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FIXED POINT THEORY FOR MULTI-VALUED OPERATORS VIA ADMISSIBLE PERTURBATION APPROACH

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In this paper, we will consider the following problem: if a fixed point inclusion $x \in B(x)$ (where B is a "bad" multi-valued operator) is replaced by another fixed point inclusion $x \in G(x)$ (where G is now a "good/better" multi-valued operator), then which properties (existence, data dependence, stability) can be obtained for our initial problem in terms of some conditions on the new operator G? We will do this by the help of the admissible perturbation technique, which will generate a "good" multi-operator G related to the "bad" multi-valued operator B by the fact that the fixed point set of B and respectively of G coincide.

1. INTRODUCTION AND PRELIMINARY NOTIONS AND RESULTS

In this paper we will follow the notations given in [**33**] and [**2**]. For the convenience of the reader we recall some of them as follows.

Let (X, d) be a metric space and $P(X)$ be the set of all nonempty subsets of X. We also define the following families of sets

$$P_{cl}(X) := \{Y \in P(X) | \ Y \text{ is closed}\}, P_{cp}(X) := \{Y \in P(X) | \ Y \text{ is compact }\},$$

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$$P_b(X) := \{Y \in P(X) |\ Y \text{ is bounded}\}, P_{b,cl}(X) := P_{cl}(X) \cap P_b(X).$$

In the case of a space X having a convexity structure, i.e., a concept of "convex set is defined" (linear space, convex metric space, generalized convex space, ...) we denote

$$P_{cv}(X) := \{Y \in P(X) |\ Y \text{ is convex }\}.$$

We also consider the following notions:

- the gap functional generated by d

$$D_d : P(X) \times P(X) \to \mathbb{R}_+,\ D_d(A, B) := \inf\{d(a, b)\ :\ a \in A,\ b \in B\}.$$

- the excess functional of A over B generated by d

$$e_d : P(X) \times P(X) \to \mathbb{R}_+ \cup \{+\infty\},\ e_d(A, B) := \sup\{D_d(a, B)\ :\ a \in A\}.$$

- the Hausdorff-Pompeiu functional generated by d

$$H_d : P(X) \times P(X) \to \mathbb{R}_+ \cup \{+\infty\},\ H_d(A, B) = \max\{e_d(A, B), e_d(B, A)\}.$$

In the same setting, if $G : X \to P(X)$ is a multi-valued operator, then $x \in X$ is called a fixed point for G if $x \in G(x)$. The set

$$Fix(G) := \{x \in X :\ x \in G(x)\}$$

is the fixed point set of G, while

$$SFix(G) := \{x \in X\ :\ G(x) = \{x\}\}$$

is the strict fixed point set of G. We also denote by

$$Graph(G) := \{(x, y) \in X \times X |\ y \in G(x)\}$$

the graph of the multi-valued operator G.
By a selection of G we understand a single-valued operator $g : X \to X$ such that

$$g(x) \in G(x),\ \text{for every } x \in X.$$

Definition 1.1. Let (X, d) be a metric space. Then, $G : X \to P(X)$ is called a multi-valued α-contraction if $\alpha \in (0, 1)$ and

$$H_d(G(x), G(y)) \leq \alpha d(x, y),\ \text{for all } (x, y) \in X \times X. \tag{1}$$

The most important metric fixed point theorem for multi-valued operators was given by Nadler in 1969 (see [**13**]), followed by a slightly refinement of Covitz and Nadler in 1970 (see [**4**]). This result is known in the literature as the Multi-valued Contraction Principle (MCP) and it says that any multi-valued contraction with nonempty and closed values on a complete metric space has at least one fixed point and there exists an iterative sequence of Picard type $\{x_n\}_{n\in\mathbb{N}} \subset X$ (i.e., $x_{n+1} \in G(x_n), n \in \mathbb{N}$), starting from arbitrary $x_0 \in X$, which converges to a fixed point g^* of G, which may depend of x_0, i.e., $g^* = g^*(x_0)$.

The following retraction-displacement notions for the case of multi-valued operators were introduced in [**21**].

Definition 1.2. Let (X, d) be a metric space, $G : X \to P(X)$ be a multi-valued operator such that $Fix(G) \neq \emptyset$ and let $r : X \to Fix(G)$ be a set retraction, i.e., $r_{|_{Fix(G)}} = 1_{|_{Fix(G)}}$. Let $\gamma : \mathbb{R}_+ \to \mathbb{R}_+$ be nondecreasing, continuous in 0 with $\gamma(0) = 0$. By definition, the condition

$$d(x, r(x)) \leq \gamma(D_d(x, G(x))), \text{ for all } x \in X \tag{2}$$

is the generalized retraction-displacement condition on G corresponding to r.
In particular, if there exists $k > 0$ such that

$$d(x, r(x)) \leq kD_d(x, G(x)), \text{ for all } x \in X, \tag{3}$$

then relation (3) is called the retraction-displacement condition on G corresponding to r.

By the proof of MCP, we can easily observe that, in a complete metric space, any multi-valued α-contraction with closed values, satisfies a retraction-displacement type condition corresponding to a set retraction, see [**13**]. Thus, the MCP given by Nadler and Covitz-Nadler can be formulated as follows.

Theorem 1.3. *Let (X, d) be a complete metric space and $G : X \to P_{cl}(X)$ be a multi-valued α-contraction. Then, for every $x_0 \in X$ there exists an iterative sequence of Picard type $\{x_n\}_{n\in\mathbb{N}}$ starting from $x_0 \in X$ which converges to a fixed point $x^*(x_0)$ of G and the following retraction-displacement condition holds*

$$d(x_0, x^*(x_0)) \leq \frac{c}{1-\alpha} D_d(x_0, G(x_0)), \text{ for all } x_0 \in X \text{ and some } c \in (0, 1).$$

For various generalizations of MCP and related notions and results see [**1**], [**3**], [**5**], [**10**], [**11**], [**12**], [**14**], [**16**], [**17**], [**18**], [**19**]

In this paper, we will consider the following problem: if a fixed point inclusion $x \in B(x)$ (where B is a "bad" multi-valued operator) is replaced by another fixed point inclusion $x \in G(x)$ (where G is now a "good/better" multi-valued operator), then which properties (existence, data dependence, stability) can be obtained for our initial problem in terms of some conditions on the new operator G ? Our results generalize, to the multi-valued case, some recent theorems given for single-valued operators in [**31**].

2. A REVIEW ON FIXED POINT RESULTS FOR MULTI-VALUED CONTRACTION TYPE OPERATORS

For the main results of the paper we will recall first a general class of multi-valued operators. More precisely, we consider the class of multi-valued operators satisfying the contraction condition (1) for every (x, y) from the graph of the multi-valued operator. This case gives a consistent extension of the MCP.

Definition 2.1.([**27, 28**]) Let (X, d) be a metric space and $G : X \to P(X)$ be a multi-valued operator. Then, G is said to be a multi-valued graph α-contraction if $\alpha \in (0, 1)$ and

$$H_d(G(x), G(y)) \le \alpha d(x, y), \text{ for all } (x, y) \in Graph(G). \tag{4}$$

It is obvious that any multi-valued α-contraction is a multi-valued graph α-contraction, but the reverse implication does not hold.

The following theorem, proved in [**19**], will be called the Multi-valued Graph Contraction Principle (MGCP).

Theorem 2.2. *Let (X, d) be a complete metric space and $G : X \to P(X)$ be a multi-valued graph α-contraction with closed graph. Then:*

(i) $Fix(G) \neq \emptyset$;

(ii) For every $x_0 \in X$ there exists an iterative sequence of Picard type $\{x_n\}_{n\in\mathbb{N}}$ starting from $x_0 \in X$ which converges to a fixed point $x^*(x_0)$ of G and the following retraction-displacement condition holds

$$d(x_0, x^*(x_0)) \le \frac{c}{1-\alpha} D_d(x_0, G(x_0)), \textit{ for all } x_0 \in X \textit{ and some } c \in (0, 1).$$

Some examples of multi-valued graph contractions are given below. For other examples see [**3**], [**12**].

Definition 2.3. Let (X, d) be a metric space and $G : X \to P(X)$ be a multi-valued operator. We say that G is a multi-valued Berinde type (α, L)-contraction (see [**1**]) if there exist $\alpha \in (0, 1)$ and $L \ge 0$ such that

$$H_d(G(x), G(y)) \le \alpha d(x, y) + LD_d(y, G(x)), \text{ for every } (x, y) \in X \times X.$$

Then, any multi-valued Berinde type (α, L)contraction (also known as multi-valued weak contraction or multi-valued almost contraction) is a multi-valued graph α-contraction.

Another example comes from Dube and Singh, see [**5**].

Definition 2.4. Let (X, d) be a metric space and $G : X \to P(X)$ be a multi-valued operator. Then, G is called a multi-valued Dube-Singh type β-contraction if there exists $\beta \in (0, \frac{1}{2})$ such that

$$H_d(G(x), G(y)) \le \beta \left(D_d(x, G(x)) + D_d(y, G(y))\right), \text{ for every } (x, y) \in X \times X.$$

Then, any multi-valued Dube-Singh type β-contraction $G : X \to P(X)$ is a multi-valued graph $\frac{\beta}{1-\beta}$-contraction.

Another significant extension of the MCP, as well as of the MGCP, was given by Y. Feng, and S. Liu in 2006. We recall first the definition of a multi-valued Feng-Liu contraction, see [**6**].

Definition 2.5. Let (X, d) be a metric space, $G : X \to P(X)$ be a multi-valued operator, $\beta \in (0, 1)$ and $x \in X$. Consider the set

$$I_\beta^x := \{y \in G(x) : \beta d(x, y) \leq D_d(x, G(x))\}.$$

Then, G is called a multi-valued Feng-Liu (α, β)-contraction if $\alpha \in (0, \beta)$ and for each $x \in X$ there is $y \in I_\beta^x$ such that

$$D_d(y, G(y)) \leq \alpha d(x, y).$$

Notice that, if $\beta \in (0, 1)$, then the set I_β^x is nonempty for every $x \in X$.

It is esy to see that any multi-valued graph α-contraction is a multi-valued Feng-Liu (α, β)-contraction, and the reverse implication does not hold, see [**6**].

The following result was essentially proved by Feng and Liu in [**6**].

Theorem 2.6. *Let (X, d) be a complete metric space and $G : X \to P_{cl}(X)$ be a multi-valued Feng-Liu (α, β)-contraction. Suppose that either the mapping $g : X \to \mathbb{R}_+$, $g(x) = D_d(x, G(x))$ is lower semi-continuous or $Graph(G)$ is closed. Then, the following conclusions hold:*

(i) $Fix(G) \neq \emptyset$;

(ii) for every $x_0 \in X$ there exists an iterative sequence $\{x_n\}_{n\in\mathbb{N}}$ of Picard type for G starting from x_0 which converges to $x^*(x_0) \in Fix(G)$ and the following relation holds

$$d(x_0, x^*(x_0)) \leq \frac{1}{1 - \frac{\alpha}{\beta}} D_d(x_0, G(x_0)).$$

In particular, the following fixed point theorem also holds.

Theorem 2.7. *Let (X, d) be a complete metric space and $G : X \to P(X)$ be a multi-valued operator with closed graph. Suppose there exists $\alpha \in (0, 1)$ such that*

$$D_d(y, G(y)) \leq \alpha d(x, y), \text{ for all } (x, y) \in Graph(G). \tag{5}$$

Then, G has at least one fixed point and for every $x_0 \in X$ there exists an iterative sequence $\{x_n\}_{n\in\mathbb{N}}$ of Picard type for G starting from x_0 which converges to $x^*(x_0) \in Fix(G)$ such that the following relation holds

$$d(x_0, x^*(x_0)) \leq \frac{1}{1 - \alpha} D_d(x_0, G(x_0)).$$

It is easy to observe that the following connection holds.

Remark 2.8. *Any multi-valued graph α-contraction satisfies the above contraction type condition (5) and any multi-valued operator satisfying (5) is a multi-valued Feng-Liu α-contraction. Indeed, if we suppose that there exists $\alpha \in (0,1)$ such that*

$$H_d(G(x), G(y)) \leq \alpha d(x,y), \text{ for all } (x,y) \in Graph(G),$$

then for $x \in X$ and $y \in G(x)$ we have $D_d(y, G(y)) \leq H_d(G(x), G(y)) \leq \alpha d(x,y)$. Notice that the reverse implication does not hold.

By the above result the following definitions occur, see also [**27**], [**28**], [**9**], [**21**].

Definition 2.9. Let (X,d) be a metric space and $G : X \to P(X)$ be a multi-valued operator. Then, G is called a multi-valued Nadler type operator (briefly mNt operator) if for each $x_0 \in X$ there exists a sequence $\{x_n\}_{n\in\mathbb{N}}$ of Picard iterates for G starting from x_0 which converges to a fixed point $x^*(x_0)$ of G.

Definition 2.10. Let (X,d) be a metric space and $G : X \to P(X)$ be an mNt operator. Define $\tilde{G} : X \to P(Fix(G))$, given by $\tilde{G}(x) := \{\ x^* \in Fix(G)\ |$ there is a sequence of Picard iterates for G starting from x that converges to $x^*\}$.

Definition 2.11. Let (X,d) be a metric space and $G : X \to P(X)$ be an mNt operator. Consider the multi-valued operator $\tilde{G}$ generated by G by Definition 2.10. A selection $\tilde{g} : X \to Fix(G)$ of $\tilde{G}$ having the property that $\tilde{g}_{|_{Fix(G)}} = 1_{|_{Fix(G)}}$ is called a selection-retraction for G.

By combining the retraction-displacement conditions with the concept of mNt operators we get the following significant subclass of the class of mNt operators, see also [**15**], [**21**].

Definition 2.12. Let (X,d) be a metric space and $G : X \to P(X)$ be an mNt operator. Then G is called a γ-multi-valued Nadler type operator (briefly γ-mNt operator) if $\gamma : \mathbb{R}_+ \to \mathbb{R}_+$ is nondecreasing, continuous in 0 with $\gamma(0) = 0$ and there exists a selection $\tilde{g}$ of $\tilde{G}$, such that

$$d(x, \tilde{g}(x)) \leq \gamma(D_d(x, G(x))), \text{ for all } x \in X. \tag{6}$$

In particular, an mNt operator for which there exists $k > 0$ and a selection $\tilde{g}$ of $\tilde{G}$, such that

$$d(x, \tilde{g}(x)) \leq kD_d(x, G(x)), \text{ for all } x \in X \tag{7}$$

is called a k-multi-valued Rus type operator (briefly k-mNt operator).

Notice that, in both cases, the selection $\tilde{g} : X \to X$ is a selection-retraction for G.

Remark 2.13 *Notice that a multi-valued operator G satisfying the assumptions of Theorem 2.6 is a $\frac{1}{1-\frac{\alpha}{\beta}}$-mNt operator, while, a multi-valued operator G satisfying the hypotheses from Theorem 2.7 is a $\frac{1}{1-\alpha}$-mNt operator. In particular, any multi-valued graph α-contraction is a $\frac{1}{1-\alpha}$-mNt operator.*

3. ADMISSIBLE PERTURBATION TECHNIQUE IN THE THEORY OF MULTI-VLUED OPERATORS

Let X be a nonempty set and $A : X \times X \to X$ be a mapping satisfying the following two conditions:

(AP1) $A(x, x) = x$, for each $x \in X$;

(AP2) if $x, y \in X$ satisfy $A(x, y) = x$, then $y = x$.

In terms of the above mapping A, the following notion was introduced in [**20**].

Definition 3.1. Let X be a nonempty set and $A : X \times X \to X$ be a mapping having the properties (AP1) and (AP2). Let $T : X \to P(X)$ be a multi-valued operator. Then, the multi-valued operator $T_A : X \to P(X)$ given by $T_A(x) := \{A(x, y) : y \in T(x)\}$ is called the admissible perturbation of T corresponding to A.

The following important properties hold.

Lemma 3.2. (see [**20**]) *If X is a nonempty set and $T : X \to P(X)$ is a multi-valued operator which admits an admissible perturbation T_A, then*

$$Fix(T) = Fix(T_A) \text{ and } SFix(T) = SFix(T_A).$$

Some examples of concrete mappings A inducing an admissible perturbation of a multi-valued operator are given now.

Example 3.3. Let $\mathcal{E}$ be a linear space, $\lambda \in \mathbb{R} \setminus \{0\}$ and $A_\lambda : \mathcal{E} \times \mathcal{E} \to \mathcal{E}$ be given by

$$A_\lambda(x, y) := \lambda x + (1 - \lambda)y.$$

If $T : \mathcal{E} \to P(\mathcal{E})$ is a multi-valued operator, then $T_{A_\lambda} : \mathcal{E} \to P(\mathcal{E})$ given by

$$T_{A_\lambda}(x) := \{\lambda x + (1 - \lambda)y : y \in T(x)\}$$

is the admissible perturbation of T corresponding to A_λ.

Example 3.4.([**7, 8**]) Let (X, F) be a convex prestructure in the sense of Gudder, i.e., X is a nonempty set and $F : [0, 1] \times X \times X \to X$ is a given mapping. If the convex prestructure (X, F) satisfies the following axioms:

(AX1) $F(\lambda, x, y) = F(1 - \lambda, y, x)$, for every $\lambda \in [0, 1]$ and each $x, y \in X$;

(AX2) $F(\lambda, x, F(\mu, y, z)) = F(\lambda + (1 - \lambda)\mu, F(\frac{\lambda}{\lambda+(1-\lambda)\mu}, x, y), z)$, for every $\lambda, \mu \in [0, 1]$ with $\lambda + (1 - \lambda)\mu \neq 0$ and for each $x, y, z \in X$;

(AX3) $F(\lambda, x, x) = x$, for every $\lambda \in [0, 1]$ and each $x \in X$;

(AX4) If for some $\lambda \in [0, 1)$ and $x \in X$ we have that $F(\lambda, x, y) = F(\lambda, x, z)$, then $y = z$;

(AX5) $F(0, x, y) = y$, for every $x, y \in X$,

then the pair (X, F) is called a convex structure in the sense of Gudder.

In this setting, a set $Y \in P(X)$ is called convex if for every $\lambda \in [0, 1]$ and every $x, y \in X$ we have that $F(\lambda, x, y) \in Y$.

Let (X, F) be a convex structure and $\lambda \in (0, 1)$. We define now the operator $A_\lambda : X \times X \to X$ by

$$A_\lambda(x, y) := F(\lambda, x, y).$$

If $T : X \to P(X)$ is a multi-valued operator, then $T_{A_\lambda} : X \to P(X)$ given by

$$T_{A_\lambda}(x) := \{F(\lambda, x, y) : y \in T(x)\}$$

is the admissible perturbation of T corresponding to A_λ.

Example 3.5. ([**35**], [**34**]) Let (X, d) be a metric space and W be a convex structure in the sense of Takahashi, i.e., $W : X \times X \times [0, 1] \to X$ satisfies the following property:

$$d(u, W(x, y, \lambda)) \leq \lambda d(u, x) + (1 - \lambda)d(u, y).$$

The triple (X, d, W), where (X, d) is a metric space and $W : X \times X \times [0, 1] \to X$ is a convex structure on X is called a convex metric space in the sense of Takahashi. In this framework, a set $Y \in P(X)$ is called convex if $W(x, y, \lambda) \in Y$, for every $\lambda \in [0, 1]$ and every $x, y \in X$.

In a convex metric space in the sense of Takahashi (X, d, W), by Proposition 3 in [**35**] we immediately get that

$$W(x, x, \lambda) = x, \text{ for every } \lambda \in [0, 1] \text{ and } x \in X. \tag{8}$$

Let (X, d, W) be a convex metric space in the sense of Takahashi. Assume that the following relation hold:

$$\lambda \in (0, 1), x, y \in X \text{ with } W(x, y, \lambda) = x \text{ imply that } y = x. \tag{9}$$

Then, for $\lambda \in (0, 1)$ let us define the operator $A_\lambda : X \times X \to X$ by

$$A_\lambda(x, y) := W(x, y, \lambda).$$

In this context, if $T : X \to P(X)$ is a multi-valued operator, then the multi-valued operator $T_{A_\lambda} : X \to P(X)$ defined by

$$T_{A_\lambda}(x) := \{W(x, y, \lambda) : y \in T(x)\}$$

is the admissible perturbation of T corresponding to A_λ.

4. FIXED POINTS FOR MULTI-VALUED OPERATORS IN TERMS OF ADMISSIBLE PERTURBATION

In this section, we will give some partial answers to the following problem: under which assumptions on the admissible perturbation T_A of a multi-valued operator T, corresponding to a mapping $A : X \times X \to X$ having the properties (AP1) and (AP2), the fixed point inclusion $x \in T(x)$ possess at least one solution and satisfy some stability properties ?

By Lemma 3.2 we immediately get the following existence result.

Theorem 4.1. *Let (X, d) be a metric space and $T : X \to P(X)$ be a multi-valued operator. Consider a mapping $A : X \times X \to X$ satisfying the conditions*

(AP1) and (AP2). Suppose that the admissible perturbation of T corresponding to A $T_A : X \to P(X), T_A(x) = \{A(x, y) : y \in T(x)\}$ is an mNt operator. Then $Fix(T) \neq \emptyset$.

Definition 4.2. Let (X, d) be a metric space, $T : X \to P(X)$ be a multi-valued operator such that $Fix(T) \neq \emptyset$ and there exists $r : X \to Fix(T)$ a set retraction. Then

$$X = \bigcup_{x^* \in Fix(T)} r^{-1}(x^*)$$

is a partition of X, called the fixed point partition of X corresponding to r.

Let us discuss now some stability and data dependence properties in terms of multi-valued admissible perturbations. The following concepts are well-known, see e.g. [**33**], [**16**], [**18**], [**20**], [**17**], [**21**].

Definition 4.3. Let (X, d) be a metric space and $T : X \to P(X)$ be a multi-valued operator. The fixed point inclusion $x \in T(x), x \in X$ is said to be Ulam-Hyers stable if there exists $c > 0$ such that, for every $\epsilon > 0$ and any $z \in X$ with

$$D_d(z, T(z)) \leq \epsilon,$$

there exists $x^* \in Fix(T)$ having the property

$$d(z, x^*) \leq c \cdot \epsilon.$$

More generally, the fixed point inclusion $x \in T(x), x \in X$ is said to be generalized Ulam-Hyers stable if there exists a function $\mu : \mathbb{R}_+ \to \mathbb{R}_+$ which is nondecreasing, continuous in 0, with $\mu(0) = 0$, such that for every $\epsilon > 0$ and any $z \in X$ with $D_d(z, T(z)) \leq \epsilon$, there exists $x^* \in Fix(T)$ satisfying the relation

$$d(z, x^*) \leq \mu(\epsilon).$$

Theorem 4.4. *Let (X, d) be a metric space and $T : X \to P(X)$ be a multi-valued operator. Consider a mapping $A : X \times X \to X$ satisfying the conditions* (AP1) *and* (AP2). *Suppose:*

(a) the admissible perturbation $T_A(x) = \{A(x, y) : y \in T(x)\}$ of T corresponding to A is a γ-mNt operator;

(b) there exists $\ell > 0$ such that $D_d(x, T_A(x)) \leq \ell D_d(x, T(x))$, for each $x \in X$.

Then the fixed point inclusion $x \in T(x), x \in X$ is generalized Ulam-Hyers stable.

Proof. Let $\epsilon > 0$ and take any $z \in X$ with $D_d(z, T(z)) \leq \epsilon$. Then, since T_A is a γ-mNt operator there exists a selection-retraction $\tilde{t}_A$ of T_A, such that

$$d(x, \tilde{t}_A(x)) \leq \gamma(D_d(x, T_A(x))), \text{ for all } x \in X.$$

Then, taking $x := z$ in the above relation, we obtain

$$d(z, \tilde{t}_A(z)) \leq \gamma(D_d(z, T_A(z))),$$

where $\tilde{t}_A(z) \in Fix(T_A)$. Thus, $\tilde{t}_A(z) \in Fix(T)$ and, by assumption (b), we get that

$$d(z, \tilde{t}_A(z)) \leq \gamma(D_d(z, T_A(z))) \leq \gamma(\ell D_d(z, T(z))) \leq \gamma(\ell\varepsilon).$$

□

We recall now the concept of well-posedness in the sense of Reich and Zaslavski.

Definition 4.5. Let (X, d) be a metric space and $T : X \to P(X)$ be a multi-valued operator such that $Fix(T) \neq \emptyset$ and let $r : X \to Fix(T)$ be a set retraction. Then the fixed point inclusion $x \in T(x), x \in X$ is said to be well-posed with respect to the fixed point partition of X corresponding to r if for each $x^* \in Fix(T)$ and for each sequence $\{u_n\}_{n\in\mathbb{N}} \subset r^{-1}(x^*)$ with $D_d(u_n, T(u_n)) \to 0$, we have that $u_n \to x^*$ as $n \to \infty$.

Our next result proves a well-posedness theorem for the fixed point inclusion $x \in T(x)$ under the assumption that the admissible perturbation T_A is a γ-mNt operator.

Theorem 4.6. *Let (X, d) be a metric space and $T : X \to P(X)$ be a multi-valued operator. Consider a mapping $A : X \times X \to X$ satisfying the conditions* (AP1) *and* (AP2). *Suppose:*

(a) the admissible perturbation $T_A(x) = \{A(x, y) : y \in T(x)\}$ of T corresponding to A is a γ-mNt operator;

(b) there exists $\ell > 0$ such that $D_d(x, T_A(x)) \leq \ell D_d(x, T(x))$, for each $x \in X$.

Then, the fixed point inclusion $x \in T(x), x \in X$ is well-posed in the sense of Reich and Zaslavski with respect to the fixed point partition of X corresponding to the selection-retraction $\tilde{t}_A$ of T_A.

Proof. Since T_A is a γ-mNt operator, there exists a selection-retraction $\tilde{t}_A$ of T_A, such that

$$d(x, \tilde{t}_A(x)) \leq \gamma(D_d(x, T_A(x))), \text{ for all } x \in X.$$

Let $x^* \in Fix(T_A)$. Then $x^* \in Fix(T)$. Take any sequence $\{u_n\}_{n\in\mathbb{N}} \subset \tilde{t}_A^{-1}(x^*)$ with $D_d(u_n, T(u_n)) \to 0$.

Then, we obtain

$$d(u_n, x^*) = d(u_n, \tilde{t}_A(u_n)) \leq \gamma(D_d(u_n, T_A(u_n))) \leq \gamma(\ell D_d(u_n, T(u_n))) \to 0, \text{ as } n \to \infty.$$

□

In what follows we will study the data dependence phenomenom for the fixed point set of a given multi-valued operator, see also [**21**]. More precisely, if $T, S : X \to P(X)$ are two multi-valued operators having nonempty fixed point sets and if there exists $\eta > 0$ such that

$$H_d(T(x), S(x)) \leq \eta, \text{ for each } x \in X,$$

we are interested under which additional conditions there exists an nondecreasing function $\chi : \mathbb{R}_+ \to \mathbb{R}_+$ which is continuous in 0 and satisfies the relation $\chi(0) = 0$, such that

$$H_d(Fix(T), Fix(S)) \leq \chi(\eta).$$

Concerning the above problem we have the following result.

Theorem 4.7. *Let (X, d) be a metric space and let $T, S : X \to P(X)$ be two multi-valued operators. Consider a mapping $A : X \times X \to X$ satisfying the conditions* (AP1) *and* (AP2). *Suppose:*

(a) the admissible perturbation $T_A(x) = \{A(x, y) : y \in T(x)\}$ of T corresponding to A is a γ-mNt operator;

(b) the admissible perturbation $S_A(x) = \{A(x, y) : y \in S(x)\}$ of S corresponding to A is a ν-mNt operator;

(c) there exists $\ell > 0$ such that $D_d(x, T_A(x)) \leq \ell D_d(x, T(x))$, for each $x \in X$;

(d) there exists $\kappa > 0$ such that $D_d(x, S_A(x)) \leq \kappa D_d(x, S(x))$, for each $x \in X$;

(e) there exists $\eta > 0$ such that $H_d(T(x), S(x)) \leq \eta$, for each $x \in X$.

Then, the fixed point set of T satisfies the data dependence phenomenon, i.e.,

$$H_d(Fix(T), Fix(S)) \leq \chi(\eta) := \max\{\nu(\kappa\eta), \gamma(\ell\eta)\}.$$

Proof. We will show that for every $u \in Fix(T)$ there exists $v \in Fix(S)$ such that $d(u, v) \leq \chi(\eta)$ and vice-versa, for every $v \in Fix(S)$ there exists $u \in Fix(T)$ such that $d(u, v) \leq \chi(\eta)$.
Let $u \in Fix(T)$. Since S_A is a ν-mNt operator, there exists a selection-retraction $\tilde{s}_A$ of S_A, such that

$$d(x, \tilde{s}_A(x)) \leq \nu(D_d(x, S_A(x))), \text{ for all } x \in X.$$

Taking $x := u$ we get that $\tilde{s}_A(u) \in Fix(S)$ and

$$d(u, \tilde{s}_A(u)) \leq \nu(D_d(u, S_A(u))) \leq \nu(\kappa D_d(u, S(u))) \leq \nu(\kappa H_d(T(u), S(u))) \leq \nu(\kappa\eta).$$

Similarly, for $v \in Fix(S)$ there exists $\tilde{t}_A(v) \in Fix(T)$ such that

$$d(v, \tilde{t}_A(v)) \leq \gamma(D_d(v, T_A(v))) \leq \gamma(\ell D_d(v, T(v))) \leq \gamma(\ell H_d(S(v), T(v))) \leq \gamma(\ell\eta).$$

By the above relations we get that

$$H_d(Fix(T), Fix(S)) \leq \max\{\nu(\kappa\eta), \gamma(\ell\eta)\}.$$

□

We will continue our study by considering the Ostrowski stability property (see also [**21**]) for a fixed point inclusion in terms of its admissible perturbation.

Definition 4.8. Let (X, d) be a metric space and $T : X \to P(X)$ be a multi-valued operator such that $Fix(T) \neq \emptyset$ and let $r : X \to Fix(T)$ be a set retraction. The fixed point inclusion $x \in T(x)$ has the Ostrowski stability property with respect to the fixed point partition of X corresponding to r if for each $x^* \in Fix(T)$ and for each sequence $\{y_n\}_{n\in\mathbb{N}} \subset r^{-1}(x^*)$ with

$$D_d(y_{n+1}, T(y_n)) \to 0 \text{ as } n \to \infty,$$

we have that $y_n \to x^*$ as $n \to \infty$.

In terms of a fixed point partition generated by a set retraction, the following notion was introduced in [**21**].

Definition 4.9. Let (X, d) be a metric space, $T : X \to P(X)$ be a multi-valued operator such that $Fix(T) \neq \emptyset$ and $r : X \to Fix(T)$ be a set retraction. Then, T is called a multi-valued k-quasicontraction with respect to the fixed point partition corresponding to r if there exists $k \in (0, 1)$ such that

$$e_d(T(x), r(x)) \leq kd(x, r(x)), \text{ for every } x \in X.$$

Our next result proves an Ostrowski stability property for the fixed point inclusion $x \in T(x)$ under the assumption that the admissible perturbation T_A is an mNt operator.

We recall first a well-known result in mathematical analysis.

Lemma 4.10.(Cauchy-Toeplitz Lemma, see e.g. [**21**]) *Let $(a_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{R}_+$, such that the series $\sum_{n\geq 0} a_n$ is convergent and let $(b_n)_{n\in\mathbb{N}} \in \mathbb{R}$ be a sequence with nonnegative terms such that $\lim_{n\to\infty} b_n = 0$. Then*

$$\lim_{n\to\infty} (\sum_{k=0}^{n} a_{n-k} b_k) = 0.$$

Theorem 4.11. *Let (X, d) be a metric space and $T : X \to P(X)$ be a multi-valued operator. Consider a mapping $A : X \times X \to X$ satisfying the conditions* (AP1) *and* (AP2). *Suppose:*

(a) the admissible perturbation $T_A(x) = \{A(x, y) : y \in T(x)\}$ of T corresponding to A is an mNt operator and there exists a selection-retraction $\tilde{t}_A$ of T_A;

(b) the admissible perturbation $T_A : X \to P(X)$ of T corresponding to A is a multi-valued k-quasicontraction;

(c) there exists $\ell > 0$ such that $D_d(x, T_A(x)) \leq \ell D_d(x, T(x))$, for each $x \in X$.

Then, the fixed point inclusion $x \in T(x), x \in X$ is Ostrowski stable with respect to the fixed point partition of X corresponding to the selection-retraction $\tilde{t}_A$ of T_A.

Proof. Since T_A is an mNt operator, the fixed point set $Fix(T)$ is nonempty. Let $\tilde{t}_A : X \to Fix(T)$ be a selection-retraction of T_A. Let $x^* \in Fix(T)$ and let $\{y_n\}_{n\in\mathbb{N}} \subset \tilde{t}_A^{-1}(x^*)$ a sequence with the property that $D_d(y_{n+1}, T(y_n)) \to 0$ as $n \to \infty$.
Then, we have

$$d(y_{n+1}, x^*) = d(y_{n+1}, \tilde{t}_A(y_n)) \leq D_d(y_{n+1}, T_A(y_n)) + e_d(T_A(y_n), \tilde{t}_A(y_n)) \leq$$

$$\ell D_d(y_{n+1},T(y_n))+kd(y_n,\tilde{t}_A(y_n))=\ell D_d(y_{n+1},T(y_n))+kd(y_n,x^*)\leq$$
$$\ell D_d(y_{n+1},T(y_n))+k[\ell D_d(y_n,T(y_{n-1}))+kd(y_{n-1},x^*)]\leq\cdots\leq$$
$$\ell D_d(y_{n+1},T(y_n))+k\ell D_d(y_n,T(y_{n-1}))+\cdots+k^n\ell D_d(y_1,T(y_0))+k^{n+1}d(y_0,x^*).$$

By the above Cauchy-Toeplitz Lemma we get the desired conclusion. □

5. EXAMPLES

5.1 The case of a Banach space

Let $(\mathcal{E},\|\cdot\|)$ be a Banach space, $\lambda\in\mathbb{R}\setminus\{0\}$ and $A_\lambda:\mathcal{E}\times\mathcal{E}\to\mathcal{E}$ be given by $A_\lambda(x,y):=\lambda x+(1-\lambda)y$. Recall that, if $T:\mathcal{E}\to P(\mathcal{E})$ is a multi-valued operator, then the multi-valued operator $T_{A_\lambda}:\mathcal{E}\to P(\mathcal{E})$ given by

$$T_{A_\lambda}(x):=\{\lambda x+(1-\lambda)y:y\in T(x)\}$$

is the admissible perturbation of T corresponding to A_λ.

Then, we have the following result.

Theorem 5.1. *Let $(\mathcal{E},\|\cdot\|)$ be a Banach space and $T:\mathcal{E}\to P_{cv}(\mathcal{E})$ be a multi-valued operator with closed graph and $\lambda\in\mathbb{R}\setminus\{0\}$. Suppose that the admissible perturbation T_{A_λ} of T, corresponding to the mapping A given above, is a multi-valued Feng-Liu (α,β)-contraction, i.e., for each $x\in\mathcal{E}$ there exists $z\in I^x_\beta:=\{z\in T_A(x):\beta\|x-z\|\leq D_{\|\cdot\|}(x,T_A(x))\}$ such that*

$$D_{\|\cdot\|}(z,\lambda z+(1-\lambda)T(z))\leq\alpha\|x-z\|.$$

Then, the following conclusions hold:

(i) T is a $\frac{\beta}{\beta-\alpha}$-mNt operator;

(ii) the fixed point inclusion $x\in T(x),x\in\mathcal{E}$ is well-posed in the sense of Reich and Zaslavski and has the Ulam-Hyers stability property.

Proof. (i) Since $Graph(T)$ is closed, we can prove that $Graph(T_{A_\lambda})$ is closed too. Indeed, let $u_n\in T_{A_\lambda}(x_n)$ such that $x_n\to x,u_n\to u$ as $n\to\infty$. We will prove that $u\in T_{A_\lambda}(x)$. Since $u_n\in T_{A_\lambda}(x_n)$ there exists $v_n\in T(x_n)$ such that $u_n=\lambda x_n+(1-\lambda)v_n$. By the fact that $u_n\to u$ and $x_n\to x$ we get that there exists $v\in\mathcal{E}$ such that $v_n\to v$ as $n\to\infty$. Taking into account that $Graph(T)$ is closed, we get (since $v_n\in T(x_n),n\in\mathbb{N}$) that $v\in T(x)$. Thus, $u=\lambda x+(1-\lambda)v\in T_{A_\lambda}(x)$. Since the admissible perturbation T_{A_λ} of T corresponding to A is a multi-valued Feng-Liu (α,β)-contraction with closed graph, by Theorem 2.6, Remark 2.13 and Theorem 4.1, we get that T_{A_λ} is a $\frac{\beta}{\beta-\alpha}$-mNt operator.

(ii) Since T has convex values, let us observe that the following relation hold

$$D_{\|\cdot\|}(x,\lambda x+(1-\lambda)T(x))\leq|\lambda|D_{\|\cdot\|}(x,x)+|1-\lambda|D_{\|\cdot\|}(x,T(x))=|1-\lambda|D_{\|\cdot\|}(x,T(x)).$$

Thus, all the assumptions of Theorem 4.4 and, respectively Theorem 4.6 are satisfied and we get the second conclusion. □

A data dependence result for the fixed point set can be also given.

Theorem 5.2. *Let $(\mathcal{E}, \|\cdot\|)$ be a Banach space and $T, S : \mathcal{E} \to P_{cv}(\mathcal{E})$ be two multi-valued operators with closed graph. Let $\lambda \in \mathbb{R} \setminus \{0\}$. Suppose that:*

(a) the admissible perturbation T_{A_λ} is a multi-valued Feng-Liu (α, β)-contraction;

(b) the admissible perturbation S_{A_λ} is a multi-valued Feng-Liu (μ, ν)-contraction.

(c) there exists $\eta > 0$ such that $H_{\|\cdot\|}(T(x), S(x)) \leq \eta$, for each $x \in X$.

Then, the following conclusions hold:

(i) T is a $\frac{\beta}{\beta-\alpha}$-mNt operator and S is a $\frac{\nu}{\nu-\mu}$-mNt operator;

(ii) the fixed point inclusion $x \in T(x), x \in \mathcal{E}$ is well-posed in the sense of Reich and Zaslavski and has the Ulam-Hyers stability property.

Then, the fixed point set of T satisfies the data dependence phenomenon, i.e.,

$$H_{\|\cdot\|}(Fix(T), Fix(S)) \leq \max\{\frac{\beta}{\beta-\alpha}|1-\lambda|\eta, \frac{\nu}{\nu-\mu}|1-\lambda|\eta\}.$$

Proof. Since the admissible perturbation T_{A_λ} of T corresponding to A is a multi-valued Feng-Liu (α, β)-contraction with closed graph, by Theorem 2.6, Remark 2.13 and Theorem 4.1, we get that T_{A_λ} is a $\frac{\beta}{\beta-\alpha}$-mNt operator. Similarly, since the admissible perturbation S_{A_λ} of S corresponding to A is a multi-valued Feng-Liu (μ, ν)-contraction with closed graph, by Theorem 2.6, Remark 2.13 and Theorem 4.1, we get that S_{A_λ} is a $\frac{\nu}{\nu-\mu}$-mNt operator.
Moreover, we also have

$$D_{\|\cdot\|}(x, \lambda x + (1-\lambda)T(x)) \leq |1-\lambda| D_{\|\cdot\|}(x, T(x))$$

and

$$D_{\|\cdot\|}(x, \lambda x + (1-\lambda)S(x)) \leq |1-\lambda| D_{\|\cdot\|}(x, S(x)).$$

Now the conclusion follows by Theorem 4.7. □

Finally, by the above proofs and Theorem 4.10, an Ostrowski stability theorem can be also stated as follows.

Theorem 5.3. *Let $(\mathcal{E}, \|\cdot\|)$ be a Banach space and $T : \mathcal{E} \to P_{cv}(\mathcal{E})$ be a multi-valued operator with closed graph. Let $\lambda \in \mathbb{R} \setminus \{0\}$. Suppose that:*

(a) the admissible perturbation T_{A_λ} is a multi-valued Feng-Liu (α, β)-contraction;

(b) the admissible perturbation T_{A_λ} is a multi-valued k-quasicontraction.

Then, the fixed point inclusion $x \in T(x), x \in \mathcal{E}$ has the Ostrowski stability property.

5.2 The case of a metric space endowed with the convex structure of S. Gudder

Let X be a nonempty set, $d : X \times X \to X$ be a metric on X. Suppose that (X, F) is the convex structure in the sense of Gudder, given in Example 3.2. For a multi-valued operator $T : X \to P(X)$ and $\lambda \in (0,1)$, we consider the admissible perturbation of T corresponding to the operator $A_\lambda : X \times X \to X$ given by $A_\lambda(x,y) := F(\lambda,x,y)$, defined by

$$T_{A_\lambda}(x) := \{F(\lambda,x,y) : y \in T(x)\}.$$

Then, we can prove the following result.

Theorem 5.4. *Let (X,d,F) be a complete metric space endowed with a convex structure in the sense of Gudder and $T : X \to P_{cv}(X)$ be a multi-valued operator with closed graph. Let $\lambda \in (0,1)$. Suppose that:*

(a) the admissible perturbation T_{A_λ} of T corresponding to the mapping A given above is a multi-valued Feng-Liu (α,β)-contraction;

(b) there exists $M > 0$ such that $D_d(x, F(\lambda,x,T(x)) \leq MD_d(x,T(x))$, for each $x \in X$ and $\lambda \in (0,1)$;

(c) if $\{x_n\}_{n\in\mathbb{N}}, \{v_n\}_{n\in\mathbb{N}}$ are two sequences in X such that $x_n \to x$ and if the sequence $u_n := F(\lambda,x_n,v_n), n \in \mathbb{N}$ is convergent in X to u, then there exists $v \in X$ such that $v_n \to v$ as $n \to \infty$ and $u = F(\lambda,x,v)$.

Then, the following conclusions hold:

(i) T is a $\frac{\beta}{\beta-\alpha}$-mNt operator;

(ii) the fixed point inclusion $x \in T(x), x \in X$ is well-posed in the sense of Reich and Zaslavski and has the Ulam-Hyers stability property.

Proof. (i) Since $Graph(T)$ is closed, we will prove again that $Graph(T_{A_\lambda})$ is closed. Let $u_n \in T_{A_\lambda}(x_n)$ such that $x_n \to x, u_n \to u$ as $n \to \infty$. We will prove that $u \in T_{A_\lambda}(x)$. Since $u_n \in T_{A_\lambda}(x_n)$ there exists $v_n \in T(x_n)$ such that $u_n = F(\lambda,x_n,v_n)$. By (c) we get that there exists $v \in X$ such that $v_n \to v$ as $n \to \infty$ and $u = F(\lambda,x,v)$. Taking into account that $Graph(T)$ is closed, we get that $v \in T(x)$. Thus, $u = F(\lambda,x,v) \in T_{A_\lambda}(x)$. So, $Graph(T_{A_\lambda})$ is a closed set.
Since the admissible perturbation T_{A_λ} of T corresponding to A is a multi-valued Feng-Liu (α,β)-contraction with closed graph, by Theorem 2.6, Remark 2.13 and Theorem 4.1, we get that T_{A_λ} is a $\frac{\beta}{\beta-\alpha}$-mNt operator.

(ii) Using (a) and (b) we observe that all the assumptions of Theorem 4.4 and, respectively Theorem 4.6 are satisfied and the proof is complete. □

5.3 The case of a complete and convex metric space in the sense of W. Takahashi

Let (X,d,W) be the convex metric space in the sense of Takahashi given in Example 3.4. Assume that the following condition (9) holds. Taking into account of (8), for given $\lambda \in (0,1)$, we consider the operator $A_\lambda : X \times X \to X$ given by $A_\lambda(x,y) := W(x,y,\lambda)$. Then, for $T : X \to P(X)$, the multi-valued operator

$$T_{A_\lambda}(x) := \{W(x,y,\lambda) : y \in T(x)\}$$

is the admissible perturbation of T corresponding to A_λ.

Open questions. It is an open question to prove some existence, data dependence and stability results for multi-valued Feng-Liu (α, β)-contractions in the framework of a complete and convex metric space in the sense of Takahashi. Another open question is to consider the case of a nonempty set endowed with two metrics, see also [**22**]. A third open question is related to obtain some approximation results following the approach proposed in [**25**].

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