Applicable Analysis and Discrete Mathematics
available online at http://pefmath.etf.rs

Appl. Anal. Discrete Math. **19** (2025), 559–576.
https://doi.org/10.2298/AADM250110030I

SHRINKING PROJECTION METHOD WITH ALLOWABLE RANGES FOR NONLINEAR MAPPINGS OF FIRMLY NONEXPANSIVE TYPE

Takanori Ibaraki

In this paper, we study a shrinking projection method with allowable ranges introduced by Takeuchi in 2019. We obtain strong convergence theorems for finding a fixed point of mappings of types (Q) and (R) in a Banach space. Using our results, we discuss the zero point problem of maximal monotone operators.

1. INTRODUCTION

In 2008, Takahashi, Takeuchi, and Kubota [20] introduced an iterative method for finding a common fixed point of some families of nonlinear mappings in a Hilbert space. We call such a method the shrinking projection method. In 2009, Kimura and Takahashi [15] improved the method. Also, in 2012, Kimura [12] considered the shrinking projection method with nonsummable errors in a geodesic space; see also [13, 14]. This method has been studied in [4–7], and others.

Recently, motivated by Kimura [12–14], Takeuchi [22] presents a shrinking projection method with allowable ranges. They obtained a strong convergence theorem for finding a fixed point of a nonlinear mapping in a Banach space.

Theorem 1 ([22]). *Let E be a reflexive, smooth, and strictly convex Banach space which has the Kadec-Klee property. Let C be a nonempty closed convex subset of*

2020 Mathematics Subject Classification. 47H05, 47H09, 47J25
Keywords and Phrases. Shrinking projection method, allowable range, type (Q), type (R), metric projection, generalized projection

E. Let $B : C \to E$ be a mapping of type (P) such that $F(B) := \{p \in C : Bp = p\}$ is nonempty. Consider an iterative procedure as below: Let $x_0 \in E$, $w_1 \in C$, $D_1 = \{y \in C : \langle Bw_1 - y, J(w_1 - Bw_1)\rangle \geq 0\}$ and $x_1 = P_{D_1}x_0$. Let $A_1 = C \setminus (D_1 \cup \{w_1\})$ and let $y_1 \in A_1$. For each $n \in \mathbb{N}$, generate $D_{n+1}, x_{n+1}, A_{n+1}$ and y_{n+1} by

$$D_{n+1} = \{y \in D_n : \langle By_n - y, J(y_n - By_n)\rangle \geq 0\}, \quad x_{n+1} = P_{D_{n+1}}x_0,$$
$$A_{n+1} = \{y \in D_n : \|x_0 - y\| \leq \|x_0 - x_{n+1}\|, y \neq y_n\}, \quad y_{n+1} \in A_{n+1}$$

where P_K is the metric projection of E onto a nonempty closed convex subset K of E. Then, either of following holds:

1. *$A_n \neq \emptyset$ for each $n \in \mathbb{N}$; the procedure is not stopped. In this case, $\{x_n\}$ and $\{y_n\}$ converge strongly to $P_{F(B)}x_0$*
2. *$A_k = \emptyset$ for some $k \in \mathbb{N}$; the procedure is stopped. In this case, $y_{k-1} \in F(B)$ or $w_1 \in F(B)$ holds.*

Note that the original result of the theorem above deals with nonlinear mappings related to mappings of type (P) in a Banach space.

On the other hand, the study of firmly nonexpansive mappings is a central topic in convex analysis since it is closely related to the theory of monotone operators. This study also has a strong connection to metric fixed point theory. Let C be a nonempty closed convex subset of a Hilbert space H and let $T : C \to H$ be a mapping. A mapping T is said to be firmly nonexpansive if

$$\langle (x - Tx) - (y - Ty), Tx - Ty\rangle \geq 0$$

for each $x, y \in C$. We know that the mapping of type (P) defined on a smooth Banach space is a generalization of firmly nonexpansive mappings on a Hilbert space. We also know two generalizations in a smooth Banach space which are called mappings of types (Q) and (R).

In this paper, we study a shrinking projection method with allowable ranges introduced by Takeuchi [22] for mappings of types (Q) and (R) in a smooth Banach space. We prove strong convergence theorems for finding a fixed point of mappings of types (Q) and (R) in a Banach space. Using our results, we discuss the zero point problem of maximal monotone operators.

2. PRELIMINARIES

$\mathbb{N}$ and $\mathbb{R}$ denote the sets of positive integers and real numbers, respectively. Let E be a real Banach space with its dual E^*. Let $\{x_n\}$ be a sequence in E. The strong and the weak convergence of $\{x_n\}$ to a point $x \in E$ are denoted by $x_n \to x$ and $x_n \rightharpoonup x$, respectively. The normalized duality mapping $J : E \to E^*$ is defined by

$$Jx = \{y^* \in E^* : \|x\|^2 = \langle x, y^*\rangle = \|y^*\|^2\}$$

for $x \in E$. A Banach space E is said to have the Kadec-Klee property if a sequence $\{x_n\}$ of E satisfying $x_n \rightharpoonup x_0$ and $\|x_n\| \to \|x_0\|$ converges strongly to x_0. We know the following properties; see, for instance, [19].

- J is monotone, that is, $\langle x - y, x^* - y^* \rangle \geq 0$ for each $(x, x^*), (y, y^*) \in J$,
- if E is smooth and strictly convex, then $\langle x - y, Jx - Jy \rangle = 0$ if and only if $x = y$,
- if E is reflexive, smooth and strictly convex, then J is surjective and the duality mapping on E^* is J^{-1},
- E is reflexive, strictly convex and has the Kadec-Klee property if and only if E^* has a Fréchet differentiable norm.

Let C be a nonempty subset of a smooth Banach space E. A mapping $T : C \to E$ is said to be of type (Q) [2, 17] if

$$\langle Tx - Ty, (Jx - JTx) - (Jy - JTy) \rangle \geq 0$$

for each $x, y \in C$. A mapping $T : C \to E$ is said to be of type (R) [2, 10] if

$$\langle (x - Tx) - (y - Ty), JTx - JTy \rangle \geq 0$$

for each $x, y \in C$. We denote by $F(T)$ the set of fixed points of T. It is clear that if $T : C \to E$ is of type (Q) and $F(T)$ is nonempty, then

$$\langle Tx - p, Jx - JTx \rangle \geq 0 \tag{1}$$

for each $x \in C$ and $p \in F(T)$. We also know the following result.

Lemma 2 ([2]). *Let E be a reflexive and strictly convex Banach space having a Fréchet differentiable norm and the Kadec-Klee property. Let C be a nonempty subset of E and let $T : C \to E$ be a mapping of type (Q). Then, T is norm to norm continuous.*

The mappings of types (Q) and (R) are strongly related to each other; it is a kind of duality in the following sense. Let E be a reflexive, smooth and strictly convex Banach space, let C be a nonempty subset of E and let $T : C \to E$ be a mapping. Define a mapping T^* as follows:

$$T^* x^* := JTJ^{-1}x^* \tag{2}$$

for each $x^* \in JC$. We know that $JF(T) = F(T^*)$; see [3, 21]. Further, we have the following result.

Lemma 3 ([2]). *Let E be a reflexive, smooth, and strictly convex Banach space, let C be a nonempty subset of E, and let $T : C \to E$ be a mapping of type (R). Let $T^* : JC \to E^*$ be a mapping defined by* (2). *Then T^* is of type (Q) in E^*.*

Let E be a reflexive and strictly convex Banach space, and let C be a nonempty closed convex subset of E. It is known that for each $x \in E$, there exists a unique point $z \in C$ such that

$$\|z - x\| = \inf_{y \in C} \|y - x\|.$$

Such a point z is denoted by $P_C x$, and P_C is called the metric projection of E onto C. Note that the metric projection of E^* onto a nonempty closed convex subset K^* of E^* is denoted by $P^*_{K^*}$. We know the following: let $z \in C$ and $x \in E$. Then, $z = P_C x$ if and only if $\langle y - z, J(z - x)\rangle \geq 0$ for each $y \in C$.

Lemma 4 ([9]). *Suppose that C is a nonempty closed convex subset of a reflexive and strictly convex Banach space E and $u \in E$. If F is a nonempty closed convex subset of C such that $P_C u \in F$, then $P_F u = P_C u$.*

Let E be a smooth Banach space and consider the following function $V : E \times E \to \mathbb{R}$ defined by

$$V(x, y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2 \tag{3}$$

for each $x, y \in E$. We know the following properties; see [1, 8, 11] for more details:

1. $(\|x\| - \|y\|)^2 \leq V(x, y) \leq (\|x\| + \|y\|)^2$ for each $x, y \in E$,
2. $V(x, y) + V(y, x) = 2\langle x - y, Jx - Jy\rangle$ for each $x, y \in E$,
3. $V(x, y) = V(x, z) + V(z, y) + 2\langle x - z, Jz - Jy\rangle$ for each $x, y, z \in E$,
4. if E is additionally assumed to be strictly convex, then $V(x, y) = 0$ if and only if $x = y$.

Let E be a reflexive, smooth, and strictly convex Banach space with its dual E^*, and consider the following function $V_* : E^* \times E^* \to \mathbb{R}$ defined by

$$V_*(x^*, y^*) = \|x^*\|^2 - 2\langle x^*, J^{-1} y^*\rangle + \|y^*\|^2$$

for each $x^*, y^* \in E^*$. We know that $V_*(Jx, Jy) = V(y, x)$ for each $x, y \in E$.

Let E be a reflexive, smooth, and strictly convex Banach space, and let C be a nonempty closed convex subset of E. It is known that for each $x \in E$, there exists a unique point $z \in C$ such that

$$V(z, x) = \min_{y \in C} V(y, x).$$

Such a point z is denoted by $\Pi_C x$, and Π_C is called the generalized projection of E onto C (see [1]). We know the following: let $z \in C$ and $x \in E$. Then, $z = \Pi_C x$ if and only if $\langle y - z, Jz - Jx\rangle \geq 0$ for each $y \in C$.

Lemma 5 ([9]). *Suppose that C is a nonempty closed convex subset of a reflexive, smooth, and strictly convex Banach space E and $u \in E$. If F is a nonempty closed convex subset of C such that $\Pi_C u \in F$, then $\Pi_F u = \Pi_C u$.*

Note that the generalized projection of E^* onto a nonempty closed convex subset K^* of E^* is denoted by $\Pi^*_{K^*}$. Let D be a nonempty subset of E such that JD is closed and convex, and let Π^*_{JD} be the generalized projection of E^* onto JD. The mapping R_D of E onto D defined by

$$R_D := J^{-1}\Pi^*_{JD}J$$

is called the sunny generalized nonexpansive retraction; see [8,16] for more details.

3. LEMMAS

In this section, we present some lemmas for obtaining our main results. We first recall the following lemma in Takeuchi [22].

Lemma 6 ([22]). *Let E be a reflexive and strictly convex Banach space. Let $x_0 \in E$ and let $\{D_n\}$ be a sequence of closed convex subsets of E satisfying $D_{n+1} \subset D_n$ for each $n \in \mathbb{N}$ and $D_0 := \cap_n D_n \neq \emptyset$. Let $x_1 = P_{D_1}x_0$. For each $n \in \mathbb{N}$, define x_{n+1}, K_n and u_n by*

$$x_{n+1} = P_{D_{n+1}}x_0, \quad K_n = \{z \in D_n : \|x_0 - z\| \leq \|x_0 - x_{n+1}\|\}, \quad u_n \in K_n.$$

Then $\{x_n\}$ and $\{u_n\}$ converge weakly to $P_{D_0}x_0$. When E has the Kadec-Klee property, $\{x_n\}$ and $\{u_n\}$ converge strongly to $P_{D_0}x_0$.

Next, we prove the following two lemmas for a mapping of type (Q). Compare these lemmas with the results in Takeuchi [22].

Lemma 7. *Let C be a nonempty closed convex subset of a reflexive, smooth, and strictly convex Banach space E. Let $T : C \to E$ be a mapping of type (Q) such that $F(T) \neq \emptyset$, and let D be a nonempty closed convex subset of C such that $F(T) \subset D$. Let $r > 0$ and let $u \in C$. Let M be a subset of E defined by*

$$M := \{z \in D : \langle Tu - z, Ju - JTu\rangle \geq 0\}.$$

Then, M is closed and convex, and $F(T) \subset M$.

Proof. Let $r > 0$ and let $u \in C$. It is clear that M is closed and convex. Set $M_E = \{z \in E : \langle Tu - z, Ju - JTu\rangle \geq 0\}$. We show $F(T) \subset M_E$. Let $p \in F(T)$. From (1), we have

$$\langle Tu - p, Ju - JTu\rangle \geq 0$$

and hence $p \in M_E$. This implies $F(T) \subset M_E$. Since $F(T) \subset D$, we have $F(T) \subset M_E \cap D = M$. □

Lemma 8. *Let E be a reflexive and strictly convex Banach space having a Fréchet differentiable norm and the Kadec-Klee property. Let C be a nonempty closed convex subset of E and let $T : C \to E$ be a mapping of type (Q) such that $F(T) \neq \emptyset$.*

Let $\{u_n\}$ be a sequence in C and let $\{D_n\}$ be a sequence of subsets of C generated by $D_1 = C$ and

$$D_{n+1} = \{z \in D_n : \langle Tu_n - z, Ju_n - JTu_n\rangle \geq 0\}.$$

for each $n \in \mathbb{N}$. Let $D_0 := \cap_{n\in\mathbb{N}} D_n$. Then the following hold:

1. *D_n is closed and convex, and $F(T) \subset D_n$ for each $n \in \mathbb{N} \cup \{0\}$.*
2. *Suppose that $\{u_n\}$ converges strongly to some $u_0 \in D_0$, then $u_0 \in F(T)$.*

Proof. We first show (1) by induction. It is clear that D_1 is closed and convex, and $F(T) \subset D_1$. Suppose that D_k is closed and convex, and $F(T) \subset D_k$ for some $k \in \mathbb{N}$. Using Lemma 7 as $u = u_k$, $D = D_k$, $M = D_{k+1}$, we see that D_{k+1} is closed and convex, and $F(T) \subset D_{k+1}$. This implies that D_n is closed and convex, and $F(T) \subset D_n$ for each $n \in \mathbb{N}$. Hence, it follows that D_0 is closed and convex, and $F(T) \subset \cap_{n\in\mathbb{N}} D_n = D_0$. Thus, we see that (1) holds.

Next, we show (2). Suppose that $\{u_n\}$ converges strongly to some $u_0 \in D_0$. From Lemma 2, we have $Tu_n \to Tu_0$. By the Fréchet differentiability of the norm, the duality mapping J is norm to norm continuous and hence we also have $Ju_n \to Ju_0$ and $JTu_n \to JTu_0$. Since $u_0 \in D_0 \subset D_{n+1}$ for each $n \in \mathbb{N}$, we have

$$\langle Tu_n - u_0, Ju_n - JTu_n\rangle \geq 0$$

for each $n \in \mathbb{N}$. Tending $n \to \infty$, we get

$$\langle Tu_0 - u_0, Ju_0 - JTu_0\rangle \geq 0.$$

Since J is monotone, we have

$$\langle Tu_0 - u_0, Ju_0 - JTu_0\rangle \leq 0$$

and hence we have $\langle Tu_0 - u_0, Ju_0 - JTu_0\rangle = 0$. From the property of J, we see that $Tu_0 = u_0$. This implies $u_0 \in F(T)$. □

Next, we prove the following two lemmas for generalized projections. Compare these lemmas with the results in Takeuchi [22]; see also [18] for Lemma 9.

Lemma 9. *Let E be a reflexive, smooth, and strictly convex Banach space. Let $x_0 \in E$, and let D_0 be a nonempty closed convex subset of E. Let $\{x_n\}$ be a sequence in E satisfying*

$$\limsup_{n\to\infty} V(x_n, x_0) \leq V(\Pi_{D_0} x_0, x_0). \tag{4}$$

Suppose every weak cluster point of $\{x_n\}$ is a point of D_0. Then, $\{x_n\}$ converges weakly to $\Pi_{D_0} x_0$. Furthermore, when E has the Kadec-Klee property, $\{x_n\}$ converges strongly to $\Pi_{D_0} x_0$.

Proof. By (4), $\{x_n\}$ is bounded, that is, $\{x_n\}$ has a weakly convergent subsequence.

Let $\{x_{n_j}\}$ be a subsequence of $\{x_n\}$ which converges weakly to $u \in D_0$. Since $\{x_{n_j}\}$ satisfies (4) and $\|\cdot\|$ is weakly lower semicontinuous, we have

$$\begin{aligned} V(u,x_0) &= \|u\|^2 - 2\langle u, Jx_0\rangle + \|x_0\|^2 \\ &\le \liminf_{j\to\infty}\left\{\|x_{n_j}\|^2 - 2\langle x_{n_j}, Jx_0\rangle + \|x_0\|^2\right\} \\ &= \liminf_{j\to\infty} V(x_{n_j}, x_0) \le \limsup_{j\to\infty} V(x_{n_j}, x_0) \\ &\le V(\Pi_{D_0}x_0, x_0) \le V(u, x_0). \end{aligned}$$

Then,

$$V(u,x_0) = V(\Pi_{D_0}x_0, x_0) = \lim_{j\to\infty} V(x_{n_j}, x_0). \tag{5}$$

Since $\Pi_D x_0$ is unique, we see $u = \Pi_{D_0}x_0$. This implies that every weakly convergent subsequence of $\{x_n\}$ converges weakly to $\Pi_{D_0}x_0$. Then, $\{x_n\}$ itself converges weakly to $\Pi_{D_0}x_0$ and satisfies (5).

For the latter part of this lemma, suppose that E has the Kadec-Klee property. From the definition of V, we have

$$\|x_n\|^2 = V(x_n, x_0) + 2\langle x_n, Jx_0\rangle - \|x_0\|^2 \tag{6}$$

for each $n \in \mathbb{N}$. From (5), (6), and $x_n \rightharpoonup \Pi_{D_0}x_0$, letting $n \to \infty$, we obtain that $\|x_n\|^2 \to \|\Pi_{D_0}x_0\|^2$ and hence

$$\|x_n\| = \sqrt{\|x_n\|^2} \to \sqrt{\|\Pi_{D_0}x_0\|^2} = \|\Pi_{D_0}x_0\|.$$

Since E has the Kadec-Klee property, by the argument as above, we immediately see that $\{x_n\}$ converges strongly to $\Pi_{D_0}x_0$. □

Remark 10. *Of course, we can replace* (4) *by the following:*

$$V(x_n, x_0) \le V(x_{n+1}, x_0) \le V(\Pi_{D_0}x_0, x_0) \tag{7}$$

for each $n \in \mathbb{N}$.

Lemma 11. *Let E be a reflexive, smooth, and strictly convex Banach space. Let $x_0 \in E$ and let $\{D_n\}$ be a sequence of closed convex subsets of E satisfying $D_{n+1} \subset D_n$ for each $n \in \mathbb{N}$ and $D_0 := \cap_n D_n \neq \emptyset$. Let $x_1 = \Pi_{D_1}x_0$. For each $n \in \mathbb{N}$, define x_{n+1}, L_n, and u_n by*

$$x_{n+1} = P_{D_{n+1}}x_0, \quad L_n = \{z \in D_n : V(z, x_0) \le V(x_{n+1}, x_0)\}, \quad u_n \in L_n.$$

Then $\{x_n\}$ and $\{u_n\}$ converge weakly to $\Pi_{D_0}x_0$. Furthermore, when E has the Kadec-Klee property, $\{x_n\}$ and $\{u_n\}$ converge strongly to $\Pi_{D_0}x_0$.

Proof. Since $\emptyset \neq D_0 \subset D_{n+1} \subset D_n$ for each $n \in \mathbb{N}$ and by the properties of the generalized projection, we know that $\{x_n\}$ satisfies (7). Then, $\{x_n\}$ has a weakly convergent subsequence. Also, every subsequence of $\{x_n\}$ satisfies (7). Let $\{x_{n_j}\}$ be a subsequence of $\{x_n\}$ which converges weakly to $u \in E$. For each $m \in \mathbb{N}$, since $\{x_{n_j}\}_{n_j \geq m} \subset D_m$ and D_m is weakly closed, $u \in D_m$ is immediate. So, $u \in D_0$. We confirmed that every weakly convergent subsequence of $\{x_n\}$ converges weakly to a point of D_0.

By $L_n \neq \emptyset$ ($x_n, x_{n+1} \in L_n$), such $\{u_n\}$ exists. We easily see that, for each $n \in \mathbb{N}$,

$$V(u_n, x_0) \leq V(x_{n+1}, x_0) \leq V(u_{n+1}, x_0) \leq V(x_{n+2}, x_0) \leq V(\Pi_{D_0} x_0, x_0).$$

Then, $\{u_n\}$ also satisfies (7). Note $u_k \in L_k \subset D_k \subset D_m$ for $k \geq m$. So, every weakly convergent subsequence of $\{u_n\}$ converges weakly to a point of D_0. From these, by Lemma 9, we immediately have the desired results. □

4. SHRINKING PROJECTION METHOD WITH ALLOWABLE RANGES BY USING THE METRIC PROJECTION

In this section, we study the shrinking projection method with allowable ranges by using the metric projection. We first prove a strong convergence theorem for a mapping of type (Q).

Theorem 12. *Let E be a reflexive and strictly convex Banach space having a Fréchet differentiable norm and the Kadec-Klee property. Let C be a nonempty closed convex subset of E and let $T : C \to E$ be a mapping of type (Q) such that $F(T) \neq \emptyset$. Consider an iterative procedure as below: Let $x_0 \in E$, $D_1 = A_1 = C$, $x_1 = P_{D_1} x_0$, and $u_1 \in A_1$. For each $n \in \mathbb{N}$, generate D_{n+1}, x_{n+1}, A_{n+1}, and u_{n+1} by*

$$D_{n+1} = \{z \in D_n : \langle Tu_n - z, Ju_n - JTu_n \rangle \geq 0\}, \quad x_{n+1} = P_{D_{n+1}} x_0,$$
$$A_{n+1} = \{y \in D_n : \|x_0 - y\| \leq \|x_0 - x_{n+1}\|, y \neq u_n\}, \quad u_{n+1} \in A_{n+1}.$$

Then, either of the following holds:

1. *$A_n \neq \emptyset$ for each $n \in \mathbb{N}$; the procedure is not stopped. In this case, $\{x_n\}$ and $\{u_n\}$ converge strongly to $P_{F(T)} x_0$.*
2. *$A_k = \emptyset$ for some $k \in \mathbb{N} \setminus \{1\}$; the procedure is stopped. In this case, $u_{k-1} \in F(T)$ holds.*

Proof. From $D_1 = A_1 = C$, D_1 and A_1 are nonempty. Let $u_1 \in A_1 (= C)$. Then we generate D_2. From Lemma 7, D_2 is nonempty, closed, and convex, and also generates x_2 and A_2. A_2 may be empty. In the case of $A_2 \neq \emptyset$, we can find $u_2 \in A_2$ and generate D_3. From Lemma 7, D_3 is nonempty, closed, and convex, and also

generates x_3 and A_3. A_3 may be empty. In the case of $A_3 \neq \emptyset$, we can find $u_3 \in A_3$ and continue this process. So, the procedure is stopped when we meet $k \in \mathbb{N} \setminus \{1\}$ satisfying $A_k = \emptyset$.

In the case of (1), we can generate sequences $\{D_n\}$, $\{x_n\}$, $\{A_n\}$, and $\{u_n\}$ inductively. By our generating method, from Lemma 8 (1), $\{D_n\}$ is a sequence of closed convex subsets of C satisfying $D_{n+1} \subset D_n$ for each $n \in \mathbb{N}$ and $\emptyset \neq F(T) \subset D_0 := \cap_{n\in\mathbb{N}} D_n$. For each $n \in \mathbb{N}$, let K_n be as in Lemma 6, that is,

$$K_n = \{z \in D_n : \|x_0 - z\| \leq \|x_0 - x_{n+1}\|\}.$$

Then, $u_{n+1} \in A_{n+1} \subset K_n$ for each $n \in \mathbb{N}$. By Lemma 6, $\{x_n\}$ and $\{u_n\}$ converge strongly to $P_{D_0}x_0$. Since $\{u_n\}$ converges strongly to $P_{D_0}x_0 \in D_0$, by Lemma 8 (2), we see $P_{D_0}x_0 \in F(T)$. From Lemma 4, we get $P_{D_0}x_0 = P_{F(T)}x_0$, which completes the proof of (1).

We show (2). Suppose that we generated u_{k-1}, D_k, x_k and $A_k = \emptyset$ for some $k \in \mathbb{N} \setminus \{1\}$. Then, D_k and D_{k-1} are nonempty closed and convex. Since $K_{k-1} = \{z \in D_{k-1} : \|x_0 - z\| \leq \|x_0 - x_k\|\}$, we see $x_{k-1}, x_k \in K_{k-1}$ and $K_{k-1} \neq \emptyset$. By $K_{k-1}\setminus\{u_{k-1}\} = A_k = \emptyset$, we see that $u_{k-1} \in K_{k-1}$ and K_{k-1} is a singleton. From these, $u_{k-1} = x_{k-1} = x_k$ holds. Thus, by $u_{k-1} = x_k \in D_k$, we see

$$0 \leq \langle Tu_{k-1} - u_{k-1}, Ju_{k-1} - JTu_{k-1}\rangle.$$

Since J is monotone, we also see

$$0 \geq \langle Tu_{k-1} - u_{k-1}, Ju_{k-1} - JTu_{k-1}\rangle.$$

and hence we have $\langle u_{k-1} - Tu_{k-1}, Ju_{k-1} - JTu_{k-1}\rangle = 0$. From the property of J, we see that $u_{k-1} = Tu_{k-1}$ and hence $u_{k-1} \in F(T)$. This completes the proof. □

Next, we prove a strong convergence theorem for a mapping type (R).

Theorem 13. *Let E be a reflexive and strictly convex Banach space having a Fréchet differentiable norm and the Kadec-Klee property. Let C be a nonempty subset of E such that JC is closed and convex, and let $T : C \to E$ be a mapping of type (R) such that $F(T) \neq \emptyset$. Consider an iterative procedure as below: Let $x_0 \in E$, $D_1 = A_1 = C$, $x_1 = J^{-1}P^*_{JD_1}Jx_0$, and $u_1 \in A_1$. For each $n \in \mathbb{N}$, generate D_{n+1}, x_{n+1}, A_{n+1}, and u_{n+1} by*

$$D_{n+1} = \{z \in D_n : \langle u_n - Tu_n, JTu_n - Jz\rangle \geq 0\}, \quad x_{n+1} = J^{-1}P^*_{JD_{n+1}}Jx_0,$$
$$A_{n+1} = \{y \in D_n : \|Jx_0 - Jy\| \leq \|Jx_0 - Jx_{n+1}\|, y \neq u_n\}, \quad u_{n+1} \in A_{n+1}.$$

Then, either of the following holds:

1. *$A_n \neq \emptyset$ for each $n \in \mathbb{N}$; the procedure is not stopped. In this case, $\{x_n\}$ and $\{u_n\}$ converge strongly to $J^{-1}P^*_{JF(T)}Jx_0$.*
2. *$A_k = \emptyset$ for some $k \in \mathbb{N} \setminus \{1\}$; the procedure is stopped. In this case, $u_{k-1} \in F(T)$ holds.*

Proof. In this setting, we know that E^* is reflexive, strictly convex, and has a Fréchet differentiable norm and the Kadec-Klee property.

Set $T^* := JTJ^{-1}$. From Lemma 3, we see that T^* is a mapping of type (Q) and $JF(T) = F(T^*)$. Put $u_n^* := Ju_n$, $x_n^* := Jx_n$, $D_n^* := JD_n$ and $D_n^* := JD_n$ for each $n \in \mathbb{N}$. Then, we have

$$\begin{aligned}
D_{n+1}^* &= \{z^* \in D_n^* : \langle J^{-1}u_n^* - J^{-1}T^*u_n^*, T^*u_n^* - z^* \rangle \geq 0\},\\
x_{n+1}^* &= P_{D_{n+1}^*}^* Jx_0,\\
A_{n+1}^* &= \{y^* \in D_n^* : \|Jx_0 - y^*\| \leq \|Jx_0 - x_{n+1}^*\|, y^* \neq u_n^*\},\\
u_{n+1}^* &\in A_{n+1}^*
\end{aligned}$$

for each $n \in \mathbb{N}$. Then $\{u_n^*\}$, $\{x_n^*\}$, $\{D_n^*\}$ and $\{A_n^*\}$ satisfy the conditions of Theorem 12 on E^*.

In the case of (1), if $A_n \neq \emptyset$ for each $n \in \mathbb{N}$, then $A_n^* \neq \emptyset$. From Theorem 12 (1), we obtain that $\{x_n^*\}$ and $\{u_n^*\}$ converge strongly to $P_{F(T^*)}^* Jx_0$. Since E is uniformly convex, E^* is uniformly smooth and hence the duality mapping J^{-1} on E^* is norm to norm continuous. Therefore, we see that

$$u_n = J^{-1}u_n^* \to J^{-1}P_{JF(T)}^* Jx_0 \quad \text{and} \quad x_n = J^{-1}x_n^* \to J^{-1}P_{JF(T)}^* Jx_0.$$

Next, in the case of (2), if $A_k = \emptyset$ for some $k \in \mathbb{N}\setminus\{1\}$, then $A_k^* = \emptyset$. From Theorem 12 (2), we have $Ju_{k-1} = u_{k-1}^* \in F(T^*) = JF(T)$. This implies $u_{k-1} \in F(T)$. □

5. SHRINKING PROJECTION METHOD WITH ALLOWABLE RANGES BY USING THE GENERALIZED PROJECTION

In this section, we study the shrinking projection method with allowable ranges by using the generalized projection. We first prove a strong convergence theorem for a mapping of type (Q).

Theorem 14. *Let E be a reflexive and strictly convex Banach space having a Fréchet differentiable norm and the Kadec-Klee property. Let C be a nonempty closed convex subset of E and let $T : C \to E$ be a mapping of type (Q) such that $F(T) \neq \emptyset$. Consider an iterative procedure as below: Let $x_0 \in E$, $D_1 = A_1 = C$, $x_1 = \Pi_{D_1}x_0$, and $u_1 \in A_1$. For each $n \in \mathbb{N}$, generate D_{n+1}, x_{n+1}, A_{n+1}, and u_{n+1} by*

$$\begin{aligned}
&D_{n+1} = \{z \in D_n : \langle Tu_n - z, Ju_n - JTu_n \rangle \geq 0\}, \quad x_{n+1} = \Pi_{C_{n+1}}x_0,\\
&A_{n+1} = \{y \in D_n : V(y, x_0) \leq V(x_{n+1}, x_0), y \neq u_n\}, \quad u_{n+1} \in A_{n+1}.
\end{aligned}$$

Then, either of the following holds:

1. *$A_n \neq \emptyset$ for each $n \in \mathbb{N}$; the procedure is not stopped. In this case, $\{x_n\}$ and $\{u_n\}$ converge strongly to $\Pi_{F(T)}x_0$.*

2. *$A_k = \emptyset$ for some $k \in \mathbb{N} \setminus \{1\}$; the procedure is stopped. In this case, $u_{k-1} \in F(T)$ holds.*

Proof. From $D_1 = A_1 = C$, D_1 and A_1 are nonempty. Let $u_1 \in A_1 (= C)$. Then we generate D_2. From Lemma 7, D_2 is nonempty, closed, and convex, and also generates x_2 and A_2. A_2 may be empty. In the case of $A_2 \neq \emptyset$, we can find $u_2 \in A_2$ and generate D_3. From Lemma 7, D_3 is nonempty, closed, and convex, and also generates x_3 and A_3. A_3 may be empty. In the case of $A_3 \neq \emptyset$, we can find $u_3 \in A_3$ and continue this process. So, the procedure is stopped when we meet $k \in \mathbb{N} \setminus \{1\}$ satisfying $A_k = \emptyset$.

In the case of (1), we can generate sequences $\{D_n\}$, $\{x_n\}$, $\{A_n\}$, and $\{u_n\}$ inductively. By our generating method, from Lemma 8 (1), $\{D_n\}$ is a sequence of closed convex subsets of C satisfying $D_{n+1} \subset D_n$ for each $n \in \mathbb{N}$ and $\emptyset \neq F(T) \subset D_0 := \cap_{n\in\mathbb{N}} D_n$. For each $n \in \mathbb{N}$, let L_n be as in Lemma 11 , that is,

$$L_n = \{z \in D_n : V(z, x_0) \leq V(x_{n+1}, x_0)\}.$$

Then, $u_{n+1} \in A_{n+1} \subset L_n$ for each $n \in \mathbb{N}$. By Lemma 11 , $\{x_n\}$ and $\{u_n\}$ converge strongly to $\Pi_{D_0} x_0$. Since $\{u_n\}$ converges strongly to $\Pi_{D_0} x_0 \in D_0$, by Lemma 8 (2), we see $\Pi_{D_0} x_0 \in F(T)$. From Lemma 5, we get $\Pi_{D_0} x_0 = \Pi_{F(T)} x_0$, which completes the proof of (1).

We show (2). Suppose that we generated u_{k-1}, D_k, x_k and $A_k = \emptyset$ for some $k \in \mathbb{N} \setminus \{1\}$. Then, D_k and D_{k-1} are nonempty, closed, and convex. Since $L_{k-1} = \{z \in D_{k-1} : V(z, x_0) \leq V(x_k, x_0)\}$, we see $x_{k-1}, x_k \in L_{k-1}$ and $L_{k-1} \neq \emptyset$. By $L_{k-1} \setminus \{u_{k-1}\} = A_k = \emptyset$, we see that $u_{k-1} \in L_{k-1}$ and L_{k-1} is a singleton. From these, $u_{k-1} = x_{k-1} = x_k$ holds. Thus, by $u_{k-1} = x_k \in D_k$, we see

$$0 \leq \langle Tu_{k-1} - u_{k-1}, Ju_{k-1} - JTu_{k-1} \rangle.$$

Since J is monotone, we also see

$$0 \geq \langle Tu_{k-1} - u_{k-1}, Ju_{k-1} - JTu_{k-1} \rangle.$$

and hence we have $\langle u_{k-1} - Tu_{k-1}, Ju_{k-1} - JTu_{k-1} \rangle = 0$. From the property of J, we see that $u_{k-1} = Tu_{k-1}$ and hence $u_{k-1} \in F(T)$. This completes the proof. □

Next, we prove a strong convergence theorem for a mapping of type (R).

Theorem 15. *Let E be a reflexive and strictly convex Banach space having a Fréchet differentiable norm and the Kadec-Klee property. Let C be a nonempty subset of E such that JC is closed and convex, and let $T : C \to E$ be a mapping of type (R) such that $F(T) \neq \emptyset$. Consider an iterative procedure as below: Let $x_0 \in E$, $D_1 = A_1 = C$, $x_1 = R_{D_1} x_0$, and $u_1 \in A_1$. For each $n \in \mathbb{N}$, generate D_{n+1}, x_{n+1}, A_{n+1}, and u_{n+1} by*

$$D_{n+1} = \{z \in D_n : \langle u_n - Tu_n, JTu_n - Jz \rangle \geq 0\}, \quad x_{n+1} = R_{D_{n+1}} x_0,$$
$$A_{n+1} = \{y \in D_n : V(x_0, y) \leq V(x_0, x_{n+1}), y \neq u_n\}, \quad u_{n+1} \in A_{n+1}.$$

Then, either of the following holds:

1. *$A_n \neq \emptyset$ for each $n \in \mathbb{N}$; the procedure is not stopped. In this case, $\{x_n\}$ and $\{u_n\}$ converge strongly to $R_{F(T)}x_0$.*
2. *$A_k = \emptyset$ for some $k \in \mathbb{N} \setminus \{1\}$; the procedure is stopped. In this case, $u_{k-1} \in F(T)$ holds.*

Proof. In this setting, we know that E^* is reflexive, strictly convex, and has a Fréchet differentiable norm and the Kadec-Klee property.

Set $T^* := JTJ^{-1}$. From Lemma 3, we see that T^* is a mapping of type (Q) and $JF(T) = F(T^*)$. Put $u_n^* := Ju_n$, $x_n^* := Jx_n$, $D_n^* := JD_n$ and $D_n^* := JD_n$ for each $n \in \mathbb{N}$. Then, we have

$$\begin{aligned}
D_{n+1}^* &= \{z^* \in D_n^* : \langle J^{-1}u_n^* - J^{-1}T^*u_n^*, T^*u_n^* - z^* \rangle \geq 0\},\\
x_{n+1}^* &= \Pi_{D_{n+1}^*}^* Jx_0,\\
A_{n+1}^* &= \{y^* \in D_n^* : V_*(y^*, Jx_0) \leq V_*(x_{n+1}^*, Jx_0), y^* \neq u_n^*\},\\
u_{n+1}^* &\in A_{n+1}^*
\end{aligned}$$

for each $n \in \mathbb{N}$. Then $\{u_n^*\}$, $\{x_n^*\}$, $\{D_n^*\}$ and $\{A_n^*\}$ satisfy the conditions of Theorem 14 on E^*.

In the case of (1), if $A_n \neq \emptyset$ for each $n \in \mathbb{N}$, then $A_n^* \neq \emptyset$. From Theorem 14 (1), we obtain that $\{x_n^*\}$ and $\{u_n^*\}$ converge strongly to $\Pi_{F(T^*)}^* Jx_0$. Since E is uniformly convex, E^* is uniformly smooth and hence the duality mapping J^{-1} on E^* is norm to norm continuous. Therefore, we see that

$$\begin{aligned}
u_n = J^{-1}u_n^* &\to J^{-1}\Pi_{JF(T)}^* Jx_0 = R_{F(T)}x_0,\\
x_n = J^{-1}x_n^* &\to J^{-1}\Pi_{JF(T)}^* Jx_0 = R_{F(T)}x_0.
\end{aligned}$$

Next, in the case of (2), if $A_k = \emptyset$ for some $k \in \mathbb{N}\setminus\{1\}$, then $A_k^* = \emptyset$. From Theorem 14 (2), we have $Ju_{k-1} = u_{k-1}^* \in F(T^*) = JF(T)$. This implies $u_{k-1} \in F(T)$. □

6. DEDUCED RESULTS

In the case where E is a Hilbert space, the mappings of types (Q) and (R) are firmly nonexpansive. We also know that the generalized projection and the sunny generalized nonexpansive retraction are the metric projection in the sense of a Hilbert space; see [1, 8] for more details: Let C be a nonempty closed convex subset of a Hilbert space H. It is known that for each $x \in H$ there exists a unique point $z \in C$ such that

$$\|z - x\| = \inf_{y \in C} \|y - x\|.$$

Such a point z is denoted by $P_C x$ and P_C is called the metric projection of H onto C. Therefore, as a direct consequence of Theorems 12, 13, 14 and 15, we obtain the following result.

Theorem 16. *Let H be a Hilbert space. Let C be a nonempty closed convex subset of E and let $T : C \to H$ be a firmly nonexpansive mapping such that $F(T) \neq \emptyset$. Consider an iterative procedure as below: Let $x_0 \in H$, $D_1 = A_1 = C$, $x_1 = P_{D_1} x_0$, and $u_1 \in A_1$. For each $n \in \mathbb{N}$, generate D_{n+1}, x_{n+1}, A_{n+1}, and u_{n+1} by*

$$D_{n+1} = \{z \in D_n : \langle Tu_n - z, u_n - Tu_n \rangle \geq 0\}, \quad x_{n+1} = P_{C_{n+1}} x_0,$$
$$A_{n+1} = \{y \in D_n : \|x_0 - y\| \leq \|x_0 - x_{n+1}\|, y \neq u_n\}, \quad u_{n+1} \in A_{n+1}.$$

Then, either of the following holds:

1. *$A_n \neq \emptyset$ for each $n \in \mathbb{N}$; the procedure is not stopped. In this case, $\{x_n\}$ and $\{u_n\}$ converge strongly to $P_{F(T)} x_0$.*
2. *$A_k = \emptyset$ for some $k \in \mathbb{N} \setminus \{1\}$; the procedure is stopped. In this case, $u_{k-1} \in F(T)$ holds.*

We next consider the problem of finding a zero of a maximal monotone operator. Let E be a reflexive, smooth, and strictly convex Banach space. Then an operator $A \subset E \times E^*$ with domain $D(A) = \{z \in E : Az \neq \emptyset\}$ is said to be monotone if $\langle x - y, x^* - y^* \rangle \geq 0$ for all $(x, x^*), (y, y^*) \in A$. A point $u \in E$ satisfying $0 \in Au$ is called a zero of A, and the set of such points is denoted by $A^{-1}0$. A monotone operator $A \subset E \times E^*$ is said to be maximal if $A = A'$ whenever $A' \subset E \times E^*$ is a monotone operator such that $A \subset A'$. Let $A \subset E \times E^*$ and $B \subset E^* \times E$ be maximal monotone operators. Then, it is known that for all $\lambda > 0$

$$R(J + \lambda A) = E^* \quad \text{and} \quad R(I + \lambda BJ) = E.$$

The following two single-valued mappings are well-defined for all $\lambda > 0$:

$$Q_\lambda = (J + \lambda A)^{-1} J : E \to D(A),$$
$$R_\lambda = (I + \lambda BJ)^{-1} : E \to D(BJ).$$

These mappings are called the resolvents of types (Q) and (R), respectively. It is known that $F(Q_\lambda) = A^{-1}0$ and $F(R_\lambda) = (BJ)^{-1}0$. We also know that Q_λ and R_λ are mappings of types (Q) and (R), respectively; see [2, 4, 6, 8] and others.

Therefore, as a direct consequence of Theorems 12 and 14, we obtain the following results.

Theorem 17. *Let E be a reflexive and strictly convex Banach space having a Fréchet differentiable norm and the Kadec-Klee property. Let $A \subset E \times E^*$ be a maximal monotone operator such that $A^{-1}0 \neq \emptyset$ and let $\lambda > 0$. Consider an iterative procedure as below: Let $x_0 \in E$, $D_1 = A_1 = E$, $x_1 = P_{D_1} x_0$, and $u_1 \in A_1$. For each $n \in \mathbb{N}$, generate D_{n+1}, x_{n+1}, A_{n+1}, and u_{n+1} by*

$$D_{n+1} = \{z \in D_n : \langle Q_\lambda u_n - z, Ju_n - JQ_\lambda u_n \rangle \geq 0\}, \quad x_{n+1} = P_{C_{n+1}} x_0,$$
$$A_{n+1} = \{y \in D_n : \|x_0 - y\| \leq \|x_0 - x_{n+1}\|, y \neq u_n\}, \quad u_{n+1} \in A_{n+1}.$$

Then, either of the following holds:

1. *$A_n \neq \emptyset$ for each $n \in \mathbb{N}$; the procedure is not stopped. In this case, $\{x_n\}$ and $\{u_n\}$ converge strongly to $P_{A^{-1}0}x_0$.*

2. *$A_k = \emptyset$ for some $k \in \mathbb{N} \setminus \{1\}$; the procedure is stopped. In this case, $u_{k-1} \in A^{-1}0$ holds.*

Theorem 18. *Let E be a reflexive and strictly convex Banach space having a Fréchet differentiable norm and the Kadec-Klee property. Let $A \subset E \times E^*$ be a maximal monotone operator such that $A^{-1}0 \neq \emptyset$ and let $\lambda > 0$. Consider an iterative procedure as below: Let $x_0 \in E$, $D_1 = A_1 = E$, $x_1 = \Pi_{D_1}x_0$ and $u_1 \in A_1$. For each $n \in \mathbb{N}$, generate $D_{n+1}, x_{n+1}, A_{n+1}$ and u_{n+1} by*

$$D_{n+1} = \{z \in D_n : \langle Q_\lambda u_n - z, Ju_n - JQ_\lambda u_n\rangle \geq 0\}, \quad x_{n+1} = \Pi_{C_{n+1}}x_0,$$
$$A_{n+1} = \{y \in D_n : V(y, x_0) \leq V(x_{n+1}, x_0), y \neq u_n\}, \quad u_{n+1} \in A_{n+1}.$$

Then, either of the following holds:

1. *$A_n \neq \emptyset$ for each $n \in \mathbb{N}$; the procedure is not stopped. In this case, $\{x_n\}$ and $\{u_n\}$ converge strongly to $\Pi_{A^{-1}0}x_0$.*

2. *$A_k = \emptyset$ for some $k \in \mathbb{N} \setminus \{1\}$; the procedure is stopped. In this case, $u_{k-1} \in A^{-1}0$ holds.*

Similarly, the following results are a direct consequence of Theorems 13 and 15.

Theorem 19. *Let E be a reflexive and strictly convex Banach space having a Fréchet differentiable norm and the Kadec-Klee property. Let $B \subset E^* \times E$ be a maximal monotone operator such that $B^{-1}0 \neq \emptyset$ and let $\lambda > 0$. Consider an iterative procedure as below: Let $x_0 \in E$, $D_1 = A_1 = E$, $x_1 = J^{-1}P^*_{JD_1}Jx_0$, and $u_1 \in A_1$. For each $n \in \mathbb{N}$, generate D_{n+1}, x_{n+1}, A_{n+1}, and u_{n+1} by*

$$D_{n+1} = \{z \in D_n : \langle u_n - R_\lambda u_n, JR_\lambda u_n - Jz\rangle \geq 0\}, \quad x_{n+1} = J^{-1}P^*_{JD_{n+1}}Jx_0,$$
$$A_{n+1} = \{y \in D_n : \|Jx_0 - Jy\| \leq \|Jx_0 - Jx_{n+1}\|, y \neq u_n\}, \quad u_{n+1} \in A_{n+1}.$$

Then, either of the following holds:

1. *$A_n \neq \emptyset$ for each $n \in \mathbb{N}$; the procedure is not stopped. In this case, $\{x_n\}$ and $\{u_n\}$ converge strongly to $J^{-1}P^*_{B^{-1}0}Jx_0$.*

2. *$A_k = \emptyset$ for some $k \in \mathbb{N} \setminus \{1\}$; the procedure is stopped. In this case, $u_{k-1} \in (BJ)^{-1}0$ holds.*

Theorem 20. *Let E be a reflexive and strictly convex Banach space having a Fréchet differentiable norm and the Kadec-Klee property. Let $B \subset E^* \times E$ be a maximal monotone operator such that $B^{-1}0 \neq \emptyset$ and let $\lambda > 0$. Consider an*

iterative procedure as below: Let $x_0 \in E$, $D_1 = A_1 = E$, $x_1 = R_{D_1}x_0$, *and* $u_1 \in A_1$. *For each* $n \in \mathbb{N}$, *generate* D_{n+1}, x_{n+1}, A_{n+1}, *and* u_{n+1} *by*

$$D_{n+1} = \{z \in D_n : \langle u_n - R_\lambda u_n, JR_\lambda u_n - Jz\rangle \geq 0\}, \quad x_{n+1} = R_{D_{n+1}}x_0,$$
$$A_{n+1} = \{y \in D_n : V(x_0, y) \leq V(x_0, x_{n+1}), y \neq u_n\}, \quad u_{n+1} \in A_{n+1}.$$

Then, either of the following holds:

1. $A_n \neq \emptyset$ *for each* $n \in \mathbb{N}$; *the procedure is not stopped. In this case,* $\{x_n\}$ *and* $\{u_n\}$ *converge strongly to* $R_{(BJ)^{-1}0}x_0$.
2. $A_k = \emptyset$ *for some* $k \in \mathbb{N} \setminus \{1\}$; *the procedure is stopped. In this case,* $u_{k-1} \in (BJ)^{-1}0$ *holds.*

7. ADDITIONAL EXPLANATIONS

In this section, we provide a remark to support the main issue. We first consider whether at some step the procedure will be continued or stopped. In Theorem 12, we consider the case where u_{k-1}, D_k, x_k, x_{k-1}, and A_k are generated for some $k \in \mathbb{N} \setminus \{1\}$. Let $\{K_n\}$ be as in Lemma 6, that is,

$$K_n = \{y \in D_n : \|x_0 - y\| \leq \|x_0 - x_{n+1}\|\}$$

for each $n \in \mathbb{N}$. Note that $A_n = K_{n-1} \setminus \{u_{n-1}\}$ for each $n \in \mathbb{N} \setminus \{1\}$. Since $x_{k-1} \in K_{k-1}$, we have $K_{k-1} \neq \emptyset$ and hence we can find a candidate w_k for u_k in K_{k-1}. If this w_k can be chosen as an element different from u_{k-1}, then this implies $A_k = K_{k-1} \setminus \{u_{k-1}\} \neq \emptyset$. That is, the procedure is continued as $u_k = w_k$. If this w_k can only be chosen as u_{k-1}, then this implies $K_{k-1} = \{u_{k-1}\}$ and hence we see $A_k = \emptyset$. That is, the procedure is stopped.

Next, we consider the setting where A_{n+1} is replaced by K_n for each $n \in \mathbb{N}$ in Theorem 12. In this setting, there are no instances where the procedure is stopped. In fact, suppose that $A_k = \emptyset$ for some $k \in \mathbb{N} \setminus \{1\}$. In the proof of Theorem 12, we see that $u_{k-1} = x_{k-1} = x_k \in F(T)$ and

$$K_{k-1} = \{x_{k-1}\}.$$

We can find $u_k = x_{k-1} \in K_{k-1}$. Therefore, the procedure is continued. Put $v_0 := x_{k-1}$. Then we generate D_{k+1}. Since $u_k = v_0 \in F(T)$, we have

$$\langle Tu_k - z, Ju_k - JTu_k\rangle = \langle v_0 - z, Jv_0 - Jv_0\rangle = \langle v_0 - z, 0\rangle = 0$$

for each $z \in D_k$. Therefore, we have $D_{k+1} = D_k$ and hence $x_{k+1} = x_k$. We also generate K_k. It is easy to see that $v_0 = x_{k-1} = x_k \in K_k$ and hence $K_k \neq \emptyset$. Since $x_{k+1} = x_k$ and the uniqueness of $x_k = P_{D_k}x_0$, we have

$$\begin{aligned} K_k &= \{y \in D_k : \|x_0 - y\| \leq \|x_0 - x_{k+1}\|\} \\ &= \{y \in D_k : \|x_0 - y\| \leq \|x_0 - x_k\|\} = \{x_k\} = \{v_0\}. \end{aligned}$$

Therefore, we can find $u_{k+1} = v_0 \in K_k$. If we repeat this argument, we obtain that

$$D_n = D_k,\ K_{n-1} = \{v_0\}, \text{ and } x_n = u_n = v_0$$

for $n \geq k$. From the above argument, if A_{n+1} is replaced by K_n for each $n \in \mathbb{N}$ in Theorem 12, we obtain the following result.

Theorem 21. *Let E be a reflexive and strictly convex Banach space having a Fréchet differentiable norm and the Kadec-Klee property. Let C be a nonempty closed convex subset of E and let $T : C \to E$ be a mapping of type (Q) such that $F(T) \neq \emptyset$. Consider an iterative procedure as below: Let $x_0 \in E$, $D_1 = C$, $x_1 = P_{D_1}x_0$, and $u_1 \in C$. For each $n \in \mathbb{N}$, generate D_{n+1}, x_{n+1}, K_n, and u_{n+1} by*

$$\begin{aligned} &D_{n+1} = \{z \in D_n : \langle Tu_n - z, Ju_n - JTu_n\rangle \geq 0\}, \quad x_{n+1} = P_{D_{n+1}}x_0, \\ &K_n = \{y \in D_n : \|x_0 - y\| \leq \|x_0 - x_{n+1}\|\}, \quad u_{n+1} \in K_n. \end{aligned}$$

Then, $\{x_n\}$ and $\{u_n\}$ converge strongly to $P_{F(T)}x_0$.

From the argument presented above, Theorem 12 explicitly states whether a fixed point is reached in a finite number of steps. However, as in Theorem 21, it can be modified to disregard the stopping condition at finite steps. We adopted the status of Theorem 12 to show the details of the behavior of $\{u_n\}$. The same arguments above can be made for Theorems 13, 14, and 15.

Acknowledgments. This work was supported by JSPS KAKENHI Grant Numbers 23K25512, 24K06807.

REFERENCES

1. Ya. I. Alber: *Metric and generalized projection operators in Banach spaces: properties and applications.* Theory and Applications of Nonlinear Operators of Accretive and Monotone Type, Dekker, New York, 1996, 15–50.
2. K. Aoyama, F. Kohsaka and W. Takahashi: *Three generalizations of firmly nonexpansive mappings: Their relations and continuity properties.* J. Nonlinear Convex Anal., **10** (2009), 131–147.
3. T. Honda, T. Ibaraki and W. Takahashi: *Duality theorems and convergence theorems for nonlinear mappings in Banach spaces and applications.* Int. J. Math. Stat., **6** (2010), 46–64.
4. T. Ibaraki: *Approximation of zero point of monotone operators with nonsummable errors.* Fixed Point Theory Appl., **2016**:48 (2016), 14 pages.
5. T. Ibaraki: *Iterative approximation with errors of zero points of maximal monotone operators in a Hilbert space.* Linear Nonlinear Anal., **3** (2017), 171–178.
6. T. Ibaraki: *Iterative approximation with errors of zero points of monotone operators in a Banach space.* Yokohama Math. J., **63** (2017), 91–109.
7. T. Ibaraki and Y. Kimura: *Approximation of a fixed point of generalized firmly nonexpansive mappings with nonsummable errors.* Linear Nonlinear Anal., **2** (2016), 301-310.
8. T. Ibaraki and W. Takahashi: *A new projection and convergence theorems for the projections in Banach spaces.* J. Approx. Theory, **149** (2007), 1–14.
9. T. Ibaraki and S. Saejung: *On shrinking projection method for cutter type mappings with nonsummable errors.* J. Inequal. Appl., **2023**:92 (2023), 20 pages.
10. T. Ibaraki and W. Takahashi: *Fixed point theorems for nonlinear mappings of nonexpansive type in Banach spaces.* J. Nonlinear Convex Anal., **10** (2009), 21–32.
11. S. Kamimura and W. Takahashi: *Strong convergence of a proximal-type algorithm in a Banach space.* SIAM J. Optim., **13** (2002), 938–945.
12. Y. Kimura: *Approximation of a fixed point of nonexpansive mapping with nonsummable errors in a geodesic space.* Proceedings of the 10th International Conference on Fixed Point Theory and Its Applications, 2012, 157–164.
13. Y. Kimura: *Approximation of a fixed point of nonlinear mappings with nonsummable errors in a Banach space.* Proceedings of the International Symposium on Banach and Function Spaces IV (Kitakyushu, Japan), (L. Maligranda, M. Kato, and T. Suzuki eds.), 2014, 303–311.

14. Y. Kimura: *Approximation of a common fixed point of a finite family of nonexpansive mappings with nonsummable errors in a Hilbert space.* J. Nonlinear Convex Anal., **15** (2014), 429–436.

15. Y. Kimura and W. Takahashi: *On a hybrid method for a family of relatively nonexpansive mappings in a Banach space.* J. Math. Anal. Appl., **357** (2009), 356–363.

16. F. Kohsaka and W. Takahashi: *Generalized nonexpansive retractions and a proximal-type algorithm in Banach spaces.* J. Nonlinear Convex Anal., **8** (2007), 197–209.

17. F. Kohsaka and W. Takahashi: *Existence and approximation of fixed points of firmly nonexpansive type mappings in Banach spaces.* SIAM J. Optim., **19** (2008), 824–835.

18. C. Martinez-Yanesa, H. K. Xu: *Strong convergence of the CQ method for fixed point iteration processes.* Nonlinear Anal., **64** (2006), 2400–2411.

19. W. Takahashi: *Nonlinear Functional Analysis - Fixed Point Theory and Its Applications.* Yokohama Publishers, 2000.

20. W. Takahashi, Y Takeuchi and R. Kubota: *Strong convergence theorems by hybrid methods for families of nonexpansive mappings in Hilbert spaces.* J. Math. Anal. Appl., **341** (2008), 276–286.

21. W. Takahashi and J.-C. Yao: *Nonlinear operators of monotone type and convergence theorems with equilibrium problems in Banach spaces.* Taiwanese J. Math., **15** (2011), 787–818.

22. Y. Takeuchi: *Shrinking projection method with allowable ranges.* J. Nonlinear Anal. Optim., **10** (2019), 83–94.

Takanori Ibaraki
Department of Mathematics Education,
Yokohama National University,
Yokohama, Japan
E-mail: *ibaraki@ynu.ac.jp*
https://orcid.org/0000-0002-7067-9763

(Received 10.01.2025.)
(Revised 07.12.2025.)