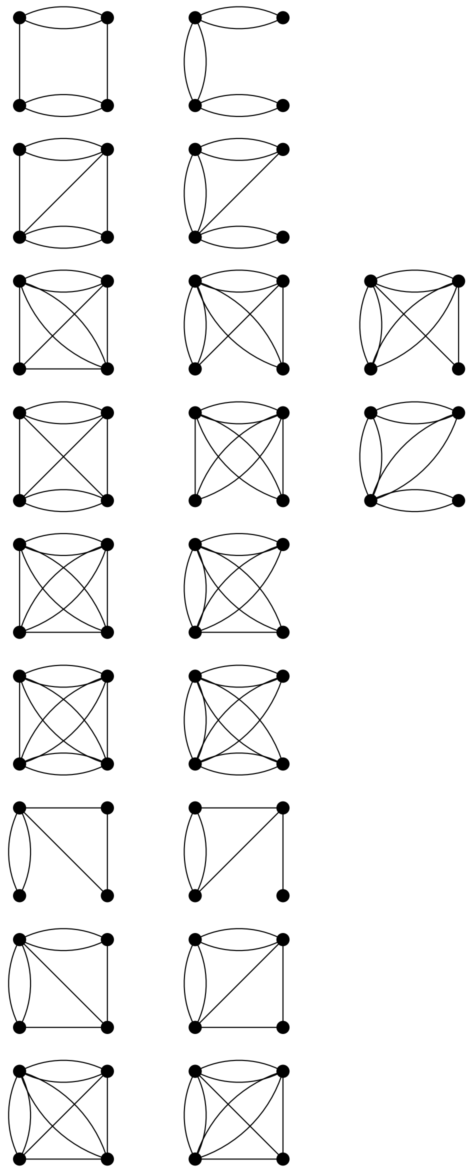

Figure 2: Connected underlying simply graphs of switching non-isomorphic simply signed graphs that produce the same signed line graph.

**REFERENCES**


1. L.W. Beineke: *Characterizations of derived graphs*, J. Combin. Theory, 9 (1970), 129–135.
2. F. Belardo, E.M. Li Marzi, S.K. Simić: *Signed line graphs with least eigenvalue* $-2$*: The star complement technique*, Discrete Appl. Math., 207 (2016), 29–38.
3. F. Belardo, Z. Stanić, T. Zaslavsky: *Total graph of a signed graph*, Ars Math. Contemp., 23 (2023), #P1.02.
4. M. Cavaleri, D. D'Angeli, A. Donno: *Characterizations of line graphs in signed and gain graphs*, European J. Combin., 102 (2022), 103479.
5. P.J. Cameron, J.M. Goethals, J.J. Seidel, E.E. Shult: *Line graphs, root systems, and elliptic geometry*, J. Algebra, 43 (1976), 305–327.
6. D. Cvetković, M. Doob, S. Simić: *Generalized line graphs*, J. Graph Theory, 5 (1981), 385–399.
7. G. Greaves, J. Koolen, A. Munemasa, Y. Sano, T. Taniguchi: *Edge-signed graphs with smallest eigenvalue greater than* $-2$, J. Combin. Theory B, 110 (2015), 90–111.
8. J. Krausz: *Demonstration nouvelle d'une theoreme de Whitney sur les reseaux*, Math. Fiz. Lapok, 50 (1943), 75–89.
9. A.C.M. van Rooij, H.S. Wilf: *The interchange graph of a finite graph* (Russian summary), Acta Math. Acad. Sci. Hungar., 16 (1965), 263–269.
10. Z. Stanić: *Spectra of signed graphs with two eigenvalues*, Appl. Math. Comput., 364 (2020), 124627.
11. G.R. Vijayakumar: *Signed graphs represented by* $D_\infty$, European J. Combin., 8 (1987), 103–112.
12. G.R. Vijayakumar, N.M. Singhi: *Some recent results on signed graphs with least eigenvalues* $\geq -2$, in D. Ray-Chaudhuri (Ed.): *Coding Theory and Design Theory, Part I*, Vol. 20 of The IMA Volumes in Mathematics and Its Applications, Springer, New York, (1990), pp. 213–218.
13. H. Whitney: *Congruent graphs and the connectivity of graphs*, American J. Math., 54 (1932), 150–168.
14. T. Zaslavsky: *Matrices in the theory of signed simple graphs*, in B.D. Acharya, G.O.H. Katona, J. Nešetřil (Eds.): *Advances in Discrete Mathematics and Applications: Mysore 2008*, Ramanujan Math. Soc., (2010), pp. 207–229.


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