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ON ZEROS OF A FAMILY OF CONE FUNCTIONS

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A concept of a (A, B)-search cone function is induced on a cone metric space, which is an essential development of an (α, β)-search functional on a metric space, induced earlier by the author. Main results of the paper concern a zero existence problem for an (A, B)-search cone function (when the cone is not normal) with consequences, and the zero existence preservation problem, for a parametric family of (A, B)-search cone functions, when changing the parameter. Possible applications and generalizations are mentioned, as well.

1. INTRODUCTION AND PRELIMINARIES

In 2009–2013 the author introduced the concept of (α, β)-search functional and proved several versions of theorems on search for zeros of such functionals in a metric space. As consequences, fixed point and coincidence theorems, for single-valued and multi-valued mappings of metric spaces, were obtained, generalizing a number of well-known results of various authors [12–15].

The aim of this article is to present a generalization of the mentioned results concerning (α, β)-search functionals, using two main ideas. The first one is to replace an usual metric space with a cone metric space, that is to consider a cone metric instead of a numerical one. The second important idea belongs to well-known Russian mathematician from Voronezh State University A.I. Perov. This idea consists of the replacing the numerical contraction coefficient, for a contractive

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mapping, with a suitable bounded linear operator. This Perov's idea appeared in his work [30] and found a broad response.

In this article we apply both of these ideas, to generalize the concept of (α, β)-search functional.

As it is noticed in some papers, many fixed point results in cone metric spaces obtained in recent years, in which the assumption is present that the underlying cone is normal and solid, can be reduced to corresponding results in metric spaces (see, for example [21]). On the other hand, it is impossible when we deal with non-normal solid cones.

Some authors claim (see, for example, [27]) that most of cone fixed point results are merely copies of classical ones and that any extension of known fixed point results to cone metric spaces is dispensable, and also that underlying Banach space with the associated cone subset is not necessary. But in fact, it is true for a small and very limited class of results, because only normal cones are required for this. So, fixed point results in cone metric spaces with non-normal cones are possibly proper extensions of the corresponding results for metric spaces.

It should be mentioned that the reducing of fixed point results from cone metric space to the usual metric space mostly concerns fixed point theorems for contractive mappings [6,21]. In this case, a contractive self-mapping on a cone metric space, having a linear operator A as the contraction coefficient, turns to a contractive self-mapping on a metric space, with the numerical contraction coefficient $||A||$. But the estimate of the distance between an initial point and the corresponding fixed point may not be the same. There are scalarization methods [8,11] which allow to transform a normed space valued cone metric into a numerical one. In addition, as it is shown in [22], many fixed point results obtained in cone metric spaces over solid non-normal cones can be reduced to the case of solid normal cones. And moreover, TVS-valued cone metric spaces over solid cones are not an essential generalization of normed space-valued cone metric spaces. It turns out that each solid cone in a TVS is normal, under a suitably defined norm. It is obtained by using the technique of Minkowski functionals [23]. But, here one have to take into account, that if the underlined cone in TVS is not normal, the topology induced by that suitably defined norm is not equivalent to the initial topology of the TVS. So, in this case, the given contractive linear operator in the TVS may turn out not contractive in the new normed topology. Nevertheless, as it is shown in the papers by E. Karapinar [24–26] and in the joint paper by W.-S. Du and E.Karapinar [9], some interesting fixed point theorems for contractive mappings using numerical contraction coefficients, may be proved in terms of usual (numerical-valued) metric spaces, with the help either of the scalarization method by W.-S. Du, when cone-values of the given cone metric in the normed space are replaced with their norms, or with the help of the scalarization using Minkowski functionals. But these methods do not always work if one uses operator contraction coefficients.

In this article, we apply Perov's idea and introduce a concept of a multivalued (A, B)-search cone function where coefficients of a numerical (α, β)-search

functional are replaced with suitable linear operators. Zero existence theorem is proved for such cone functions. As its consequences, fixed point and coincidence point existence theorems are obtained for multivalued mappings between cone metric spaces. In addition, the zero existence preservation theorem is proved, for a parametric family of multivalued (A, B)-search cone functions. The results of this article concerning cone metric spaces can not be derived from analogous results in metric spaces.

At first, below we give necessary definitions and the basic zero existence theorem, for a multi-valued (α, β)-search functional on a metric space.

Definition 1.1. Let (X, d) be a metric space, and $\alpha, \beta \in \mathbb{R}_+, 0 \leq \beta < \alpha$. A multi-valued functional $\varphi : X \rightrightarrows \mathbb{R}_+ = \{x \in \mathbb{R} | x \geq 0\}$ is called (α, β)-*search on* X, if for any point $(x_0, c_0) \in Graph(\varphi) = \{(x, c) \in X \times \mathbb{R}_+ | c \in \varphi(x)\}$, there exists a point $(x', c') \in Graph(\varphi)$ such that $d(x_0, x') \leq c_0$, and $c' \leq \frac{\beta}{\alpha} c_0$.

Definition 1.2. The graph $Graph(\varphi)$ of a multi-valued functional $\varphi : X \rightrightarrows \mathbb{R}_+$ is called $\{0\}$-*complete (*$\{0\}$-*closed)* if and only if for any fundamental sequence $(x_n, c_n) \in Graph(\varphi)$, with $c_n \to 0$, it is true that the sequence $\{x_n\}$ converges to an element $\xi \in X$ (if and only if the graph $Graph(\varphi)$ contains all limit points of the form $(\xi, 0)$).

Theorem 1.3. *Let on a metric space (X, d) a multi-valued (α, β)-search functional $\varphi : X \rightrightarrows \mathbb{R}_+$ be given, for some $\alpha, \beta \in \mathbb{R}_+, 0 \leq \beta < \alpha$. Let also either the space (X, d) be complete and $Graph(\varphi)$ be $\{0\}$-closed, or $Graph(\varphi)$ be $\{0\}$-complete. Then, for any initial point $(x_0, c_0) \in Graph(\varphi)$, there is a point $(\xi, 0) \in Graph(\varphi)$ (that is, $0 \in \varphi(\xi)$) such that $d(x_0, \xi) \leq \frac{c_0}{\alpha - \beta}$.*

The complete proof of this statement is contained in [14].

Definition 1.4. *A fixed point of a multi-valued mapping $F : X \rightrightarrows X$* is a point $\xi \in X$ such that $\xi \in F(\xi)$. Given two metric spaces (X, d), (Y, ρ), *a coincidence point* of multi-valued mappings $F, G :\rightrightarrows Y$ is a point $\xi \in X$ such that $F(\xi) \cap G(\xi) \neq \emptyset$.

One can consider a functional $\varphi : X \rightrightarrows \mathbb{R}_+$ such that its zero subset $Nil(\varphi) := \{\xi \in X | 0 \in \varphi(\xi)\}$ coincides either with the fixed point set $Fix(F) := \{x \in X | x \in F(x)\}$ of the mapping $F : X \rightrightarrows X$, or with the coincidence set $Coin(F, G) := \{x \in X | F(x) \cap G(x) \neq \emptyset\}$ of $F, G : X \rightrightarrows Y$. When applying Theorem 1.3 to such a functional, one can obtain fixed point and coincidence point theorems for multi-valued mappings.

Such type theorems are demonstrated by the author in [12–14]. In [14,15] the preimage existence problem is investigated. That is to search for the preimage $G^{-1}(Z) := \{x \in X | G(x) \cap Z \neq \emptyset$ of a closed subset Z of the space Y, under the mapping $G : X \rightrightarrows Y$. If $Z = \{c\} \subset Y$ is a singleton, one can obtain the root existence theorem, for the inclusion: $c \in F(x)$.

Coincidence points of a number of multi-valued mappings $F_1, \cdots, F_n : X \rightrightarrows Y$ can be considered as the preimage points of the diagonal $\Delta_n \subset Y^n$, under the mapping $F : X \rightrightarrows Y^n$, where $F = (F_1, \cdots, F_n)$.

Similarly, common fixed point theorems for the mappings $F_1, \cdots, F_n : X \rightrightarrows X$ are obtained, in the particular case when $X = Y$.

In this way, in [12–15] several results are obtained generalizing some known statements of different authors.

Now, let us introduce possible counterparts of (α, β)-search functional in a cone metric space. At first we consider the case where a cone K in a Banach space S is normal. Below we give necessary definitions.

Let X be a nonempty set and $(S, \|\cdot\|)$ be a Banach space. Let K be a positive convex closed acute cone, that is K be a closed convex subset satisfying the following conditions: 1) $0 \in K$; 2) $\forall a \in K, a \neq 0 \Longrightarrow \forall \mu > 0, \mu a \in K, -\mu a \notin K$; 3) $\forall a, b \in K \Longrightarrow \forall t \in [0; 1], ta + (1-t)b \in K$.

The presence of the cone K in the set S defines a partial order $\leq_K$ on S. Namely, for any $x, y \in S$ we say $x \leq_K y$ if and only if $y - x \in K$. Thus, $(S, \leq_K)$ is a partially ordered set.

Definition 1.5. The *cone metric* on X associated with the cone K is a mapping satisfying the axioms of a metric, that is for any $x, y \in S$, the following conditions hold:

1. $d_K(x, y) \geq_K 0; (d_K(x, y) = 0) \Longleftrightarrow (x = y)$;
2. $d_K(x, y) = d_K(y, x)$;
3. for any $z \in X$ it is true that $d_K(x, y) \leq_K d_K(x, z) + d_K(z, y)$.

Suppose that $X \neq \emptyset$ is equipped with a cone metric d_K as described above. The space (X, d_K) with the cone metric is called *a cone metric space*.

In 1964 A.I. Perov proved in [30] a generalization of the well-known contraction mapping principle by Banach-Caccioppoli, in spaces with a metric taking values in the positive cone $\mathbb{R}^n_+$ (see also [31]). A positive linear operator with a spectral radius smaller than unity was used as a contraction coefficient.

After these Perov's works, many papers of different authors have appeared with results generalizing the Perov's fixed point theorem. We would like to mention here a few such papers, for instance [1–4,6–8,10] and [19,20,29,30]. In these works the authors consider various generalizations of cone metric spaces and obtain more general fixed point theorems than Perov's one. Different applications of Perov's fixed point theorem are also presented to the theory of differential equations, in particular, to semilinear systems of equations, to fractional differential equations and others. In the paper [29], the authors introduce a new class of (Q, β)-condensing multivalued mappings, where β is a vector measure of the noncompactness and Q is a bounded positive linear operator whose spectral radius $r(Q) < 1$. Some properties of such multivalued mappings are studied. In particular, it is shown that the topological degree theory and classical fixed point theorems can be extended to this class of multivalued mappings.

So, the development of the Perov's idea is realized in several theoretical and applicable directions.

In addition, in 2018 this useful natural idea by A.I. Perov — to replace a number contraction coefficient with a linear bounded operator, was employed (without any reference to Perov's works) by E.S. Zhukovskiy and E.A. Panasenko in [34], to characterize contraction mappings in a cone metric space.

In this paper, we use the Perov's idea of operator coefficients in order to develop the method of search for zeros of (α, β)-search functionals, to extend it to cone metric spaces, and to demonstrate its applications to the fixed point and coincidence theory.

The following definitions 1.6–1.8, 1.11 and theorems 1.9, 1.10, 1.12, 1.13 of this section were presented in [18] (in its original versions).

Assume that the norm in S is monotone with respect to the partial order $\leq_K$, i.e., $(a \leq_K b) \Longrightarrow (\| a \| \leq \| b \|)$. In this case, K is also called *a normal cone* with the normal constant equal to 1. In addition, we assume that K is a generating cone, i.e. $S = K - K$. In other words, any element $a \in S$ is represented as a difference of some elements of K.

In the set $\mathcal{L}(S) := \{L : S \to S\}$ of all bounded linear operators on S, one can consider the subset $\mathcal{L}_+$ of linear operators leaving K invariant, i.e. $\mathcal{L}_+ := \{P \in \mathcal{L}(S) | P(K) \subseteq K\}$. Following [30], we notice that as K is a generating cone *with a monotone norm*, it is easy to see that $\mathcal{L}_+$ is also a positive convex closed acute cone in $\mathcal{L}(S)$. And this cone $\mathcal{L}_+$ defines a partial order on $\mathcal{L}(S)$. Namely, we say that for $F, G, \in \mathcal{L}(S)$ it is true that $F \leq G$ if $G - F \in \mathcal{L}_+$.

Now, we introduce the concept of an (A, B)-search cone function with operator coefficients A, B. Multivalued mappings will be denoted by double arrows $\rightrightarrows$.

The following concept is a cone analog of an (α, β)-search functional.

Definition 1.6. A multivalued mapping $\varphi : X \rightrightarrows K$ is called (A, B)-*search cone function (on X)*, if the following conditions hold:
(i) $A, B \in \mathcal{L}_+(S)$ are bounded commuting linear operators such that A is invertible, $A^{-1} \in \mathcal{L}_+$, and the composition $A^{-1}B : K \to K$ has a spectral radius $\lambda = \lambda(A^{-1}B) < 1$;
(ii) for any $x \in X$ and any $c \in \varphi(x) \subset K$, there exists an element $x' \in X$, such that $d_K(x, x') \leq_K A^{-1}(c)$, and there exists a value $c' \in \varphi(x')$, such that $c' \leq_K A^{-1}B(c)$.

The graph of an (A, B)-search cone function φ is denoted by $Graph(\varphi) = \{(x, c) | x \in X, c \in \varphi(x)\} \subseteq X \times K$.

The Cauchy property and convergence of sequences in $X \times S$ (in particular, in $Graph(\varphi)$) are considered relative to the componentwise metric $D = d_K \times \nu$, where $\nu(a, b) := \| a - b \|$.

Definition 1.7. We say the graph $Graph(\varphi)$ of an (A, B)-search cone function is called $\{0\}$-*complete* if any Cauchy sequence $\{(x_n, c_n)\} \subset Graph(\varphi)$, with $c_n \to 0$, as $n \to \infty$, converges to an element of the graph. The graph $Graph(\varphi)$ is called $\{0\}$-*closed* if all its limit elements of the form $(\xi, 0)$ are contained in it.

Definition 1.8. A sequence $\{(x_n, c_n)\}_{n=0,1,\cdots} \subset Graph(\varphi)$ is called (A, B)-*search*

sequence if the following conditions hold, $n \in \mathbb{N}$:

$$d_K(x_{n-1}, x_n) \leq_K A^{-1}(c_{n-1}), c_n \leq_K A^{-1}B(c_{n-1}). \tag{1}$$

We say the graph $Graph(\varphi)$ of an (A, B)-search cone function is called (A, B)-*search-closed* if it contains limits of (A, B)-search sequences.

For the completeness, we give the following theorem with the complete proof.

Theorem 1.9. *Let (X, d_K) be a complete cone metric space, with the cone $K \subset S$, and the norm in S be monotone with respect to the order $\leq_K$. Let $\varphi : X \rightrightarrows K$ be a multivalued (A, B)-search cone function with operator coefficients $A, B : K \to K$. Assume that the graph $Graph(\varphi)$ is (A, B)-search-closed. Then, for any point $x_0 \in X$ and any value $c_0 \in \varphi(x_0)$, there exists a point $x_* = x_*(x_0, c_0) \in X$, such that $0 \in \varphi(x_*)$ and the cone distance $d_K(x_0, x_*)$ satisfies the following estimate $d_K(x_0, x_*) \leq_K A^{-1}(I - A^{-1}B)^{-1}(c_0)$.*

Proof. The properties of the multivalued (A, B)-search cone function imply that, starting from an arbitrary initial pair $(x_0, c_0) \in Graph(\varphi)$, it is possible to construct an (A, B)-search sequence $\{(x_n, c_n)\} \subset Graph(\varphi)$ with the properties (1). We need to show that $\{x_n\}_{n=0,1,\dots}$ is a Cauchy sequence. The assumption $\lambda(A^{-1}B) < 1$ implies the invertibility of the operator $I - A^{-1}B : K \to K$. And, the operator $(I - A^{-1}B)^{-1}$ is equal to the sum of the iterative series $\sum\limits_{i=0}^{\infty}(A^{-1}B)^i$. This series consists of the operators $(A^{-1}B)^i \in \mathcal{L}_+$. As the cone $\mathcal{L}_+$ is closed, it follows that $(I - A^{-1}B)^{-1} \in \mathcal{L}_+$, moreover, for any $N \in \mathbb{N}$, we have $\sum\limits_{i=0}^{N}(A^{-1}B)^i \leq (I - A^{-1}B)^{-1}$.

From the inequalities (1) and the properties of the cone metric d_K we obtain in the standard way that

$$d_K(x_n, x_{n+m}) \leq_K \sum_{j=0}^{m-1} d_K(x_{n+j}, x_{n+j+1}) \leq_K$$

$$\leq_K \sum_{j=0}^{m-1} A^{-1}(c_{n+j}) \leq_K \sum_{j=0}^{m-1} A^{-1}(A^{-1}B)^{n+j}(c_0) = \sum_{j=0}^{m-1}(A^{-1}B)^{n+j}A^{-1}(c_0) =$$

$$= (A^{-1}B)^n \sum_{j=0}^{m-1}(A^{-1}B)^j A^{-1}(c_0) \leq_K (A^{-1}B)^n(I - A^{-1}B)^{-1}A^{-1}(c_0) =$$

$$= (A^{-1}B)^n[A^{-1}(I - A^{-1}B)^{-1}(c_0)]. \tag{2}$$

As the spectral radius $\lambda(A^{-1}B) < 1$, we have $(A^{-1}B)^n \underset{n\to\infty}{\longrightarrow} 0$ in $\mathcal{L}_+$.

For properties of the spectral radius and the sufficient conditions for the inequality $\lambda(A^{-1}B) < 1$, see [31].

Using the property of the monotonicity of the norm (in other words, the property of the normality of the cone K) we conclude from (2) that

$$\| d(x_n, x_{n+m}) \| \leq_K \| (A^{-1}B)^n(I - A^{-1}B)^{-1}A^{-1}(c_0) \| \underset{n\to\infty}{\longrightarrow} 0.$$

It means that $\{x_n\}_{n=0,1,\dots}$ is a Cauchy sequence.

As the cone metric space (X, d_K) is assumed to be complete, there exists a point $x_* = \lim\limits_{n\to\infty} x_n \in X$. In view of the continuity of the metric d_K (in the corresponding topology), we have

$$d_K(x_0, x_*) = \lim_{n\to\infty} d_K(x_0, x_n) \leq_K \lim_{n\to\infty} \sum_{j=0}^{n-1} d_K(x_j, x_{j+1}) \leq_K \lim_{n\to\infty} \sum_{j=0}^{n-1} A^{-1}(c_j) \leq_K$$

$$\leq_K A^{-1} \lim_{n\to\infty} \sum_{j=0}^{n-1} (A^{-1}B)^j(c_0) \leq_K A^{-1}(I - A^{-1}B)^{-1}(c_0).$$

Note that $c_n \leq_K (A^{-1}B)^n(c_0) \underset{n\to\infty}{\longrightarrow} 0$. By the theorem condition, $Graph(\varphi)$ is (A, B)-search-closed, therefore $(x_*, 0) \in Graph(\varphi)$, i.e. $0 \in \varphi(x_*)$. $\square$

The result of Theorem 1.9 could not be derived from analogous results on metric spaces by scalarization methods, since our conditions for an (A, B)-search cone function contain bounded linear operators A, B instead of number coefficients, and the considered cone functions are multivalued. The same is true for all the following results of this article.

Consider the following consequences from the latter theorem.

Theorem 1.10. *Let (X, d_K) be a complete cone metric space and $F : X \rightrightarrows X$ be a multivalued mapping with closed bounded images. Let a mapping $\varphi : X \rightrightarrows K$, where $\varphi(x) = \{d \in K | \exists z \in F(x), d = d_K(x, z)\}$, be an (A, B)-search cone function for linear bounded operators $A, B \in \mathcal{L}_+$. Suppose that $Graph(\varphi)$ is (A, B)-search-closed. Then, for any point $x_0 \in X$ and any $y_0 \in F(x_0)$, there exists a fixed point $\xi = \xi(x_0, y_0)$ of the mapping F, that is $\xi \in F(\xi)$, and the following estimate holds:*

$$d_K(x_0, \xi) \leq_K A^{-1}(I - A^{-1}B)^{-1}(d_K(x_0, y_0)).$$

Definition 1.11. Let (X, d_K) be a cone metric space, $F : X \rightrightarrows X$ be a multivalued mapping, and $Q : K \to K$ be a bounded linear operator with a spectral radius $\lambda(Q) < 1$. The mapping F is called *Q-contraction (contraction with the operator coefficient Q)* if for any $x_1, x_2 \in X$ and any $y_1 \in F(x_1)$ there exists $y_2 \in F(x_2)$, for which $d_K(y_1, y_2) \leq_K Q(d_K(x_1, x_2))$.

One can consider the particular case of Theorem 1.10, when the mapping F is a Q-contraction.

The following cone metric analogue of the well-known Nadler fixed point theorem is presented in [34]. We show that this result follows from the previous Theorem 1.10.

Theorem 1.12. *([34], Theorem 1). Let (X, d_K) be a complete cone metric space, the norm in S be monotone with respect to the partial order $\leq_K$ induced by the cone K. Let $F : X \rightrightarrows X$ be a multivalued Q-contracting mapping with closed images and an operator contraction coefficient $Q \in \mathcal{L}_+$ having a spectral radius $\lambda(Q) < 1$. Then, for any $x_0 \in X$ and any $y_0 \in F(x_0)$, the mapping F has a fixed point $\xi \in F(\xi)$ such that the following estimate holds: $d_K(x_0, \xi) \leq_K (I - Q)^{-1}(d_K(x_0, y_0))$.*

Proof. We show that this result follows from Theorem 1.10. In fact, this is a particular case of Theorem 1.10. It is easy to see that under the conditions of Theorem 1.12 the mapping $\varphi : X \rightrightarrows K$, where $\varphi(x) = \{c \in K | \exists y \in F(x), c = d_K(x, y)\}$, is a multivalued (I, Q)-search cone function, where $I = Id_X$. Indeed, for any pair $(x_0, c_0) \in Graph(\varphi)$ where $c_0 = d_K(x_0, y_0)$, for some $y_0 \in F(x_0)$, there is a pair $(x_1, c_1) \in Graph(\varphi)$, where $x_1 = y_0$, and (by definition of the contraction mapping F) there is a point $y_1 \in F(x_1) = F(y_0)$, such that $c_1 = d_K(x_1, y_1) \in \varphi(x_1)$, $c_1 = d_K(y_0, y_1) \leq_K Q(d_K(x_0, y_0) = Q(c_0)$. As the mapping F is Q-contraction, its graph is closed. It implies that the $Graph(\varphi)$ is also closed, and in particular, it is (A, B)-search-closed. So, all conditions of Theorem 1.10 are fulfilled, for the mapping F. Therefore, the statement follows from Theorem 1.10. □

One more important consequence of Theorem 1.9 is the following coincidence existence theorem.

Theorem 1.13. *Let $F, G : X \rightrightarrows Y$ be two multivalued mappings with bounded closed images, and $\psi : X \rightrightarrows K$, where*

$$\psi(x) := \{c \in K | \exists y \in F(x), \exists z \in G(x), \rho_K(y, z) = c\}, x \in X.$$

Assume that ψ is an (A, B)-search cone function, with respect to linear operators $A, B \in \mathcal{L}_+$. In addition, assume that $Graph(\psi)$ is (A, B)-search-closed. Then, for any $x_0 \in X$ and any points $y_0 \in F(x_0)$, $z_0 \in G(x_0)$, the mappings F, G have a coincidence point $\xi = \xi(x_0, y_0, z_0) \in X$, that is $F(\xi) \cap G(\xi) \neq \emptyset$, and the following estimate holds:

$$d_K(x_0, \xi) \leq_K A^{-1}(I - A^{-1}B)^{-1}(\rho_K(y_0, z_0)). \quad \square$$

The proof of this theorem follows from Theorem 1.9. It is just sufficient to take into account that $Coin(F, G) = Nil(\varphi)$.

2. MAIN RESULTS.

It should be noticed that the statements of previous theorems 1.9, 1.10, 1.12, 1.13 may be presented under more general conditions.

Now, let us consider (A, B)-search cone functions in the case when the underline cone K is solid and non-normal.

Let $F : S \to S$ be a bounded linear operator on S that is $F \in L(S)$, and $\lambda(F) = \lim\limits_{n\to\infty} \| F^n \|^{1/n} = \inf_n \| F^n \|^{1/n}$ be the spectral radius of F. It is known (see also [7]) that if $\lambda(F) < 1$, then the series $\sum\limits_{i=0}^{\infty} F^n$ is absolutely convergent, the operator $I - F$ is invertible in $L(S)$ and $\sum\limits_{i=0}^{\infty} F^n = (I - F)^{-1}$. Like as above, let S be a Banach space with a cone $K \subset S$, $\leq_K$ be the partial order induced by cone K on S, and for a nonempty set X, let $d_K : X \times X \to K$ be a cone metric. So, we have a cone metric space (X, d_K).

We assume now that the cone K is solid (that is its interior $Int(K) \neq \emptyset$), but not necessary normal. We say $0 \ll c$ if and only if $c \in Int(K)$. It is known that under these conditions, $0 \leq_K x_n \to 0$, as $n \to \infty$, means that for any $c, 0 \ll c$, there exists $n_0 \in \mathbb{N}$, such that $0 \leq_K x_n \ll c$, for any $n > n_0$. And, if for any $c, 0 \ll c$, there exists $n_0 \in \mathbb{N}$, such that $0 \leq_K d_K(x_n, x_m) \ll c$, for any $m, n, m > n > n_0$, it means that $\{x_n\}_n$ is a Cauchy sequence. It is clear that if $F \in L(S)$, then F is monotone (that is $a \leq_K b \Longrightarrow F(a) \leq_K F(b)$) if and only if $F(K) \subseteq K$. (See [7], for the detail proofs of these facts).

The next theorem is a version of Theorem 1.9, without normality condition of the cone K.

Theorem 2.14. *Let (X, d_K) be a complete cone metric space, with solid cone $K \subset S$, which is not necessary normal with respect to the order $\leq_K$. Let $\varphi : X \rightrightarrows K$ be a multivalued (A, B)-search cone function with operator coefficients $A, B : K \to K$. Assume that the graph $Graph(\varphi)$ is (A, B)-search-closed. Then, for any point $x_0 \in X$ and any value $c_0 \in \varphi(x_0)$, there exists a point $x_* = x_*(x_0, c_0) \in X$, such that $0 \in \varphi(x_*)$ and the cone distance $d_K(x_0, x_*)$ satisfies the estimate $d_K(x_0, x_*) \leq_K A^{-1}(I - A^{-1}B)^{-1}(c_0)$.*

Proof. When repeating the proof of Theorem 1.9, we come to a search sequence $\{(x_n, c_n)\} \subset Graph(\varphi)$ starting from an arbitrary initial pair $(x_0, c_0) \in Graph(\varphi)$, and then obtain the estimate (2). In particular, we obtain the inequality

$$d_K(x_n, x_{n+m}) \leq_K (A^{-1}B)^n \sum_{j=0}^{m-1} (A^{-1}B)^j A^{-1}(c_0) \leq_K (A^{-1}B)^n \sum_{j=0}^{\infty} (A^{-1}B)^j A^{-1}(c_0),$$

for any $m, n \in \mathbb{N}$, $m > n > 0$. As the spectral radius $\lambda(A^{-1}B) < 1$, we have $(A^{-1}B)^n \underset{n\to\infty}{\longrightarrow} 0$ in $\mathcal{L}_+$. Consequently,

$$(A^{-1}B)^n \sum_{j=0}^{\infty} (A^{-1}B)^j A^{-1}(c_0) = (A^{-1}B)^n [A^{-1}(I - A^{-1}B)^{-1}](c_0) \to 0.$$

It follows that the sequence $\{x_n\}_{n=0,1,\ldots}$ is a Cauchy sequence.

As the cone metric space (X, d_K) is assumed to be complete, there exists a point $x_* = \lim\limits_{n\to\infty} x_n \in X$. Note that $c_n \leq_K (A^{-1}B)^n(c_0) \underset{n\to\infty}{\longrightarrow} 0$. By the theorem condition, $Graph(\varphi)$ is (A, B)-search-closed, therefore $(x_*, 0) \in Graph(\varphi)$, i.e. $0 \in \varphi(x_*)$. $\square$

The following theorem is a consequence from Theorem 2.14 concerning the fixed point problem. It is a generalization of Theorem 1.10 given above, for the case when the considered cone is not normal.

Theorem 2.15. *Let (X, d_K) be a complete cone metric space where the cone $K \subset S$ is solid but not necessary normal. Let $F : X \rightrightarrows X$ be a multivalued mapping with closed bounded images. Consider the following cone function $\varphi : X \rightrightarrows K$, where $\varphi(x) := \{d \in K | \exists z \in F(x), d = d_K(x, z)\}$. Let φ be an (A, B)-search cone function,*

for linear bounded operators $A, B \in \mathcal{L}_+$. Suppose that $Graph(\varphi)$ is (A,B)-search-closed. Then, for any point $x_0 \in X$ and any $y_0 \in F(x_0)$, there exists a fixed point $\xi = \xi(x_0, y_0)$ of the mapping F, that is $\xi \in F(\xi)$, and the following estimate holds:

$$d_K(x_0, \xi) \leq_K A^{-1}(I - A^{-1}B)^{-1}(d_K(x_0, y_0)). \quad \square$$

The next theorem is also a consequence of Theorem 2.14. It concerns the coincidence existence problem. This result generalizes Theorem 1.13 given above, for the case when the considered cone K is not normal.

Theorem 2.16. *Let $F, G : X \rightrightarrows Y$ be multivalued mappings with bounded closed images, $\psi : X \rightrightarrows K$, where $\psi(x) := \{c \in K | \exists y \in F(x), \exists z \in G(x), \rho_K(y,z) = c\}$. Assume that ψ is an (A,B)-search cone function, with respect to linear operators $A, B \in \mathcal{L}_+$. In addition, assume that $Graph(\psi)$ is (A,B)-search-closed. Then, for any $x_0 \in X$ and any points $y_0 \in F(x_0)$, $z_0 \in G(x_0)$, the mappings F, G have a coincidence point $\xi = \xi(x_0, y_0, z_0) \in X$, that is $F(\xi) \cap G(\xi) \neq \emptyset$, and the following estimate holds:*

$$d_K(x_0, \xi) \leq_K A^{-1}(I - A^{-1}B)^{-1}(\rho_K(y_0, z_0)). \quad \square$$

We do not give proofs of Theorems 2.15, 2.16. To prove these theorems, it is enough to note that the set $Fix(F)$ is equal to the set $Nil(\varphi)$, and the set $Coin(F,G) = Nil(\psi)$, and to repeat the proof of Theorem 2.14.

The following theorem is presented in [7]. We give it in suitable denotations. In this theorem in [7], the cone K in a Banach space S is supposed to be solid but not normal. It is clear that Theorem 2.17 is a generalization of the above Theorem 1.12 (see [34]), in which the cone K is supposed to be normal (since the norm in S is monotone relative to the order $\leq_K$).

Theorem 2.17. *([7], Theorem 2.3) Let (X, d_K) be a complete cone metric space, $d_K : X \times X \to K$, and let T be a multivalued d_K-Perov contractive mapping from X into itself (i.e. there exists $Q \in L(S)$, such that $\lambda(Q) < 1$, $Q(K) \subseteq K$ and for any $x_1, x_2 \in X$ and any $y_1 \in T(x_1)$ there is $y_2 \in T(x_2)$ with $d(y_1, y_2) \leq_K Q(d_K(x_1, x_2))$) such that for any $x \in X$, $T(x)$ is a nonempty closed subset of X. Then there exists $x_0 \in X$ such that $x_0 \in T(x_0)$, i.e., x_0 is a fixed point of T.*

Proof. We show that Theorem 2.17 follows from Theorem 2.14. Consider the multivalued cone function $\eta : X \rightrightarrows K$, where $\eta(x) := \{c \in K | \exists y \in T(x), d_K(x,y) = c\}$. We need to show that the function η satisfies all conditions of Theorem 2.14. By Theorem 2.17 conditions, for any $x_0 \in X$ and any $y_0 = x_1 \in T(x_0)$, there is a point $y_1 = x_2 \in T(y_0)$ such that $d_K(y_0, y_1) = d_K(x_1, x_2) = c_1 \leq_K Q(d_(x_0, y_0)) = Q(c_0)$. It follows that one can construct a sequence $\{x_n\}_n \subset X$, and hence a search sequence $(x_n, c_n) \in Graph(\eta)$, with the properties

$$d_K(x_{n+1}, x_n) \leq_K c_n, c_{n+1} \leq_K Q(c_n), n = 0, 1, \cdots \tag{3}$$

One can see from (3) that the function η is a (I,Q)-search function where $I = Id_X$ is the identical mapping on X. Further, we need to show that the sequence

$\{x_n\}_{n=0,1,\cdots} \subset X$ is Cauchy, and the graph $Graph(\eta)$ is search-closed that is the limit of any convergent search sequence belongs to the graph.

Similarly to the proof of Theorem 2.14, we have

$$d_K(x_n, x_{n+m}) \leq_K (Q)^n \sum_{j=0}^{m-1}(Q)^j(c_0) \leq_K (Q)^n \sum_{j=0}^{\infty}(Q)^j(c_0) = Q^n(I-Q)^{-1}(c_0),$$

for any $m, n \in \mathbb{N}$, $m > n > 0$.

As the spectral radius $\lambda(Q) < 1$, we have $(Q)^n \underset{n\to\infty}{\longrightarrow} 0$ in $\mathcal{L}_+$. Consequently,

$$d_K(x_n, x_{n+m}) \leq_K Q^n(I-Q)^{-1}(c_0) \to 0.$$

It follows that the sequence $\{x_n\}_{n=0,1,\cdots}$ is Cauchy sequence.

As the cone metric space (X, d_K) is assumed to be complete, there exists a point $x_* = \lim\limits_{n\to\infty} x_n \in X$.

Now, we need to show that $0 \in \eta(x_*)$. To do this, one can follow the corresponding consideration in the proof of Theorem 2.3 in [7].

By the theorem conditions, for the points $x_{n-1}, x_* \in X$, and $x_n \in Q(x_{n-1})$, there is a point $v_n \in Q(x_*)$ such that $d_K(x_n, v_n) \leq_K Q(d_K(x_{n-1}, x_*)) \leq_K Q^n(I - Q)^{-1}(c_0)$.

As $x_n \to x_*$, for any $\varepsilon \gg 0$, there is $n_0 \in \mathbb{N}$, such that $d_K(x_n, x_*) \ll \varepsilon, \forall n > n_0$.

Then we have $d_K(x_n, x_*) \leq_K d_K(x_n, x_m) + d_K(x_m, x_*) \leq_K Q^n(I-Q)^{-1}(c_0) + \varepsilon$, for $m > \max\{n, n_0\}$. Since ε is arbitrary, it follows that $d_K(x_n, x_*) \leq_K Q^n(I - Q)^{-1}(c_0)$, for any $n \geq 1$.

Then, $d_K(v_n, x_*) \leq_K d_K(v_n, x_n) + d_K(x_n, x_*) \leq_K 2Q^n(I-Q)^{-1}(c_0) \to 0$, as $n \to \infty$. As $T(x_*)$ is closed, it follow that $x_* \in T(x_*)$ that is $0 \in \eta(x_*)$. In other words, $Graph(\eta)$ is search-closed. So, all conditions of Theorem 2.14 are fulfilled. □

Now we are ready to consider the problem of the zero existence preservation, under the parameter variation, for a parameter family of (A, B)-search cone functions, on an open subset U of a cone metric space (X, d_K).

Denotation. For any subset $Z \subseteq X$, and a family $\Phi = \{\Phi_t : Y \rightrightarrows K\}_{t\in[0;1]}$ of multivalued (A, B)-search functions, we introduce the following denotation:

$$M_Z(\Phi) := \{(x, t) \in Z \times [0; 1] \mid 0 \in \Phi_t(x)\}.$$

Definition 2.18. We say that the family Φ is *θ-continuous on the set $M_Z(\Phi)$*, if for any pair $(x, t) \in M_Z(\Phi)$ and any $t' > t$ there is $c \in \Phi_{t'}(x)$ such that $c \leq_K \theta(t') - \theta(t)$.

Definition 2.19. For a subset $Y \subset X$, we say that a multivalued cone function $\varphi : Y \rightrightarrows K$ is *(A, B)-search on Y* if it meets all conditions of the Definition 1.6, for the cone metric subspace (Y, d_K). In other words, for any point $y \in Y$ and any $c \in \varphi(y)$, there exist a point $y' \in Y$ and a value $c' \in \varphi(y')$ such that $d_K(y, y') \leq_K A^{-1}(c)$, $c' \leq_K A^{-1}B(c)$.

On the space $X \times [0;1]$ (in particular, on $Y \times [0;1]$), we consider the metric $D : (X \times [0;1])^2 \to \mathbb{R}_+$ defined by the rule $D((x',t'),(x'',t'')) = d_K(x',x'') + |t'-t''|$, for all $x', x'' \in X$ and any $t', t'' \in [0;1]$. The convergence in this metric is obviously component-wise.

The following Proposition is a useful auxiliary statement.

Proposition 2.20. *Let (X, d_K) be a cone metric space, $U \subset X$ be an open subset in X, and let $\theta : [0;1] \to K$ be a continuous increasing function. Let, on $\overline{U}$, there be given a one-parameter family $\Phi = \{\Phi_t : \overline{U} \rightrightarrows K\}_{t\in[0;1]}$ of multivalued functions which are (A,B)-search on U. Suppose that the family Φ is θ-continuous on the set $M_U(\Phi)$, and for each $t \in [0;1]$, the graph $Graph(\Phi_t)$ is $\{0\}$-closed. Then, $M_U(\Phi)$ is closed if $M_{\partial U}(\Phi) = \varnothing$.*

Proof. Let a sequence $\{(x_n,t_n)\} \subseteq M_U(\Phi)$ converge to an element $(\xi,t) \in \overline{U \times [0;1]}$. Let us show that $(\xi,t) \in M_U(\Phi)$. The sequence $\{x_n\}$ converges to ξ. Since $0 \in \Phi_{t_n}(x_n)$, in view of θ-continuity, there exists an element $c_n \in \Phi_t(x_n)$ such that $|c_n - 0| \le |\theta(t) - \theta(t_n)|$. Therefore, $c_n = |c_n - 0| \le |\theta(t) - \theta(t_n)| \to 0$, and hence the sequence $(x_n, c_n) \subseteq Graph(\Phi_t)$ converges to some element $(\xi, 0)$. By virtue of the $\{0\}$-closedness of the graph of the function Φ_t, we have $(\xi,0) \in Graph(\Phi_t)$, i.e., $0 \in \Phi_t(\xi)$. Since $M_{\partial U}(\Phi) = \varnothing$, it follows that $\xi \notin \partial U$. This means that $(\xi,t) \in M_U(\Phi)$. $\square$

Theorem 2.21. *Let (X, d_K) be a complete cone metric space (as described above), let $U \subset X$ be an open subset of X, and let $\theta : [0;1] \to K$ be a continuous increasing cone function. Let there be given a family $\Phi = \{\Phi_t : \overline{U} \rightrightarrows K\}_{t\in[0;1]}$ of multivalued (A,B)-search cone functions on $\overline{U}$, with $\{0\}$-closed graphs $Graph(\Phi_t) = \{(x,c) \in \overline{U} \times K | c \in \Phi_t(x)\}$, $t \in [0;1]$. As well, we suppose that $M_{\partial U}(\Phi) = \varnothing$, $M = M_U(\Phi) \neq \varnothing$, and the family Φ is θ-continuous on $M = M_U(\Phi)$.*

Then, if there exists an element of the form $(x_0, 0) \in M$, it follows that there exists an element of the form $(x_1, 1) \in M$.

Proof. We shall proceed similarly to the corresponding reasoning in [33]. Consider the partial order $\preceq$ on M defined by the following rule:

$$(x',t') \preceq (x'',t'') \Leftrightarrow \begin{cases} t' \le t'' \\ d(x',x'') \le A^{-1}(I - A^{-1}B)^{-1}(\theta(t'') - \theta(t')). \end{cases}$$

We show that under these conditions, there is a maximal element in the partially ordered set $(M, \preceq)$. For this, we need to show that any chain in $(M, \preceq)$ has an upper bound. Then, by the well-known Kuratowski-Zorn lemma, there exists a maximal element in M.

Let $\mathcal{V} = \{(x,t)\} \subseteq M$ be an arbitrary chain in M. Consider the set $T_{\mathcal{V}} = \{t \in [0;1] | \exists (x,t) \in \mathcal{S}\} \subseteq [0;1]$. It is clear that the set $T_{\mathcal{V}}$ is bounded above. Then, there exists its least upper bound $\bar{t} = \sup T_{\mathcal{V}}$ and a nondecreasing sequence $\{t_n\} \subseteq T_{\mathcal{V}}$ such that $t_n \le \bar{t}$. Take a sequence $\{(x_n,t_n)\} \subseteq \mathcal{V}$ corresponding to the sequence $\{t_n\}$, such that $(x_n,t_n) \preceq (x_{n+1},t_{n+1})$ for all $n \in \mathbb{N}$.

Such sequences do exist, because, for any natural number n, it is true that $t_n \le t_{n+1}$, and in the chain $\mathcal{V}$ all elements are pairwise comparable.

In addition, for any $n \in \mathbb{N}$, we have $d(x_n, x_{n+1}) \le A^{-1}(I-A^{-1}B)^{-1}(\theta(t_{n+1})-\theta(t_n))$. This inequality implies that, for any $n, m \in \mathbb{N}$ the following condition holds: $d(x_n, x_{n+m}) \le$

$$\le d(x_n, x_{n+1}) + \cdots + d(x_{n+m-1}, x_{n+m}) \le A^{-1}(I - A^{-1}B)^{-1}(\theta(t_{n+m}) - \theta(t_n)) \quad (4)$$

As the function θ is continuous, the sequence $\{\theta(t_n)\}$ converges to $\theta(\bar{t})$, which means that it is a Cauchy sequence. In view of inequality (4), it is clear that the sequence $\{x_n\}$ is also a Cauchy sequence. In addition, by virtue of the condition $0 \in \Phi_{t_n}(x_n)$ and the property of θ-continuity of the family Φ on the set $M = M_U(\Phi)$, there exists a value $c_n \in \Phi_{\bar{t}}(x_n)$ such that

$$c_n = c_n - 0 \le \theta(\bar{t}) - \theta(t_n) \to 0.$$

As $\Phi_{\bar{t}}$ has $\{0\}$-closed graph, it follows that there exists a $\xi \in \overline{U}$ such that $x_n \to \xi$, and $0 \in \Phi_{\bar{t}}(\xi)$. By the theorem condition, the set M is closed. Since $\{(x_n, t_n)\} \subseteq \mathcal{V} \subseteq M$ and $(x_n, t_n) \to (\xi, \bar{t})$, it follows that $(\xi, \bar{t}) \in M$.

Now, we show that the element $(\xi, \bar{t})$ is an upper bound of the chain $\mathcal{V}$.

Passing to the limit in the inequality (4), as $m \to \infty$, we obtain $d(x_n, \xi) \le A^{-1}(I - A^{-1}B)^{-1}(\theta(\bar{t}) - \theta(t_n))$. In view of the inequality $t_n \le \bar{t}$ we see that, for all $n \in \mathbb{N}$ it is true that $(x_n, t_n) \preceq (\xi, \bar{t})$.

Now, one may consider an arbitrary element $(x, t) \in \mathcal{V}$. As $t_n \to \bar{t}$, it follows that, for $\varepsilon = \bar{t} - t$, there exists an $N \in \mathbb{N}$ such that $\bar{t} - t_N \le \bar{t} - t$. Therefore, $t \le t_N$. But, since arbitrary pairs (x, t) and (x_N, t_N) are comparable in the chain $\mathcal{V}$, we have $(x, t) \preceq (x_N, t_N)$. Hence, by transitivity, $(x, t) \preceq (\xi, \bar{t})$. Therefore, $(\xi, \bar{t})$ is an upper bound of the chain $\mathcal{V}$.

So, we have obtained that any chain $\mathcal{V}$ in M has an upper bound. Then, by the well-known Kuratowski-Zorn lemma, there exists a maximal element in M. Furthermore, if there is an element $(x_0, 0) \in M$, it follows that there is a maximal element $(\tilde{x}, \tilde{t}) \in M$ such that $(x_0, 0) \preceq (\tilde{x}, \tilde{t})$.

Let us show that $\tilde{t} = 1$. On the contrary, one can take $t', \tilde{t} < t' < 1$. As $(\tilde{x}, \tilde{t}) \in M$, we have $0 \in \Phi_{\tilde{t}}(\tilde{x})$. Then, due to the condition of θ-continuity of the family Φ on M, there is a value $c' \in \Phi_{t'}(\tilde{x})$ such that $c' \le_K \theta(t') - \theta(\tilde{t})$. If $t' - \tilde{t}$ is enough small, and due to Theorem 1.9, there is a point $x' \in U$ with $0 \in \Phi_{t'}(x')$, such that $d_K(\tilde{x}, x') \le_K A^{-1}(I - A^{-1}B)^{-1}(c') \le_K A^{-1}(I - A^{-1}B)^{-1}(\theta(t') - \theta(\tilde{t}))$. So, we have $(\tilde{x}, \tilde{t}) \preceq (x', t')$. It means that the element $(\tilde{x}, \tilde{t})$ is not maximal.

It follows that the maximal element is of the form $(x_1, 1)$. □

The main results of this paper imply consequences on the fixed point (coincidence) existence preservation when changing the parameter, for a family of multivalued mappings of cone metric spaces with a normed space valued cone metrics. We don't give here these results.

Above we mentioned various directions in which Perov's theorem is generalized in many papers of different authors.

As far as one can see, the further development of the results obtained here may be realized in similar directions concerning generalizations of cone metric spaces and considerations of some new applications.

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