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IMPACT OF KANNAN CONTRACTION ON KHAN CONTRACTION AND ITS GENERALIZATIONS

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Several concepts of rational contractions originated during the last 50 years with vivid discussion on possible applications and practical examples. The main topic of this article is a Khan contraction, its modifications and generalizations, with the emphasis on its relation to some well-known classes of contractive mappings. We prove that a Khan contraction is an example of Bianchini and, consequently, a Ćirić contraction. Further, some of its generalizations present a special type of Kannan contraction.

1. INTRODUCTION AND PRELIMINARIES

The rational contractive condition was introduced by Das and Gupta [**10**] and was further widely investigated and modified in several ways. It has been shown that the idea of rational contraction of Das and Gupta is independent from the concept of Banach contraction, but its relation with some other classes of contractive mappings was not discussed in details. It is interesting to mention that Jaggi [**16**] also introduced the idea of rational contractive condition in 1977. with a slight modification comparing to [**10**]. The same idea of contractive condition by Jaggi was investigated on a Banach space by Ray and Singh [**20**]. Khan's approach to rational contractive condition originated in 1978. and it was partially influenced by the work of Chatterjea [**6**]. The Khan contraction was improved by Fisher [**11**]

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and further extended in several ways in [**2, 3, 17, 18, 19, 21**] among others.
The idea of comparing rational contractions with Kannan contraction can be justified with an example in the sequel which verifies that rational contractions, and specifically Khan contractions, are not necessary continuous.
The study of Kannan contraction started one decade prior to rational contractions, in 1960. in [**13**]. Kannan contractive condition determines a class of mappings on a complete metric spaces that do have a unique fixed point, but it is independent from the class of Banach contractions. Several examples of Kannan contractions which are not continuous are given in the literature. It is important to emphasize the result of Subrahmanyam [**27**] that the class of Kannan contractions characterizes the completeness of a metric space.
Yet another approach to modifying Kannan contractive condition was presented by Bianchini [**5**] where the maximum of distances first was included in contractive condition. In that manner, Bianchini generalized Kannan contracion. The approach of Bianchini was further extended by Ćirić [**7**] who unified the results of [**4, 5, 6, 12, 13, 22, 23, 24**].
The main aim of this article is to prove that a Khan contraction and some of its generalizations belong to some of the previously mentioned classes of contractive mappings as Bianchini contraction or Kannan contraction.

We collect some well-known fixed point results in the sequel assuming that X is a non-empty set.

Theorem 1. *[**4**] Let (X, d) be a complete metric space and $T : X \mapsto X$ a contraction. There exists a unique fixed point $x^* \in X$ and for any $x \in X$ the iterative sequence $(T^n x)$ converges to x^*.*

Continuity is not assumed in the statement of Theorem 1, although a contraction is inherently a continuous mapping. Kannan [**13**] addressed the question of whether continuity can be omitted for a contractive mapping while still guaranteeing a unique fixed point.

Theorem 2. *[**13**] Let (X, d) be a complete metric space and $T : X \to X$ be a mapping. If there exists $q \in [0, \frac{1}{2})$ such that*

$$d(Tx, Ty) \leq q(d(x, Tx) + d(y, Ty)), \tag{1.1}$$

for all $x, y \in X$, then T possesses a unique fixed point.

It is possible to observe an equivalent form of Kannan contraction.

Theorem 3. *Let (X, d) be a complete metric space and $T : X \to X$ be a mapping. If there exist $\alpha, \beta \in [0, 1)$ fulfilling $\alpha + \beta < 1$ such that for any $x, y \in X$,*

$$d(Tx, Ty) \leq \alpha d(x, Tx) + \beta d(y, Ty), \tag{1.2}$$

then T possesses a unique fixed point.

Obviously, (1.2) already contains (1.1), but reverse also holds. Due to symmetry, by interchanging x and y, we have

$$d(Ty, Tx) \leq \alpha d(y, Ty) + \beta d(x, Tx),$$

which combined with (1.2) gives (1.1) for $q = \frac{\alpha+\beta}{2} \in [0, \frac{1}{2})$. Recently, question of iterates of a mapping satisfying a Kannan contractive condition for a mapping itself was studied in [**9**].

Theorem 4. *[**9**] Let (X, d) be a complete metric space and $T : X \to X$ be a mapping such that T^k is continuous mapping for some $k \in \mathbb{N}$. If there exist $q \in [0, \frac{1}{2})$ and $m \in \mathbb{N}$ such that the inequality*

$$d(T^m x, T^m y) \leq q\left(d(x, Tx) + d(y, Ty)\right), \tag{1.3}$$

holds for all $x, y \in X$, then T possesses a unique fixed point.

The contractive condition was further modified in several different ways, Reich, Rus and Ćirić [**8, 22, 23, 24, 25**] combined Banach and Kannan contraction.

Theorem 5. *[**8, 22, 25**] If (X, d) is a complete metric space and $T : X \to X$ is a mapping such that the inequality*

$$d(Tx, Ty) \leq \alpha d(x, y) + \beta\left(d(x, Tx) + d(y, Ty)\right) \tag{1.4}$$

holds for some $\alpha, \beta \in [0, 1)$ with $\alpha + 2\beta < 1$ and all $x, y \in X$, then the mapping T possesses a unique fixed point in X.

One equivalent of (1.4) is given by the inequality

$$d(Tx, Ty) \leq \alpha d(x, y) + \beta d(x, Tx) + \gamma d(y, Ty)$$

fulfilled for certain $\alpha, \beta, \gamma \geq 0$ with $\alpha + \beta + \gamma < 1$ and any $x, y \in X$.

Different choice of distances was given by Chatterjea in [**6**]

Theorem 6. *[**6**] If (X, d) is a complete metric space and $T : X \to X$ is a mapping such that*

$$d(Tx, Ty) \leq q\left(d(x, Ty) + d(y, Tx)\right) \tag{1.5}$$

holds for some $q \in [0, \frac{1}{2})$ and all $x, y \in X$, then the mapping T possesses a unique fixed point in X.

In the same manner we can observe an inequality equivalent to (1.5)

$$d(Tx, Ty) \leq \alpha d(x, Ty) + \beta d(y, Tx)$$

for some $\alpha, \beta \in [0, 1)$ with $\alpha + \beta < 1$, as well as its extension with an additional $d(x, y)$ term

$$d(Tx, Ty) \leq \alpha d(x, y) + \beta d(x, Ty) + \gamma d(y, Tx)$$

for some $\alpha, \beta, \gamma \in [0, 1)$ with $\alpha + \beta + \gamma < 1$.
The result of Hardy and Rogers [**12**] unified Banach, Reich-Rus-Ćirić and Chatterjea contraction.

Theorem 7. *[**12**] If (X,d) is a complete metric space, $T:X\to X$ is a mapping and numbers $\alpha,\beta,\gamma,\delta,\eta\geq 0$ with $0\leq\alpha+\beta+\gamma+\delta+\eta<1$ are such that*

$$d(Tx,Ty)\leq\alpha d(x,y)+\beta d(x,Tx)+\gamma d(y,Ty)+\delta d(x,Ty)+\eta d(y,Tx) \tag{1.6}$$

holds for any $x,y\in X$, then the mapping T possesses a unique fixed point in X.

Bianchini [**5**] replaced the linear combination of distances in (1.3) with their maximum creating a less strict condition possible defining a wider class of contractive mappings.

Theorem 8. *[**5**] If (X,d) is a complete metric space and $T:X\to X$ is a mapping such that*

$$d(Tx,Ty)\leq q\max\{d(x,Tx),d(y,Ty)\} \tag{1.7}$$

holds for some $q\in[0,1)$ and all $x,y\in X$, then the mapping T possesses a unique fixed point in X.

Sehgal extended (1.7) in [**26**].

Theorem 9. *[**26**] If (X,d) is a complete metric space and $T:X\to X$ is a mapping such that*

$$d(Tx,Ty)\leq q\max\{d(x,y),d(x,Tx),d(y,Ty)\} \tag{1.8}$$

holds for some $q\in[0,1)$ and all $x,y\in X$, then the mapping T possesses a unique fixed point in X.

Ćirić presented a maximum analogue to the contractive condition of Hardy and Rogers in [**7**].

Theorem 10. *[**7**] If (X,d) is a complete metric space, $T:X\to X$ is a mapping and $q\in[0,1)$ such that*

$$d(Tx,Ty)\leq q\max\{d(x,y),d(x,Tx),d(y,Ty),d(x,Ty),d(y,Tx)\} \tag{1.9}$$

holds for any $x,y\in X$, then the mapping T possesses a unique fixed point in X.

It is significant to mention that in Ćirić's paper is included an example of mapping which is a quasi-contraction, but not Hardy-Rogers contraction. In that way, we can talk about the independence of a maximum condition from previously obtained ones and also observe it as their extension.

The idea of rational contractive condition was introduced by Das and Gupta in [**10**].

Theorem 11. *[**10**] If (X,d) is a complete metric space, $T:X\to X$ is a mapping and $\alpha,\beta\in[0,1)$ with $\alpha+\beta<1$ are such that the inequality*

$$d(Tx,Ty)\leq\alpha\frac{(1+d(x,Tx))\,d(y,Ty)}{1+d(x,y)}+\beta d(x,y) \tag{1.10}$$

holds for any $x,y\in X$, then the mapping T possesses a unique fixed point in X.

Jaggi proved another existence and uniqueness result for rational contractions, but it appears to be more strict in some important cases in comparison with (1.10).

Theorem 12. *[16] If (X,d) is a complete metric space, $T:X\to X$ is a mapping and $\alpha,\beta\in[0,1)$ with $\alpha+\beta<1$ are such that the inequality*

$$d(Tx,Ty)\leq \alpha\frac{d(x,Tx)d(y,Ty)}{d(x,y)}+\beta d(x,y) \tag{1.11}$$

holds for all mutually distinct $x,y\in X$, then the mapping T possesses a unique fixed point in X.

Khan [**15**] in 1975. defined a new class of mappings determined by a rational contractive condition.

Definition 1. *[15] If $T:X\mapsto X$ is a mapping on a metric space (X,d) such that the inequality*

$$d(Tx,Ty)\leq q\frac{d(x,Tx)d(x,Ty)+d(y,Ty)d(y,Tx)}{d(x,Ty)+d(y,Tx)} \tag{1.12}$$

holds for some $q\in[0,1)$ and all $x,y\in X$, then T is a Khan contraction.

The main result of [**15**] claims existence and uniqueness of a fixed point for a Khan contraction on a complete metric space.

Theorem 13. *[15] If (X,d) is a complete metric space and $T:X\mapsto X$ a mapping satisfying (1.12) for some $q\in[0,1)$ and all $x,y\in X$, then it possesses a unique fixed point in X.*

The flaw in Khan's definition was observed and corrected by Fisher [**11**]. We will use Fisher's definition in the sequel when addressing the Khan contraction.

Definition 2. *[11] Let $T:X\mapsto X$ be a mapping on a metric space (X,d) such that*

$$d(Tx,Ty)\leq\begin{cases}q\frac{d(x,Tx)d(x,Ty)+d(y,Ty)d(y,Tx)}{d(x,Ty)+d(y,Tx)}, & d(x,Ty)+d(y,Tx)\neq 0\\ 0, & d(x,Ty)+d(y,Tx)=0\end{cases} \tag{1.13}$$

for some $q\in[0,1)$ and all $x,y\in X$, then T is a Khan contraction.

Theorem 14. *[11] Khan contraction on a complete metric space (X,d) possesses a unique fixed point in X.*

Note that if we assume that $d(x,Ty)+d(y,Tx)=0$, then

$$d(x,Tx)d(x,Ty)+d(y,Ty)d(y,Tx)=0$$

and for that purpose the change was made as in 1.13 by Fisher. Additionally, this case leads to the conclusion $x=y$ and $Tx=x$, so the fixed point of the mapping T is obtained. The motivation for Fisher's modification can be seen through the following example.

Example 1. *Let $X = \{0, 1\}$ be equipped with the Euclidean distance $d : X \times X \mapsto [0, \infty)$ defined by $d(x, y) = |x - y|$ for $x, y \in X$. If $T : X \mapsto X$ is a mapping defined by $T0 = 1$ and $T1 = 0$, then it has no fixed point and $d(0, T1) + d(1, T0) = 0$, so the original inequality given in [**15**] is fulfilled.*

Several generalizations of Khan contraction were presented in [**2, 3, 17, 18, 19, 21**]

2. MAIN RESULTS

Theorem 15. *If (X, d) is a metric space and $T : X \mapsto X$ a Khan contraction on X fulfilling (1.13) for some $q \in [0, 1)$, then*

$$d(Tx, Ty) \leq q \max \{d(x, Tx), d(y, Ty)\}, \tag{2.14}$$

for all $x, y \in X$.

Proof. Let (X, d) be a metric space and $T : X \mapsto X$ a mapping such that (1.13) holds for some $q \in [0, 1)$ and for all $x, y \in X$. Observe that in order to prove this statement it is sufficient to show

$$\frac{d(x, Tx)d(x, Ty) + d(y, Ty)d(y, Tx)}{d(x, Ty) + d(y, Tx)} \leq \max \{d(x, Tx), d(y, Ty)\}$$

for any $x, y \in X$ such that $Tx \neq y$ and $Ty \neq x$, while the case $d(x, Ty) + d(y, Tx) = 0$ will be discussed separately.
If $d(x, Ty) + d(y, Tx) = 0$, then $Tx = y$ and $Ty = x$ while from (1.13) $Tx = Ty$, so (2.14) evidently holds in this case.
In the sequel, we will claim that $d(x, Ty) + d(y, Tx) \neq 0$. Assume contrary of what we intend to prove that there exist some $x_0, y_0 \in X$ such that

$$\frac{d(x_0, Tx_0)d(x_0, Ty_0) + d(y_0, Ty_0)d(y_0, Tx_0)}{d(x_0, Ty_0) + d(y_0, Tx_0)} > \max \{d(x_0, Tx_0), d(y_0, Ty_0)\}.$$

Then,

$$\begin{aligned} d(x_0, Tx_0)d(x_0, Ty_0) + d(y_0, Ty_0)d(y_0, Tx_0) &> d(x_0, Ty_0)d(x_0, Tx_0) \\ &\quad + d(y_0, Tx_0)d(x_0, Tx_0) \end{aligned}$$

implies

$$d(y_0, Ty_0)d(y_0, Tx_0) \geq d(y_0, Tx_0)d(x_0, Tx_0),$$

and due to the presumption that $Tx_0 \neq y_0$ and $Ty_0 \neq x_0$, it follows that $d(y_0, Ty_0) > d(x_0, Tx_0)$.
Likewise,

$$\begin{aligned} d(x_0, Tx_0)d(x_0, Ty_0) + d(y_0, Ty_0)d(y_0, Tx_0) &> d(x_0, Ty_0)d(y_0, Ty_0) \\ &\quad + d(y_0, Tx_0)d(y_0, Ty_0), \end{aligned}$$

implies

$$d(x_0, Tx_0)d(x_0, Ty_0) > d(x_0, Ty_0)d(y_0, Ty_0),$$

and further $d(x_0, Tx_0) > d(y_0, Ty_0)$.
Hence, the contradiction is obtained and the assumption must be false. Consequently,

$$\frac{d(x,Tx)d(x,Ty) + d(y,Ty)d(y,Tx)}{d(x,Ty) + d(y,Tx)} \leq \max\{d(x,Tx), d(y,Ty)\}$$

for any $x, y \in X$, and (2.14) holds for any $x, y \in X$. □

Corollary 16. *Khan contraction on a metric space (X, d) is a Bianchini contraction.*

Corollary 17. *Khan contraction on a metric space (X, d) is a Ćirić contraction (quasi-contraction).*

Proof. Evidently,

$$\max\{d(x,Tx), d(y,Ty)\} \leq \max\{d(x,y), d(x,Tx), d(y,Ty), d(x,Ty), d(y,Tx)\},$$

for any $x, y \in X$. □

Example 2. *Let $X = [-\frac{1}{2}, \infty]$ and $d : X \times X \mapsto \mathbb{R}$ Euclidean metric, i.e.,*

$$d(x, y) = |x - y|, \ x, y \in X.$$

Note that (X, d) is a complete metric space.
Define a mapping $T : X \mapsto X$ such that

$$Tx = \begin{cases} 0, & x \in [-\frac{1}{2}, 2] \\ -\frac{1}{2}, & x > 2 \end{cases}.$$

Then $d(Tx, Ty) = 0$ except for $x \in [-\frac{1}{2}, 2]$ and $y > 2$ or vice versa. Thus, without loss of generality, assume that $x \in [-\frac{1}{2}, 2]$ and $y > 2$ since the inequality (1.13) is symmetric. Further, let $q = \frac{1}{2}$, then

$$d(Tx, Ty) = \frac{1}{2} \leq \frac{1}{2}\frac{|x|(x + \frac{1}{2}) + y(y + \frac{1}{2})}{x + \frac{1}{2} + y}$$

holds for any $x \in [-\frac{1}{2}, 2]$ and $y > 2$. Hence, T is a Khan contraction.
Evidently, taking $x = 2$ and $y = \frac{9}{4}$, testifies that T is not a Banach contraction.
If we take $q = \frac{2}{5}$, it can be seen that (1.1) is satisfied for any $x, y \in X$.
Moreover, the mapping T is a Bianchini contraction due to

$$\max\{d(x,Tx), d(y,Ty)\} = \max\{|x|, y + \frac{1}{2}\} = y + \frac{1}{2} \geq \frac{5}{2} \geq 5d(Tx, Ty)$$

for $x \in [-\frac{1}{2}, 2]$ *and* $y > 2$ *or vice versa. Consequently,*

$$d(Tx, Ty) \leq \frac{1}{5} \max\{d(x, Tx), d(y, Ty)\}$$

for all $x, y \in X$*.*

Theorem 18. *Let* $T : X \mapsto X$ *be a mapping on a metric space* (X, d) *such that*

$$d(Tx, Ty) \leq \begin{cases} q\frac{d(x,Tx)d(x,Ty)+d(y,Ty)d(y,Tx)}{max\{d(x,Ty),d(y,Tx)\}}, & d(x, Ty) + d(y, Tx) \neq 0 \\ 0, & d(x, Ty) = d(y, Tx) = 0 \end{cases} \tag{2.15}$$

for some $q \in [0, 1)$ *and all* $x, y \in X$*, then there exists some* $m \in \mathbb{N}$ *such that (1.3) holds.*

Proof. Assume that (2.15) holds for a mapping $T : X \mapsto X$ on a metric space (X, d). If $d(x, Ty) = d(y, Tx) = 0$, then $Tx = x = y = Ty$ and

$$d(Tx, Ty) \leq \max\{d(x, Tx), d(y, Ty)\}.$$

Otherwise, let $d(x, Ty) + d(y, Tx) \neq 0$. The inequality

$$\begin{aligned} max\{d(x, Ty), d(y, Tx)\}\,(d(x, Tx) + d(y, Ty)) \geq\ & d(x, Tx)d(x, Ty) \\ & + d(y, Ty)d(y, Tx) \end{aligned}$$

further implies

$$\begin{aligned} d(Tx, Ty) &\leq q\frac{d(x, Tx)d(x, Ty) + d(y, Ty)d(y, Tx)}{max\{d(x, Ty), d(y, Tx)\}} \\ &\leq q\frac{max\{d(x, Ty), d(y, Tx)\}(d(x, Tx) + d(y, Ty))}{max\{d(x, Ty), d(y, Tx)\}} \\ &= qd(x, Tx) + d(y, Ty)). \end{aligned}$$

We intend to prove that

$$d(T^n x, T^n y) \leq q^n\,(d(x, Tx) + d(y, Ty)) \tag{2.16}$$

for any $n \in \mathbb{N}$. In order to achieve that, we will use the principle of mathematical induction twice, first to show the auxiliary result that $d(T^n x, T^{n+1} x) \leq q^n d(x, Tx)$ for any $n \in \mathbb{N}$.
Evidently,

$$\begin{aligned} d(Tx, T^2 x) &\leq q\frac{d(x, Tx)d(x, T^2 x)}{d(x, T^2 x)} \\ &= qd(x, Tx) \end{aligned}$$

assuming that $T^2x \neq x$. Otherwise $x = Tx = T^2x$ and the inequality $d(Tx, T^2x) \leq qd(x, Tx)$ holds.
Assume that $d(T^{n-1}x, T^nx) \leq q^{n-1}d(x, Tx)$ holds for some $n > 2$. Consequently,

$$\begin{aligned} d(T^nx, T^{n+1}x) &\leq q\frac{d(T^{n-1}x, T^nx)d(T^{n-1}x, T^{n+1}x)}{d(T^{n-1}x, T^{n+1}x)} \\ &= qd(T^{n-1}x, T^nx) \\ &\leq q^nd(x, Tx). \end{aligned}$$

Previous estimates were made under the assumption that $d(T^{n-1}x, T^{n+1}x) \neq 0$, while otherwise same comment is applicable as for $n = 1$.
Hence,

$$d(T^nx, T^{n+1}x) \leq q^nd(x, Tx), \tag{2.17}$$

for any $n \in \mathbb{N}$ and $x \in X$.
Recalling the inequality (2.16), we observe that the basis for $n = 1$ is already acquired. Therefore, the assumption that (2.16) holds for some $n > 1$ leads to

$$\begin{aligned} d(T^{n+1}x, T^{n+1}y) &\leq \frac{d(T^nx, T^{n+1}x)d(T^nx, T^{n+1}y) + d(T^ny, T^{n+1}y)d(T^ny, T^{n+1}x)}{\max\{d(T^nx, T^{n+1}y), d(T^ny, T^{n+1}x)\}} \\ &\leq q^{n+1}\frac{d(x, Tx)d(T^nx, T^{n+1}y) + d(y, Ty)d(T^ny, T^{n+1}x)}{\max\{d(T^nx, T^{n+1}y), d(T^ny, T^{n+1}x)\}} \\ &\leq q^{n+1}\left(d(x, Tx) + d(y, Ty)\right). \end{aligned}$$

The principle of mathematical induction asserts that (2.16) is fulfilled for any $n \in \mathbb{N}$ and $x, y \in X$.
As $q \in [0, 1)$, there exists some $m \in \mathbb{N}$ starting from which $q^m < \frac{1}{2}$. For chosen m, the inequality (1.3) is fulfilled. □

Corollary 19. *Let $T : X \mapsto X$ be a mapping on a metric space (X, d) such that (1.13) holds for some $q \in [0, 1)$ and all $x, y \in X$, then there exists some $k \in \mathbb{N}$ such that (1.3) is fulfilled for all $x, y \in X$.*

Proof. Assume that $T : X \mapsto X$ is a Khan contraction on a metric space (X, d). Note that conditions (1.13) and 2.15 coincide in the case $d(x, Ty) = d(y, Tx) = 0$. Hence, we observe the alternative, if $d(x, Ty) + d(y, Tx) \neq 0$. Then,

$$\frac{d(x, Tx)d(x, Ty) + d(y, Ty)d(y, Tx)}{d(x, Ty) + d(y, Tx)} \leq \frac{d(x, Tx)d(x, Ty) + d(y, Ty)d(y, Tx)}{max\{d(x, Ty), d(y, Tx)\}}$$

for any such $x, y \in X$, so Khan contraction fulfills the generalized Khan contractive condition (2.15) and assumptions of Theorem 18 hold, so the conclusion is directly obtained from the proof of Theorem 18. □

The result of [**2**] for the setting of a complete metric space is the following.

Theorem 20. *[2] Let $T : X \mapsto X$ be a mapping on a metric space (X, d) such that*
(2.18)
$$d(Tx, Ty) \leq \begin{cases} \alpha d(x, y) + \beta \frac{d(x,Tx)d(x,Ty)+d(y,Ty)d(y,Tx)}{d(x,Ty)+d(y,Tx)}, & d(x, Ty) + d(y, Tx) \neq 0 \\ 0, & d(x, Ty) = d(y, Tx) = 0 \end{cases}$$

for some $\alpha, \beta \in [0, 1)$ such that $\alpha + \beta < 1$ and all $x, y \in X$, then there exists some $k \in \mathbb{N}$ such that (1.3) holds.

We will prove that directly from Theorem 15 or Corollary 16 we obtain a conclusion that a mapping fulfilling the inequality (2.18) is a quasi-contraction.

Theorem 21. *If (X, d) is a metric space and $T : X \mapsto X$ a mapping fulfilling (2.18) on X for some $\alpha, \beta \geq 0$ with $\alpha + \beta < 1$, then*

$$d(Tx, Ty) \leq q \max\{d(x, y), d(x, Tx), d(y, Ty)\} \tag{2.19}$$

for any $x, y \in X$ and $q = \alpha + \beta \in [0, 1)$.

Proof. Assuming that the presumptions of Theorem 21 hold, recall that from the proof of Theorem 15 we get

$$\frac{d(x, Tx)d(x, Ty) + d(y, Ty)d(y, Tx)}{d(x, Ty) + d(y, Tx)} \leq \max\{d(x, Tx), d(y, Ty)\}$$

for any $x, y \in X$. Therefore, if $d(x, Ty) + d(y, Tx) \neq 0$, then

$$\begin{aligned} d(Tx, Ty) &\leq \alpha d(x, y) + \beta \frac{d(x, Tx)d(x, Ty) + d(y, Ty)d(y, Tx)}{d(x, Ty) + d(y, Tx)} \\ &\leq \alpha d(x, y) + \beta \max\{d(x, Tx), d(y, Ty)\} \\ &\leq (\alpha + \beta) \max\{d(x, y), d(x, Tx), d(y, Ty)\}. \end{aligned}$$

Otherwise, $x = Ty$ and $y = Tx$ imply $y = Tx = Ty$ since $y \neq Ty$ implies

$$\begin{aligned} d(x, y) = d(Ty, T^2y) &\leq \alpha d(y, Ty) + \beta \frac{d(y, Ty)d(y, T^2y)}{d(y, T^2y)} \\ &= (\alpha + \beta)d(y, Ty) \\ &= (\alpha + \beta)d(y, x) \end{aligned}$$

which is in contradiction with initial assumption. Hence, (2.19) evidently holds for arbitrary $q \in [0, 1)$. Hence, the inequality (2.19) holds for any $x, y \in X$ and $q = \alpha + \beta$. □

Corollary 22. *If (X, d) is a metric space and $T : X \mapsto X$ a mapping fulfilling (2.18) on X for some $\alpha, \beta \geq 0$ with $\alpha + \beta < 1$, then T is a quasi-contraction on a metric space X.*

Proof. Taking into the account Theorem 21 and the fact that

$$\max\{d(x,y),d(x,Tx),d(y,Ty)\} \leq \max\{d(x,y),d(x,Tx),d(y,Ty),d(x,Ty), d(y,Tx)\},$$

it is obvious that a mapping T satisfying (2.18) is a quasi-contraction on X for a contractive constant $q=\alpha+\beta\in[0,1)$.

□

In [**18**] was presented an example of a mapping fulfilling (2.15), but not (1.13) on a complete metric space.

Example 3. *If* $X=\{\frac{1}{n}\mid n\in\mathbb{N}\}\cup\{0\}$ *and* $d:X\times X\mapsto[0,\infty)$ *is defined by*

$$d(x,y)=\begin{cases}0, & x=y\\ 1, & (x,y)\in\{(0,\frac{1}{n+1}),(\frac{1}{n+1},0)\mid n\in\mathbb{N}\}\\ \frac{3}{2}, & (x,y)\in\{(\frac{1}{2},1),(1,\frac{1}{2})\}\\ 2, & (x,y)\in\{(0,1),(1,0)\}\\ \max\{x,y\}, & (x,y)\in\{(\frac{1}{m},\frac{1}{n})\mid m\neq n,m,m\in\mathbb{N}\}\setminus\{(\frac{1}{2},1),(1,\frac{1}{2})\}\end{cases},$$

then (X,d) *is a complete metric space.*
Let $T:X\mapsto X$ *be a mapping defined by*

$$Tx=\begin{cases}\frac{1}{2}, & x=1\\ 0, & x\in X\setminus\{1\}\end{cases}.$$

Such mapping is not a Khan contraction, since it does not satisfy (1.13) on X *as can be seen for* $x=0$ *and* $y=1$

$$d(Tx,Ty)=1=\frac{\frac{3}{2}\cdot 2}{1+2}=\frac{d(x,Tx)d(x,Ty)+d(y,Ty)d(y,Tx)}{d(x,Ty)+d(y,Tx)}$$

Conversely, it is a generalized Khan contraction on X*. Indeed, we need to discuss on two different cases and their symmetric equivalents, if* $x=0$ *and* $y=1$ *or if* $x=\frac{1}{n+1}$ *and* $y=1$ *for some* $n\in\mathbb{N}$*. In the sequel, let* $q=\frac{2}{3}$*.*
Assuming that $x=0$ *and* $y=1$*, we obtain*

$$\begin{aligned}q\frac{d(x,Tx)d(x,Ty)+d(y,Ty)d(y,Tx)}{max\{d(x,Ty),d(y,Tx)\}} &= q\frac{3}{2}\\ &= d(Tx,Ty).\end{aligned}$$

Otherwise, if $x=\frac{1}{n+1}$ *and* $y=1$*, then*

$$\begin{aligned}q\frac{d(x,Tx)d(x,Ty)+d(y,Ty)d(y,Tx)}{max\{d(x,Ty),d(y,Tx)\}} &\geq q\frac{\frac{1}{2}+3}{2}\\ &\geq d(Tx,Ty).\end{aligned}$$

Evidently, T *fulfills (1.3) for* $k=2$*, so it is a* 2*-Kannan contraction on* X*.*

3. CONCLUSION

As noted, Khan contraction along with some of its modifications is related to some of the well-known classes of mappings, such as Kannan contraction, Bianchini contraction or quasi-contraction. Therefore, the existence and uniqueness results for this type of rational contractions are implicitly already obtained. As the open question, we foreseen the relation of more general classes of rational contractions to mentioned classical contractive mappings.

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