Applicable Analysis and Discrete Mathematics
available online at http://pefmath.etf.rs

Appl. Anal. Discrete Math. **19** (2025), 548–558.
https://doi.org/10.2298/AADM250119027M

NEW CONTRIBUTIONS TO COMMON FIXED POINT THEORY IN VECTOR METRIC SPACES

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In this paper, we will prove several new existence and stability results for the common fixed point problem $f(x) = x = g(x), x \in X$, where $f, g : X \to X$ are two operators and X is an $\mathbb{E}$-complete vector metric space (where $\mathbb{E}$ is a Riesz space).

1. INTRODUCTION AND PRELIMINARY NOTIONS AND RESULTS

Let E be a Riesz space and denote by $\preceq$ its partial ordering. Denote by E_+ the positive cone, i.e., $E_+ = \{c \in E : c \succeq 0\}$. We denote by $\mathbb{N}$ the set of natural numbers including 0. In this context, we denote $c_n \downarrow c$ if $(c_n)_{n\in\mathbb{N}}$ is a decreasing sequence in E such that $\inf c_n = c$. Also, for $c \in E$ we denote $\mid c \mid = \sup\{c, -c\}$. Recall that b is a supremum of a set $F \subset E$ if b is a least upper bound for F. Similar for infimum, which is the greatest lower bound for F.

Definition 1.1. Let E be a Riesz space.

- A sequence $(c_n)_{n\in\mathbb{N}}$ is said to be order convergent to c if there is a sequence $(b_n)_{n\in\mathbb{N}}$ in E with $b_n \downarrow 0$ and $\mid c_n - c \mid \preceq b_n$, for all n.
- A sequence $(c_n)_{n\in\mathbb{N}}$ is called order-Cauchy if there exists a sequence $(b_n)_{n\in\mathbb{N}}$ in E with $b_n \downarrow 0$ and $\mid c_n - c_{n+p} \mid \preceq b_n$, for all n and p.

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2020 Mathematics Subject Classification. 47H10, 54H25.
Keywords and Phrases. vector metric space, fixed point, common fixed point, local fixed point theorem, stability properties.

Definition 1.2. A Riesz space is called Archimedean if $n^{-1}c \downarrow 0$, for every $c \in E_+$.

For other notions and results concerning Riesz spaces see [**7**].

The following concepts are considered in [**6**].

Definition 1.3. Let X be a non-empty set and let E be a Riesz space. The mapping $d : X \times X \to E_+$ is called a vector metric (or an E-metric) if the following properties hold:

(i) $d(x, y) = 0$ if and only if $x = y$;

(ii) $d(x, y) \preceq d(x, z) + d(y, z)$, for all $x, y, z \in X$.

Under the above assumptions, the pair (X, d) is called vector metric space.

Remark 1.4. ([**6**]) If d is a vector metric on X, then $d(x, y) = d(y, x)$, for each $x, y \in X$.

Some useful notions are now presented.

Definition 1.5. Let (X, d) be a vector metric space.

- A sequence $(x_n)_{n\in\mathbb{N}} \subset X$ is E-convergent (or vectorial convergent) to an element $x \in X$ (and we will write $x_n \xrightarrow{d,E} x$) if there exists $(a_n)_{n\in\mathbb{N}} \subset E$, $a_n \downarrow 0$ and $d(x_n, x) \preceq a_n$, for all n.
- A sequence $(x_n)_{n\in\mathbb{N}} \subset X$ is E-Cauchy sequence if there exists $(a_n)_{n\in\mathbb{N}} \subset E$, $a_n \downarrow 0$ and $d(x_n, x_{n+p}) \preceq a_n$, for all n, p.
- A vector metric space X is E-complete if each E-Cauchy sequence in X is E-convergent to a limit in X.
- A subset Y of a vector metric space X is said to be E-closed whenever $(x_n)_{n\in\mathbb{N}} \subseteq Y$ and $x_n \xrightarrow{d,E} x$ imply $x \in Y$.

For other notions and results concerning vector metric spaces see [**6**].

Let (X, d) be a vector metric space. We will consider $f, g : X \to X$ two operators and the fixed point equations:

$$x = f(x), \tag{1}$$

respectively

$$x = f(x) = g(x). \tag{2}$$

In the above context, we will denote by $Fix(f) = \{x \in X \mid f(x) = x\}$ the fixed point set of f and by

$$CFP(f, g) = \{x \in X \mid x = f(x) = g(x)\}$$

the common fixed point set of f and g.

Remark 1.6. It is easy to see that $CFP(f,g) = Fix(f) \cap Fix(g)$.

Let $f : X \to X$ be an operator. We will denote $f^n := f \circ f \circ ... \circ f$ (n-times).

Definition 1.7. Let (X, d) be a vector metric space. We say that $f : X \to X$ is a graphic k-contraction if there exists $k \in (0, 1)$ such that

$$d(f(x), f^2(x)) \preceq kd(x, f(x)), \text{ for all } x \in X.$$

Definition 1.8. Let (X, d) be a vector metric space. Then, by definition, $f : X \to X$ is called a weakly Picard operator if $Fix(f) \neq \emptyset$ and, for each $x \in X$, the sequence $(f^n(x))_{n\in\mathbb{N}}$ E-converges to $x^*(x) \in Fix(f)$. Any weakly Picard operator with a unique fixed point (i.e., $Fix(f) = \{x^*\}$) is called a Picard operator.

Definition 1.9. Let (X, d) be a vector metric space and $f : X \to X$ be a weakly Picard operator. If there exists $C > 0$ such that

$$d(x, x^*(x)) \preceq Cd(x, f(x)), \text{ for all } x \in X, \tag{3}$$

then f is called weakly C-Picard operator.
In the case of a unique fixed point, the above condition give rise to the concept of C-Picard operator.

Remark 1.10. The relation (3) is called the retraction-displacement condition corresponding to f.

The purpose of this work is prove several new results for the common fixed point problem $f(x) = x = g(x), x \in X$, where $f, g : X \to X$ are two operators and X is an $\mathbb{E}$-complete vector metric space (where $\mathbb{E}$ is a Riesz space, which is a richer structure than a cone metric space, but poorer than that of a real Banach space). Stability results are also given. Our results extend some theorems from [**6**], including the study of the fixed point problem from the following perspectives: existence, uniqueness, data dependence and stability of some iteration processes.

2. MAIN RESULTS

We start this section by a fixed point theorem for graphic contractions in E-complete vector metric spaces. For the recommended terminology in fixed point theory see [**1**]. For related results and applications in classical metric spaces see [**12**].

Theorem 2.1. *Let (X, d) be an E-complete vector metric space such that E is Archimedean. Let $f : X \to X$ be a graphic k-contraction. Suppose that the set $Graph(f) := \{(x, f(x)) : x \in X\}$ is E-closed in $X \times X$.*
Then the following conclusions hold:

(a) $Fix(f) \neq \emptyset$;

(b) for each $x \in X$ the sequence $(f^n(x))_{n\in\mathbb{N}}$ E-converges to an element $x^*(x) \in Fix(f)$;

(c) f is a weakly $(1-k)^{-1}$-Picard operator, i.e., the following retraction-displacement condition holds

$$d(x, x^*(x)) \preceq \frac{1}{1-k} d(x, f(x)), \text{ for all } x \in X.$$

(d) if $k < \frac{1}{3}$, then f is a $\frac{k}{1-2k}$-quasi-contraction, i.e.,

$$d(f(x), x^*(x)) \preceq \frac{k}{1-2k} d(x, x^*(x)), \text{ for every } x \in X.$$

Proof. (a)-(b) We will show that f is a weakly Picard operator. Let x be an arbitrary element of X and define $x_0 = x$ and $x_n := f^n(x)$, for $n \in \mathbb{N}$ with $n \geq 1$. Since, for every $n \in \mathbb{N}$ we have

$$d(x_{n+1}, x_n) = d(f(x_n), f(x_{n-1})) \preceq kd(x_n, x_{n-1}) \preceq \cdots \preceq k^n d(f(x), x),$$

it follows that

$$d(x_n, x_{n+p}) \preceq k^n(1-k^p)(1-k)^{-1} d(f(x), x) \preceq k^n(1-k)^{-1} d(x, f(x)),$$

for each $n, p \in \mathbb{N}, p \geq 1$. Since E is Archimedean, the sequence $(x_n)_{n\in\mathbb{N}}$ is E-Cauchy in X and hence it is E-convergent to an element $x^*(x) \in X$. By the relation $x_{n+1} = f(x_n)$, for $n \in \mathbb{N}$, since f has closed graph, we get that $x^*(x) \in Fix(f)$.

(c) Using the relation

$$d(x_n, x_{n+p}) \preceq k^n(1-k)^{-1} d(f(x), x),$$

and letting $p \to \infty$ we obtain that

$$d(x_n, x^*(x)) \preceq k^n(1-k)^{-1} d(f(x), x), \text{ for each } n \in \mathbb{N}.$$

(d) We suppose $k < \frac{1}{3}$. By (c), it follows

$$d(f(x), x^*(x)) \preceq \frac{1}{1-k} d(f(x), f^2(x)) \preceq \frac{k}{1-k} d(x, f(x))$$

$$\preceq \frac{k}{1-k} \left(d(x, x^*(x)) + d(f(x), x^*(x)) \right).$$

Hence,

$$d(f(x), x^*(x)) \preceq \frac{k}{1-2k} d(x, x^*(x)), \text{ for all } x \in X.$$

□

Example 2.2. *Let $E = \mathbb{R}^2$ with the coordinatwise ordering and $X = \{(x, 0) : x \in [0, 1]\} \cup \{(0, y) : y \in [0, 1]\} \cup \{(1, y) : y \in [0, 1]\}$. We define on X the metric d by:*

$$d((x, 0), (y, 0)) = (\frac{4}{3}|x-y|, |x-y|), d((0, x), (0, y)) = (|x-y|, \frac{2}{3}|x-y|),$$

$$d((x,0),(0,y)) = (\frac{4}{3}x + y, x + \frac{2}{3}y), d((1,x),(1,y)) = (|x-y|, \frac{4}{3}|x-y|),$$

$$d((x,0),(1,y)) = (|1-y|, |x-1|), d((0,x),(1,y)) = (y, |x-y|).$$

Let $f : X \to X$ be given by

$$f(x,0) = (0,x), f(0,y) = (\frac{y}{2}, 0), f(1,y) = (1,1).$$

Then f is a graphic $\frac{3}{4}$-contraction with closed graph. Moreover, $Fix(f) = \{(0,0),(1,1)\}$ and f is not a contraction.

Proof. We consider $(x,0) \in X$. Thus,

$$d(f(x,0), f^2(x,0)) = d((0,x), f(0,x)) = d((0,x), (\frac{x}{2}, 0))$$

$$= d((\frac{x}{2},0),(0,x)) = (\frac{5x}{3}, \frac{7x}{6}) \preceq \frac{3}{4}(\frac{7x}{3}, \frac{5x}{3})$$

$$= \frac{3}{4}d((x,0),(0,x)) = \frac{3}{4}d((x,0), f(x,0)).$$

Let now $(0,y) \in X$. We obtain

$$d(f(0,y), f^2(0,y)) = d((\frac{y}{2},0), f(\frac{y}{2},0)) = d((\frac{y}{2},0),(0,\frac{y}{2}))$$

$$= (\frac{7y}{6}, \frac{5y}{6}) \preceq \frac{3}{4}(\frac{5y}{3}, \frac{7y}{6})$$

$$= \frac{3}{4}d((\frac{y}{2},0),(0,y)) = \frac{3}{4}d((0,y), f(0,y)).$$

Similarly, we have

$$d(f(1,y), f^2(1,y)) = d((1,1), f(1,1)) = d((1,1),(1,1)) = 0$$

$$\preceq \frac{3}{4}d((1,y), f(1,y)), \text{ for all } (1,y) \in X.$$

We conclude that f is a graphic $\frac{3}{4}$-contraction.

Let $(x,x) \in X$. From $f(x,x) = (x,x)$, it follows $(x,x) = (0,0)$, $(x,x) = (1,1)$, i.e. $Fix(f) = \{(0,0),(1,1)\}$. □

In particular, another retraction-displacement result for graphic contractions with a unique fixed point holds.

Theorem 2.3. *Let (X,d) be a vector metric space and $f : X \to X$ be a graphic k-contraction such that $Fix(f) = \{x^*\}$ and, for each $x \in X$, the sequence $(f^n(x))_{n\in\mathbb{N}}$ is E-convergent to x^*. Then*

$$d(x, x^*) \preceq \frac{1}{1-k}d(x, f(x)), \text{ for all } x \in X.$$

Proof. Let $x \in X$ be arbitrary and denote $x_n := f^n(x)$, $n \in \mathbb{N}$. We successively obtain

$$d(x,x^*) \preceq d(x,f(x)) + d(x^*,f(x)) \preceq d(x,f(x)) + d(x^*,f^2(x)) + d(f(x),f^2(x))$$

$$\preceq (1+k)d(x,f(x)) + d(x^*,f^2(x)) \preceq ... \preceq (1+k+...+k^{n-1})d(x,f(x)) + d(x^*,f^n(x))$$

$$\preceq \frac{1}{1-k}d(x,f(x)) + d(x^*,f^n(x)) = \frac{1}{1-k}d(x,f(x)) + d(x^*,x_n).$$

Since $(x_n)_{n\in\mathbb{N}}$ is E-convergent to x^*, there exists a sequence $(a_n)_{n\in\mathbb{N}}$ in E such that $a_n \downarrow 0$ and $d(x_n,x^*) \preceq a_n$, for every $n \in \mathbb{N}$. Hence, we get that

$$d(x,x^*) \preceq \frac{1}{1-k}d(x,f(x)) + a_n.$$

In conclusion, we get that $d(x,x^*) \preceq \frac{1}{1-k}d(x,f(x))$. □

Some stability properties for some general classes of mappings in vector metric spaces are established in the next part of this section.

First, the well-posedness in the sense of Reich and Zaslavski, adapted to the context of a vector metric space, is presented. For related considerations see [**14**].

Definition 2.4. Let (X,d) be a vector metric space and $f : X \to X$ be an operator with a unique fixed point $x^* \in X$. If, for any sequence $(u_n)_{n\in\mathbb{N}} \subset X$ such that $(d(u_n,f(u_n)))_{n\in\mathbb{N}}$ is order convergent to 0 we have that $(u_n)_{n\in\mathbb{N}}$ is E-convergent to x^*, then we say that the fixed point equation $x = f(x)$ is well-posed in the sense of Reich and Zaslavski.

Theorem 2.5. *Let (X,d) be a vector metric space and $f : X \to X$ be a C-Picard operator. Denote by x^* its unique fixed point. Then, the fixed point equation $x = f(x)$ is well-posed in the sense of Reich and Zaslavski.*

Proof. We consider $(u_n)_{n\in\mathbb{N}} \subset X$, such that $d(u_n,f(u_n))$ is order convergent to 0. Then, there exists a sequence $(a_n)_{n\in\mathbb{N}}$ in E such that $a_n \downarrow 0$ and $d(u_n,f(u_n)) \preceq a_n$, for every $n \in \mathbb{N}$. Then, by the C-Picard assumption on f, it follows that

$$d(u_n,x^*) \preceq cd(u_n,f(u_n)) \preceq ca_n, \text{ for all } n \in \mathbb{N}.$$

Thus, $d(u_n,x^*) \preceq ca_n \downarrow 0$ and so, the sequence $(u_n)_{n\in\mathbb{N}}$ is E-convergent to x^*. □

For our next stability result, we will need the following variant of the classical Cauchy-Toeplitz Lemma in a Riesz space E.

Lemma 2.6. *Let E be an Archimedean Riesz space. Let $\gamma \in (0,1)$ and $(b_n)_{n\in\mathbb{N}}$ be a sequence in E_+ order convergent to 0. Then $c_n := \sum_{k=0}^{n} \gamma^{n-k} b_k \downarrow 0$.*

Proof. Since the sequence $(b_n)_{n\in\mathbb{N}}$ from E_+ is order convergent to 0, there exists a sequence $(a_n)_{n\in\mathbb{N}}$ in E_+, such that $a_n \downarrow 0$ and $b_n \preceq a_n$ for every $n \in \mathbb{N}$. Thus, $a_n \succeq 0$ for every $n \in \mathbb{N}$ and $\inf\limits_{n\in\mathbb{N}} a_n = 0$. As a consequence, $(a_n)_{n\in\mathbb{N}}$ is decreasing and for every $\varepsilon \succeq 0$ there exists $n_\varepsilon \in \mathbb{N}$ such that $a_{n_\varepsilon} \preceq \varepsilon$. Thus $a_n \preceq \varepsilon$, for every $n \geq n_\varepsilon$. Then, we have

$$c_n \preceq \sum_{k=0}^{n} \gamma^{n-k} a_k = \sum_{k=0}^{n_\varepsilon} \gamma^{n-k} a_k + \sum_{k=n_\varepsilon+1}^{n} \gamma^{n-k} a_k \preceq$$

$$\gamma^n \sum_{k=0}^{n_\varepsilon} \gamma^{-k} a_k + \varepsilon \sum_{k=n_\varepsilon+1}^{n} \gamma^{n-k} \downarrow 0.$$

□

Next, the Ostrovski stability property for the fixed point equation is proved.

Definition 2.7. Let (X, d) be a vector metric space and $f : X \to X$ be an operator with a unique fixed point $x^* \in X$. If, for any sequence $(v_n)_{n\in\mathbb{N}} \subset X$ such that $(d(v_{n+1}, f(v_n)))_{n\in\mathbb{N}}$ is order convergent to 0 it follows that $(v_n)_{n\in\mathbb{N}}$ is E-convergent to x^*, then we say that the fixed point equation $x = f(x)$ has the Ostrowski stability property.

Theorem 2.8. *Let (X, d) be a vector metric space with respect to an Archimedean Riesz space E. Let $f : X \to X$ be a γ-quasi-contraction, i.e., there exists $\gamma \in (0, 1)$ such that*

$$d(f(x), x^*) \preceq \gamma d(x, x^*), \text{ for every } x \in X.$$

Suppose that $Fix(f) = \{x^*\}$. Then, the fixed point equation $x = f(x)$ has the Ostrowski stability property.

Proof. We consider $(v_n)_{n\in\mathbb{N}} \subset X$ such that $(d(v_{n+1}, f(v_n)))_{n\in\mathbb{N}}$ is order convergent to 0. Since the operator f is a γ-quasi-contraction, it follows

$$d(v_{n+1}, x^*) \preceq d(v_{n+1}, f(v_n)) + d(x^*, f(v_n))$$

$$\preceq d(v_{n+1}, f(v_n)) + \gamma d(x^*, v_n)$$

$$\preceq d(v_{n+1}, f(v_n)) + \gamma d(v_n, f(v_{n-1})) + \gamma^2 d(x^*, v_{n-1}) \preceq ...$$

$$\preceq d(v_{n+1}, f(v_n)) + \gamma d(v_n, f(v_{n-1})) + ... + \gamma^n d(v_1, f(v_0)) + \gamma^{n+1} d(x^*, v_0).$$

By the above Cauchy-Toeplitz Lemma in Riesz spaces (see Lemma 2.6, applied here for $b_n := d(v_{n+1}, f(v_n))$), we get that $(v_n)_{n\in\mathbb{N}}$ is E-convergent to x^*. □

We can summarize the above results in our next theorem.

Theorem 2.9. *Let (X, d) be an E-complete vector metric space such that E is Archimedean. Let $f : X \to X$ be a graphic k-contraction with $Fix(f) = \{x^*\}$. Suppose that the set $Graph(f) := \{(x, f(x)) : x \in X\}$ is E-closed in $X \times X$.*
Then the following conclusions hold:

(i) f is $(1-k)^{-1}$-Picard operator;

(ii) the fixed point equation $x = f(x)$ is well-posed in the sense of Reich and Zaslavski

(iii) if additionally $k < \frac{1}{3}$, then the fixed point equation $x = f(x)$ has the Ostrowski stability property.

For related results and open questions see [**11**].

By a similar approach, Ulam-Hyers stability of the fixed point equation can be established. Recall that, if (X, d) is a vector metric space and $f : X \to X$ is an operator with $Fix(f) \neq \emptyset$, then the fixed point equation $x = f(x)$ is called Ulam-Hyers stable if there exists $c \in E_+ \setminus \{0\}$ such that for every $\epsilon \in E_+ \setminus \{0\}$ and any $z \in X$ with $d(z, f(z)) \preceq \epsilon$ there exists $x^* \in X$ such that $d(x^*, z) \preceq c\epsilon$. Then, it is easy to prove that for any weakly C-Picard operator $f : X \to X$ the fixed point equation $x = f(x)$ is Ulam-Hyers stable. We refer to [**8**] and [**15**] for similar results and details.

We will continue our study by considering the common fixed point problem for two operators $f, g : X \to X$ in a vector metric space (X, d). We have the following existence, uniqueness and approximation result.

Theorem 2.10. *Let (X, d) be a E-complete metric space such that E is Archimedean and let $f, g : X \to X$ be two operators for which there exists $k \in (0, \frac{1}{2})$ such that:*

$$d(f(x), g(y)) \preceq k[d(x, f(x)) + d(y, g(y))], \text{ for all } x, y \in X. \tag{4}$$

Then, the following assertions hold:

(a) *for every $x_0 \in X$, the sequence $(x_n)_{n\in\mathbb{N}}$ given by*

$$x_{2n+1} = f(x_{2n}), \quad x_{2n} = (g \circ f)^n(x_0), \quad \forall\, n \in \mathbb{N},$$

is E-convergent to an element $x^* \in Fix(f)$.

(b) *for every $y_0 \in X$, the sequence $(y_n)_{n\in\mathbb{N}}$ given by*

$$y_{2n+1} = g(y_{2n}), \quad y_{2n} = (f \circ g)^n(y_0), \quad \forall\, n \in \mathbb{N},$$

is E-convergent to an element $y^* \in Fix(g)$.

(c) *$Fix(f) = Fix(g) = \{x^*\}$ and thus $CFP(f, g) = \{x^*\}$;*

(d) *the operators f and g are graphic $\frac{k}{1-k}$-contractions;*

(e) *the operators f and g are $\frac{k}{1-k}$-quasi-contractions;*

(f) *the operators f and g are $\frac{1-k}{1-2k}$-Picard operators;*

(g) *the fixed point problem for f and the fixed point problem for g are well-posed in the sense of Reich and Zaslavski;*

(h) *if, additionally $k < \frac{1}{4}$, then the fixed point problem for f and the fixed point problem for g have the Ostrowski property.*

Proof. (a) We consider $x_0 \in X$ arbitrary chosen and $(x_n)_{n\in\mathbb{N}}$ as in (a). We have

$$d(x_1, x_2) = d(f(x_0), g(x_1)) \preceq k[d(x_0, f(x_0)) + d(x_1, g(x_1))]$$

and then

$$d(x_1, x_2) \preceq \frac{k}{1-k} d(x_0, x_1).$$

Inductively, we get

$$d(x_n, x_{n+1}) \preceq \left(\frac{k}{1-k}\right)^n d(x_0, x_1), \quad \forall n \in \mathbb{N}. \tag{5}$$

It follows that $(x_n)_{n\in\mathbb{N}}$ is a E-Cauchy sequence. Then there exists $(a_n)_{n\in\mathbb{N}} \subset E$, $a_n \downarrow 0$ and $d(x_n, x^*) \preceq a_n$, for every $n \in \mathbb{N}$.
Thus,

$$\begin{aligned} d(x^*, f(x^*)) &\preceq d(x^*, x_{2n}) + d(x_{2n}, f(x^*)) \\ &\preceq d(x^*, x_{2n}) + d(g(x_{2n-1}), f(x^*)) \\ &\preceq d(x^*, x_{2n}) + k[d(x^*, f(x^*)) + d(x_{2n-1}, x_{2n})]. \end{aligned}$$

We deduce $d(x^*, f(x^*)) = 0$ and hence $x^* \in Fix(f)$. Now, from (5), we obtain

$$d(x_n, x_{n+p}) \preceq \frac{\left(\frac{k}{1-k}\right)^n}{1 - \frac{k}{1-k}} d(x_0, x_1), \quad \forall\, n \in \mathbb{N},\ p \in \mathbb{N}^*.$$

For $p \to +\infty$ and $n = 0$ it results:

$$d(x_0, x^*) \preceq \frac{1-k}{1-2k} d(x_0, f(x_0)). \tag{6}$$

(b) Is similar with (a) we will obtain that the sequence $(y_n)_{n\in\mathbb{N}}$ given by (b) is E-convergent to $y^* \in Fix(g)$ and the following retraction displacement condition holds

$$d(y_0, y^*) \preceq \frac{1-k}{1-2k} d(y_0, g(y_0)). \tag{7}$$

(c) We justify $Fix(f) = Fix(g)$. Let $x^* \in Fix(f)$. From (4), it results

$$d(x^*, g(x^*)) = d(f(x^*), g(x^*)) \preceq k d(x^*, g(x^*))$$

and then $x^* \in Fix(g)$.
Let now $x^*, y^* \in Fix(f) \cap Fix(g)$. Using (4), we obtain

$$d(x^*, y^*) = d(f(x^*), g(y^*)) \preceq k[d(x^*, f(x^*)) + d(y^*, g(y^*))].$$

Thus $d(x^*, y^*) = 0$, which implies $x^* = y^*$.

(d) We consider $x \in X$. Then

$$d(f^2(x), f(x)) \preceq d(f^2(x), g(x^*)) + d(f(x), g(x^*))$$

$$\preceq k[d(f(x), f^2(x)) + d(x^*, g(x^*))] + k[d(x, f(x)) + d(x^*, g(x^*))]$$

$$= k[d(f^2(x), f(x)) + d(x, f(x))].$$

Thus,

$$d(f^2(x), f(x)) \preceq \frac{k}{1-k} d(x, f(x)) \tag{8}$$

and we deduce that f is graphic contraction. Similarly, we obtain that g is a graphic contraction too.

(e) We justify that f is quasi-contraction. We have

$$d(f(x), x^*) = d(f(x), g(x^*)) \preceq kd(x, f(x))$$

$$\preceq k[d(x, x^*) + d(x^*, f(x))],$$

which implies

$$d(f(x), x^*) \preceq \frac{k}{1-k} d(x, x^*), \text{ for all } x \in X.$$

From (4) we obtain that g is also quasi-contraction.

(f) the conclusion follows by (6) and (7).

(g) and (h) follow by Theorem 2.9. □

Remark 2.11. For related results in cone metric spaces see [**2, 3, 4, 5**]. For other related results and developments see [**10**], [**13**]. For the multivalued case see [**9**].

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(Received 19.01.2025.)
(Revised 05.12.2025.)

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