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COMMON FIXED POINTS FOR ENRICHED g-CONTRACTION MAPPINGS

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The aim of this paper is to introduce the notion of enriched g-contraction and enriched g-nonexpansive mappings of the second kind and study the existence and uniqueness of common fixed points of such mappings under weak commutativity conditions. We apply our results to establish the existence of common fixed points in the set of best approximations and obtain the existence and uniqueness of the common solution of the Volterra integral equations.

1. INTRODUCTION AND PRELIMINARIES

Fixed point theory has made significant contributions to the development of nonlinear analysis. This is because many nonlinear problems can be reformulated as solving operator equations, which can be further reduced to the problem of finding the fixed points of a single mapping or common fixed points of two or more mappings. A primary objective in this area is to find the minimal set of conditions on the mapping and on the space that guarantees the existence of fixed points for various classes of contractive mappings such as enriched contractive mappings [**7**]. In addition, fixed point theory provides constructive methods for approximating fixed points of such mappings and, occasionally, for studying the error estimates or convergence rates of the iterative methods used. Moreover, fixed point techniques

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are also widely applied in solving problems in best approximations, image deblurring, graph theory, nonlinear integral and differential equations, etc. For more details on the topic, we refer the reader to ([**1, 3, 24, 13, 9, 19, 26, 28, 29, 30**]).

In this paper, we address a limitation in Marchiş' definition of (b,θ)-enriched Jungck-contraction mappings [**15**], we refer to here as (b,θ)-enriched g-contraction mappings of the first kind. Specifically, the definition could not be applied in studying the convergence of sequences generated by the Jungck–Krasnoselskij iteration to the unique common fixed point of f and g. To overcome this limitation, we introduce a new class of (b,θ)-enriched g-contraction mappings of the second kind. Using this definition, we establish the convergence of the Jungck–Krasnoselskij iteration to the unique common fixed point of f and g under weak commutativity conditions.

Before proceeding further, we recall some essential definitions and symbols relevant to this study. We use $\mathbb{N} := \{1, 2, 3, \dots\}$ and $\mathbb{W} := \{0, 1, 2, \dots\}$ to denote the set of natural numbers and non-negative integers, respectively.

Definition 1. *[**1**] Let f and g be self-mappings of a set $\mathcal{X}$:*

(i) If $z = fu = gu$ for some $u \in \mathcal{X}$, then u is called a coincidence point of f and g and z is called a point of coincidence of f and g. The set of coincidence points of f and g is denoted by $C(f,g)$;

(ii) If $u = fu = gu$, then $u \in \mathcal{X}$ is a common fixed point of f and g. The set of common fixed points of f and g is denoted by $Fix(f,g)$.

Remark 1. *[**17**] The following relationships hold:*

(i) $Fix(f,g) \subseteq C(f,g)$;

(ii) $Fix(f) \cap C(f,g) = Fix(g) \cap C(f,g) = Fix(f,g)$.

In 2013, Lin and Takahashi [**14**] studied the concept of attractive points of nonlinear mappings in the Banach space setting.

Definition 2. *Let $\mathcal{M}$ be a nonempty subset of a normed space $\mathcal{X}$ and $f : \mathcal{M} \to \mathcal{X}$. The set*

$$A(f) = \{z \in \mathcal{X} : \|fu - z\| \leq \|u - z\|, \, \forall u \in \mathcal{M}\},$$

is called the set of attracitve points of f.

In 2018, Khan [**12**] gave the concept of common attractive points for two mappings f and g.

Definition 3. *Let $\mathcal{M}$ be a nonempty subset of a normed space $\mathcal{X}$ and $f, g : \mathcal{M} \to \mathcal{X}$. The set*

$$CAP(f,g) = \{z \in \mathcal{X} : \max\{\|fu - z\|, \|gu - z\|\} \leq \|u - z\|, \, \forall u \in \mathcal{M}\},$$

is called the set of common attractive points of f and g.

Note that $CAP(f,g) = A(f)$ when $f = g$.

Definition 4. *[1] Let $\mathcal{M}$ be a nonempty subset of a normed space $\mathcal{X}$ and $f : \mathcal{M} \to \mathcal{M}$. Then*

(i) the set $\mathcal{M}$ is called q-starshaped if there exists $q \in \mathcal{M}$ such that $ku+(1-k)q \in \mathcal{M}$ for all $u \in \mathcal{M}$ and all $k \in [0,1]$;

(ii) the set $\mathcal{M}$ is called convex if $ku + (1-k)v \in \mathcal{M}$ for all $u, v \in \mathcal{M}$ and all $k \in [0,1]$;

(iii) the set $\mathscr{B}_{\mathcal{M}}(p) = \{u \in \mathcal{M} : \|u - p\| = \operatorname{dist}(p, \mathcal{M})\}$ is called the set of best $\mathcal{M}$-approximants to $p \in \mathcal{X}$, where $\operatorname{dist}(p, \mathcal{M}) = \inf_{z \in \mathcal{M}} \|z - p\|$. If $\mathcal{M}$ is closed and convex, then $\mathscr{B}_{\mathcal{M}}(p)$ is also closed and convex;

(iv) the mapping f is called affine if $\mathcal{M}$ is convex and

$$f(ku + (1-k)v) = kfu + (1-k)fv,$$

for all $u, v \in \mathcal{M}$ and all $k \in [0,1]$.

In 1968, Goebel [**10**] established the following coincidence result.

Theorem 1. *Let $\mathcal{M}$ be a subset of a metric space $(\mathcal{X}, d)$ and $f, g : \mathcal{M} \to \mathcal{M}$. If $f(\mathcal{M}) \subseteq g(\mathcal{M})$, $g(\mathcal{M})$ is a complete subspace of $\mathcal{X}$ and f is a g-contraction, i.e., there exists $\theta \in [0,1)$ such that*

$$d(fu, fv) \leq \theta d(gu, gv), \forall u, v \in \mathcal{M},$$

then f and g have a coincidence point.

In 1976, Jungck [**11**] obtained the following common fixed point result for commuting mappings as an extension of the Banach fixed point theorem.

Theorem 2. *Let $(\mathcal{X}, d)$ be a complete metric space, and let f and g be commuting self-mappings of $\mathcal{X}$. If $f(\mathcal{X}) \subseteq g(\mathcal{X})$, f is a g-contraction, and g is continuous, then f and g have a unique common fixed point.*

Jungck defined the following iteration scheme

$$gu_{n+1} = fu_n, \forall n \in \mathbb{W}.$$

If $g = I$, where I is the identity mapping, the above iteration reduces to the Picard iteration, which has inspired some researchers, including the development of the following Jungck–Krasnoselskij iteration (see [**8, 25**]).

Definition 5. *Let $\mathcal{X}$ be a normed space and $f, g : \mathcal{X} \to \mathcal{X}$ such that $f(\mathcal{X}) \subseteq g(\mathcal{X})$ and $u_0 \in \mathcal{X}$. The sequence $\{gu_n\}$, given by*

$$gu_{n+1} = (1-\lambda)gu_n + \lambda fu_n, \forall n \in \mathbb{W},$$

for some $0 < \lambda < 1$, is called the Jungck–Krasnoselskij iteration.

After that, Pant [**18**] introduced the notion of R-weakly commuting mappings.

Definition 6. *The self-mappings f and g of a normed space $\mathcal{X}$ are R-weakly commuting if there exists $R > 0$ such that $\|fgu - gfu\| \leq R\|fu - gu\|$, $\forall u \in \mathcal{X}$.*

Note that if $R = 1$, then f and g are weakly commuting mappings. Moreover, weakly commuting mappings are R-weakly commuting mappings, and the converse holds only when $R \leq 1$ (see [**1, 21**]).

Shahzad [**23**] proved the following result on R-weakly commuting mappings.

Theorem 3. *Let $\mathcal{M}$ be a closed subset of a metric space $\mathcal{X}$ and $f, g : \mathcal{M} \to \mathcal{M}$ such that $f(\mathcal{M}) \subseteq g(\mathcal{M})$. If f is g-contraction and continuous, $\overline{f(\mathcal{M})}$ is complete, and f and g are R-weakly commuting mappings, then $Fix(f, g)$ is a singleton.*

Subsequently, Al-Thagafi and Shahzad [**2**] introduced the notion of occasionally weakly compatible mappings.

Definition 7. *The self-mappings f and g of a subset $\mathcal{M}$ of a metric space $\mathcal{X}$ are occasionally weakly compatible if $gfu = fgu$ for some $u \in C(f, g)$.*

Observe that the classes of commuting, weakly commuting and R-weakly commuting self-mappings are subclasses of the class of occasionally weakly compatible self-mappings.

Also, Al-Thagafi and Shahzad [**2**] defined the notion of ultraoccasionally weakly compatible mappings.

Definition 8. *Let $\mathcal{M}$ be a q-starshaped subset of a normed space $\mathcal{X}$, $f, g : \mathcal{M} \to \mathcal{M}$ and $f_k : \mathcal{M} \to \mathcal{M}$ where $f_k u := kfu + (1-k)q$ for every $u \in \mathcal{M}$ and $k \in [0, 1]$. Then f and g are called ultraoccasionally weakly compatible if f_k and g are occasionally weakly compatible for every $k \in [0, 1]$.*

Notice that the class of ultraoccasionally weakly compatible self-mappings is a proper subset of the class of occasionally weakly compatible self-mappings.

Definition 9. *[**6**] Let f be a self-mapping of a convex subset $\mathcal{M}$ of a vector space $\mathcal{X}$, then, for any $\lambda \in (0, 1]$, the averaged mapping f_λ is defined as*

$$f_\lambda u = (1 - \lambda)u + \lambda fu, \ \forall u \in \mathcal{M}.$$

Note that $Fix(f_\lambda) = Fix(f)$.

Recently, Berinde and Păcurar [**6**] introduced the class of (b, θ)-enriched contraction mappings, which contains contraction mappings as a special case.

Definition 10. *Let $\mathcal{X}$ be a normed space. A mapping $f : \mathcal{X} \to \mathcal{X}$ is a (b, θ)-enriched contraction if there exist $b \in [0, \infty)$ and $\theta \in [0, b + 1)$ such that*

$$\|b(u - v) + fu - fv\| \leq \theta\|u - v\|, \ \forall u, v \in \mathcal{X}.$$

In 2021, Marchiş [**15**] introduced the class of (b,θ)-enriched Jungck-contraction mappings, which we call here the class of (b,θ)-enriched g-contraction mappings of the first kind.

Definition 11. *Let $\mathcal{X}$ be a normed space and $f,g:\mathcal{X}\to\mathcal{X}$. A mapping f is a (b,θ)-enriched g-contraction of the first kind if there exist $b\in[0,\infty)$ and $\theta\in[0,b+1)$ such that*

$$\|b(u-v)+fu-fv\|\leq\theta\|gu-gv\|,\ \forall u,v\in\mathcal{X}. \tag{1}$$

Remark 2. *If $b=0$ in (1), then f is a $(0,\theta)$-enriched g-contraction mapping (or f is a g-contraction mapping).*

Theorem 4. *[**15**] Let $\mathcal{X}$ be a Banach space and $f,g:\mathcal{X}\to\mathcal{X}$ for which f is a (b,θ) enriched g-contraction of the first kind. If f_λ and g are continuous and weakly commuting mappings and $f_\lambda(\mathcal{X})\subseteq g(\mathcal{X})$, then*

(i) $Fix(f,g)=\{a\}$, for some $a\in\mathcal{X}$;

(ii) there exists $\lambda\in(0,1]$ such that the iteration $\{gu_n\}_{n=0}^{\infty}$, given by

$$gu_{n+1}=(1-\lambda)\,u_n+\lambda fu_n,\ \forall n\in\mathbb{W},$$

converges strongly to a.

2. ENRICHED g-CONTRACTION MAPPINGS OF THE SECOND KIND

We introduce the following notion of (b,θ)-enriched g-contraction mappings of the second kind.

Definition 12. *Let $\mathcal{M}$ be a subset of a normed space $\mathcal{X}$ and $f,g:\mathcal{M}\to\mathcal{M}$. A mapping f is called a (b,θ)-enriched g-contraction of the second kind if there exist $b\in[0,\infty)$ and $\theta\in[0,b+1)$ such that*

$$\|b(gu-gv)+fu-fv\|\leq\theta\,\|gu-gv\|,\ \forall u,v\in\mathcal{M}. \tag{2}$$

Example 1. *Consider $\mathcal{X}=[0,\frac{1}{2}]$ equipped with the usual norm, and let $f,g:\mathcal{X}\to\mathcal{X}$ be defined by*

$$fu=\tfrac{1}{2}-u^2\ \text{and}\ gu=u^2,\ \forall u\in\mathcal{X}.$$

Then f is a (b,θ)-enriched g-contraction mapping of the second kind. To verify this, consider

$$\|b(gu-gv)+fu-fv\|\leq\theta\|gu-gv\|,\ \forall u,v\in\mathcal{X},$$

which implies

$$|b-1||u^2-v^2| \leq \theta|u^2-v^2|, \forall u, v \in \mathcal{X}.$$

This, for any $u \neq v$, leads to $|b-1| \leq \theta$. Hence, for any $b > 1$ and $\theta \geq b-1$, the inequality holds.

We generalize the definition of the averaged mappings as follows:

Definition 13. *Let f and g be self-mappings of a convex subset $\mathcal{M}$ of a vector space $\mathcal{X}$ and then we define, for any $\lambda \in (0,1]$, the averaged g-mapping (or averaged mapping with respect to g) as*

$$f_{g,\lambda}u = (1-\lambda)gu + \lambda fu, \forall u, v \in \mathcal{M}.$$

Note that $Fix(f_{g,\lambda}, g) = Fix(f, g)$.

Remark 3. *Let us denote $\lambda = \frac{1}{b+1}$, where $\lambda \in (0,1)$ in inequality (2). Then we have*

$$\left\|(\frac{1}{\lambda}-1)gu + fu - (\frac{1}{\lambda}-1)gv - fv\right\| \leq \theta \left\|gu - gv\right\|, \forall u, v \in \mathcal{M},$$

which can be written as

$$\left\|f_{g,\lambda}u - f_{g,\lambda}v\right\| \leq c \left\|gu - gv\right\|, \forall u, v \in \mathcal{M},$$

where $c = \lambda\theta \in [0,1)$.

Now, we prove our first main result.

Theorem 5. *Let $\mathcal{X}$ be a normed space, and let $f, g : \mathcal{X} \to \mathcal{X}$ for which f is a (b,θ)-enriched g-contraction mapping of the second kind. If $f(\mathcal{X}) \subseteq g(\mathcal{X})$, $g(\mathcal{X})$ is complete, and $g(\mathcal{X})$ is convex (or, in particular, g is affine), then f and g have a unique coincidence point in $\mathcal{X}$. Moreover, if f and g are occasionally weakly compatible, then*

(i) $Fix(f, g) = \{a\}$, for some $a \in \mathcal{X}$;

(ii) there exists $\lambda \in (0,1]$ such that the Jungck–Krasnoselskij iteration $\{gu_{n+1}\}$, given by

$$gu_{n+1} = (1-\lambda)\, gu_n + \lambda f u_n, \forall n \in \mathbb{W}, \tag{3}$$

converges strongly to the unique common fixed point a of f and g, for any $u_0 \in \mathcal{X}$;

(iii) the following error estimate holds

$$\left\|gu_{n+i-1} - a\right\| \leq \frac{c^i}{1-c} \left\|gu_n - gu_{n-1}\right\|, \forall n, i \in \mathbb{N}, \tag{4}$$

where $c = \lambda\theta$.

Proof. Let $u, v \in \mathcal{X}$. We consider two cases.

Case 1. Assume that $b > 0$. In this case, given that f is a (b, θ)-enriched g-contraction mapping of the second kind, it follows that there exists a constant $b \in (0, \infty)$ such that inequality (2) is satisfied for all $u, v \in \mathcal{X}$. Fix $\lambda = \frac{1}{b+1} \in (0, 1)$. Thus, inequality (2) is equivalent to

$$\|f_{g,\lambda}u - f_{g,\lambda}v\| \leq c\,\|gu - gv\|\,, \forall u, v \in \mathcal{X}, \tag{5}$$

where $c = \lambda\theta$ and $f_{g,\lambda}u = (1-\lambda)gu + \lambda fu$, $\forall u \in \mathcal{X}$. Since $\theta \in (0, b+1)$, it follows that $c \in (0, 1)$ and so $f_{g,\lambda}$ is a g-contraction mapping.

Let u_0 be an arbitrary point in $\mathcal{X}$. Since $f(\mathcal{X}) \subseteq g(\mathcal{X})$ and $g(\mathcal{X})$ is convex (or, in particular, g is affine), it follows that $(1-\lambda)gu_0 + \lambda fu_0 \in g(\mathcal{X})$. Consequently, there exists $u_1 \in \mathcal{X}$ such that $gu_1 = (1-\lambda)gu_0 + \lambda fu_0$. Continuing in the same way, we can choose a sequence $\{u_n\}$ in $\mathcal{X}$ such that

$$gu_n = f_{g,\lambda}u_{n-1}, \forall n \in \mathbb{N}.$$

Letting $u = u_n$ and $v = u_{n-1}$ in (5), we obtain

$$\|gu_{n+1} - gu_n\| = \|f_{g,\lambda}u_n - f_{g,\lambda}u_{n-1}\| \leq c\,\|gu_n - gu_{n-1}\|\,, \forall n \in \mathbb{N}. \tag{6}$$

By induction, we obtain

$$\|gu_{n+m} - gu_{n+m-1}\| \leq c^m\,\|gu_n - gu_{n-1}\|\,, \forall n, m \in \mathbb{N}.$$

For $m > i$, we get

$$\|gu_{n+m} - gu_{n+i-1}\| \leq \frac{c^i(1-c^{m-i+1})}{1-c}\,\|gu_n - gu_{n-1}\|\,, \forall n, i \in \mathbb{N}, \tag{7}$$

and so, for $i = 1$, we get

$$\|gu_{n+m} - gu_n\| \leq \frac{c(1-c^m)}{1-c}\,\|gu_n - gu_{n-1}\|\,, \forall n \in \mathbb{N}.$$

Thus,

$$\begin{aligned}\|gu_{n+m} - gu_n\| &\leq \frac{c(1-c^m)}{1-c}\,\|gu_n - gu_{n-1}\| \\ &\leq \ldots \leq \frac{c^n(1-c^m)}{1-c}\,\|gu_1 - gu_0\|\,, \forall n \in \mathbb{N},\end{aligned}$$

and since $0 \leq c < 1$, we get

$$\|gu_{n+m} - gu_n\| \leq \frac{c^n}{1-c}\|gu_1 - gu_0\|, \forall n \in \mathbb{N}.$$

Hence, $\|gu_{n+m} - gu_n\| \to 0$ as $n \to \infty$, which shows that $\{gu_n\}$ is a Cauchy sequence. Since $g(\mathcal{X})$ is complete, there exists an $a \in g(\mathcal{X})$ such that $gu_n \to a$ as

$n \to \infty$. Thus, for some $z \in \mathcal{X}$, we have $a = gz$. Also, we have $f_{g,\lambda}u_n \to a$ as $n \to \infty$. Notice that

$$\begin{aligned}\|a - f_{g,\lambda}z\| &\leq \|a - f_{g,\lambda}u_n\| + \|f_{g,\lambda}u_n - f_{g,\lambda}z\| \\ &\leq \|a - f_{g\lambda}u_n\| + c\|gu_n - gz\|, \ \forall n \in \mathbb{N}.\end{aligned}$$

This implies that $\|a - f_{g,\lambda}z\| \to 0$ as $n \to \infty$. Thus, $a = f_{g,\lambda}z = gz$, i.e., a is a point of coincidence of $f_{g,\lambda}$ and g and hence, a is a point of coincidence of f and g.

To show that a is the unique point of coincidence of f and g, assume that there exists $w \in \mathcal{X}$ with $z \neq w$ such that $z \in C(f,g) = C(f_{g,\lambda}, g)$ and $w \in C(f,g) = C(f_{g,\lambda}, g)$, then, by (5), we get

$$\|gz - gw\| = \|f_{g,\lambda}z - f_{g,\lambda}w\| \leq c\|gz - gw\|,$$

which leads to a contradiction. Thus, $gz = gw = a$ which shows that $C(f,g)$ is nonempty and a is unique.

Since f and g are occasionally weakly compatible mappings, $C(f,g)$ is nonempty, and a is unique, there exists $v \in C(f,g)$, i.e., $fv = gv = f_{g,\lambda}v = a$, such that $ga = gfv = fgv = fa$. Thus,

$$\begin{aligned}\|a - f_{g,\lambda}a\| &= \|f_{g,\lambda}v - f_{g,\lambda}a\| \\ &\leq c\|gv - ga\| \\ &= c\|a - f_{g,\lambda}a\|,\end{aligned}$$

which leads to a contradiction. Hence, $Fix(f,g) = \{a\}$.

Case 2. Assume that $b = 0$. In this case, $\lambda = 1, c = \theta$, and the proof is similar to Case 1 but instead of $f_{g,\lambda}$, we have $f = f_{g,1}$ and the iteration reduces to the Jungck iteration $gu_n = fu_{n-1}$, $\forall n \in \mathbb{N}$.

By letting $m \to \infty$ in (7), the error estimate (4) holds. □

Remark 4. *Notice that if f and g are occasionally weakly compatible mappings on a normed space $\mathcal{X}$ or on a convex set $\mathcal{M}$ and g is affine, then $f_{g,\lambda}$ and g are occasionally weakly compatible mappings.*

Corollary 1. *[**6**] Let $\mathcal{X}$ be a Banach space and $f : \mathcal{X} \to \mathcal{X}$. If f is a (b,θ)-enriched contraction mapping, then*

(i) $Fix(f) = \{a\}$, for some $a \in \mathcal{X}$;

(ii) there exists $\lambda \in (0,1]$ such that the iterative method $\{u_{n+1}\}$, given by

$$u_{n+1} = (1-\lambda)u_n + \lambda f u_n, \ \forall n \in \mathbb{W},$$

converges strongly to the unique fixed point a, for any $u_0 \in \mathcal{X}$;

(iii) the following error estimate holds

$$\|u_{n+i-1} - a\| \leq \frac{c^i}{1-c}\|u_{n-1} - u_n\|, \ \forall n, i \in \mathbb{N},$$

where $c = \lambda\theta$.

Theorem 6. *Let $\mathcal{M}$ be a convex subset of a normed space $\mathcal{X}$ and $f, g : \mathcal{M} \to \mathcal{M}$. Assume that f is a (b, θ)-enriched g-contraction mapping of the second kind and one of the following conditions holds:*

1. *There exists $\lambda = \frac{1}{b+1} \in (0, 1]$ such that $\overline{f_{g,\lambda}(\mathcal{M})} \subseteq g(\mathcal{M})$ and $\overline{f_{g,\lambda}(\mathcal{M})}$ is complete;*

2. *$f(\mathcal{M}) \subseteq g(\mathcal{M})$, $g(\mathcal{M})$ is complete and $g(\mathcal{M})$ is convex (or, in particular, g is affine),*

then f and g have a unique coincidence point in $\mathcal{M}$. Moreover, if f and g are occasionally weakly compatible, then

(i) $Fix(f, g) = \{a\}$, for some $a \in \mathcal{X}$;

(ii) there exists $\lambda \in (0, 1]$ such that the Jungck–Krasnoselskij iteration $\{gu_{n+1}\}$, given by (3), converges strongly to the unique common fixed point a of f and g, for any $u_0 \in \mathcal{M}$;

(iii) the error estimate (4) holds, where $c = \lambda\theta$.

Proof. Let $u, v \in \mathcal{M}$. We consider two cases.

Case 1. Assume that $b > 0$. In this case, given that f is a (b, θ)-enriched g-contraction mapping of the second kind, it follows that there exists a constant $b \in (0, \infty)$ such that inequality (2) is satisfied for all $u, v \in \mathcal{M}$. Fix $\lambda = \frac{1}{b+1} \in (0, 1)$. Thus, inequality (2) is equivalent to

$$\|f_{g,\lambda}u - f_{g,\lambda}v\| \leq c\|gu - gv\|, \forall u, v \in \mathcal{M},$$

where $c = \lambda\theta$ and $f_{g,\lambda}u = (1 - \lambda)gu + \lambda fu$, $\forall u \in \mathcal{M}$. Since $\theta \in (0, b + 1)$, it follows that $c \in (0, 1)$ and so $f_{g,\lambda}$ is a g-contraction mapping.

1. Let u_0 be an arbitrary point in $\mathcal{M}$. As $f_{g,\lambda}(\mathcal{M}) \subseteq \overline{f_{g,\lambda}(\mathcal{M})} \subseteq g(\mathcal{M})$, we can choose $u_1 \in \mathcal{M}$ such that $gu_1 = f_{g,\lambda}u_0$. Inductively, choose a sequence $\{u_n\}$ in $\mathcal{M}$ such that

$$gu_n = f_{g,\lambda}u_{n-1}, \forall n \in \mathbb{N}.$$

Following the same process as in Theorem 5, we deduce that

$$\|gu_{n+m} - gu_n\| \to 0,$$

as $n \to \infty$, which shows that $\{gu_n\}$ is a Cauchy sequence. Consequently, $\{f_{g,\lambda}u_n\}$ is a Cauchy sequence. Since $\overline{f_{g,\lambda}(\mathcal{M})}$ is complete, then, as $n \to \infty$,

$$f_{g,\lambda}u_n \to a \in \overline{f_{g,\lambda}(\mathcal{M})} \subseteq g(\mathcal{M}) \subseteq \mathcal{M}.$$

Hence, $gu_n \to a$ as $n \to \infty$, where $a = gz$ for some $z \in \mathcal{M}$. Since

$$\begin{aligned} \|a - f_{g,\lambda}z\| &\leq \|a - f_{g,\lambda}u_n\| + \|f_{g,\lambda}u_n - f_{g,\lambda}z\| \\ &\leq \|a - f_{g\lambda}u_n\| + c\,\|gu_n - gz\|\,, \forall n \in \mathbb{N}, \end{aligned}$$

we have $\|a - f_{g,\lambda}z\| \to 0$ as $n \to \infty$. Thus, $a = f_{g,\lambda}z = gz$, i.e., a is a point of coincidence of $f_{g,\lambda}$ and g and hence, a is a point of coincidence of f and g. Therefore, by using the same process as Theorem 5, we get $Fix(f,g) = \{a\}$.

2. The proof is similar to that of Theorem 5 with some minor modifications.

Case 2. Assume that $b = 0$. In this case, $\lambda = 1, c = \theta$, and the proof is similar to Case 1 but instead of $f_{g,\lambda}$, we have $f = f_{g,1}$ and the iteration reduces to the Jungck iteration $gu_n = fu_{n-1}$, $\forall n \in \mathbb{N}$.

By letting $m \to \infty$ in (7), the error estimate (4) holds. □

Corollary 2. *Let $\mathcal{M}$ be a convex subset of a normed space $\mathcal{X}$ and $f, g : \mathcal{M} \to \mathcal{M}$. Assume that f is a (b,θ)-enriched g-contraction mapping of the second kind and one of the following conditions holds:*

1. *There exists $\lambda = \frac{1}{b+1} \in (0,1]$ such that $\overline{f_{g,\lambda}(\mathcal{M})} \subseteq g(\mathcal{M})$ and $\overline{f_{g,\lambda}(\mathcal{M})}$ is complete;*
2. *$f(\mathcal{M}) \subseteq g(\mathcal{M})$, $g(\mathcal{M})$ is complete, and $g(\mathcal{M})$ is convex (or, in particular, g is affine).*

Then f and g have a unique coincidence point in $\mathcal{M}$. Moreover, if f and g are R-weakly commuting mappings, then

(i) $Fix(f,g) = \{a\}$, for some $a \in \mathcal{X}$;

(ii) there exists $\lambda \in (0,1]$ such that the Jungck–Krasnoselskij iteration $\{gu_{n+1}\}$, given by (3), converges strongly to the unique common fixed point a of f and g, for any $u_0 \in \mathcal{M}$;

(iii) the error estimate (4) holds, where $c = \lambda\theta$.

3. ENRICHED g-NONEXPANSIVE MAPPINGS OF THE SECOND KIND

Berinde ([**4, 5**]) introduced the following notion of enriched nonexpansive mappings.

Definition 14. *Let $\mathcal{M}$ be a nonempty subset of a normed space $\mathcal{X}$ and $f : \mathcal{M} \to \mathcal{M}$. A mapping f is called a b-enriched nonexpansive if there exists $b \in [0,\infty)$ such that*

$$\|b(u - v) + fu - fv\| \leq (b+1)\,\|u - v\|\,, \forall u, v \in \mathcal{M}.$$

We introduce the following notion of b-enriched g-nonexpansive mappings of the second kind.

Definition 15. *Let $\mathcal{M}$ be a subset of a normed space $\mathcal{X}$ and $f, g : \mathcal{M} \to \mathcal{M}$. A mapping f is called a b-enriched g-nonexpansive of the second kind if there exists $b \in [0, \infty)$ such that*

$$\|b(gu - gv) + fu - fv\| \leq (b+1)\|gu - gv\|, \forall u, v \in \mathcal{M}. \tag{8}$$

Definition 16. *[**27**] Let $\mathcal{M}$ be a subset of a Banach space $\mathcal{X}$. A mapping $f : \mathcal{M} \to \mathcal{X}$ is called hemicompact if each bounded sequence $\{u_n\} \subseteq \mathcal{M}$ has a convergent subsequence whenever $\|u_n - fu_n\| \to 0$ as $n \to \infty$.*

Definition 17. *[**20**] Let $\mathcal{M}$ be a subset of a Banach space $\mathcal{X}$. A mapping $f : \mathcal{M} \to \mathcal{X}$ is called demicompact if each bounded sequence $\{u_n\} \subseteq \mathcal{M}$ has a convergent subsequence whenever $\{u_n - fu_n\}$ is convergent.*

Remark 5. *Every demicompact operator is a hemicompact operator but the converse is not true in general, as shown in [**27**].*

Theorem 7. *Let $\mathcal{M}$ be a convex subset of a normed space $\mathcal{X}$, $f, g : \mathcal{M} \to \mathcal{M}$, $f(\mathcal{M}) \subseteq g(\mathcal{M})$, and $g(\mathcal{M})$ complete. Assume that $q \in Fix(g)$, $g(\mathcal{M})$ is convex (or, in particular, g is affine), $f(\mathcal{M})$ is bounded, f is a hemicompact and b-enriched g-nonexpansive mapping of the second kind, and f and g are continuous and ultraoccasionally weakly compatible, then $Fix(f, g)$ is nonempty.*

Proof. Choose a fixed sequence $\{k_n\} \subseteq (0, 1)$ such that $k_n \to 1$ as $n \to \infty$. For all $n \in \mathbb{N}$ and $k_n(b+1) < 1$ for some $b \in [0, \infty)$, define f_n by $f_n u = (1 - k_n)q + k_n fu$, for all $u \in \mathcal{M}$.

Since g is a self-mapping of $\mathcal{M}$, $\mathcal{M}$ is convex, $g(\mathcal{M})$ is convex, $q \in Fix(g)$, and $f(\mathcal{M}) \subseteq g(\mathcal{M})$, it follows that each f_n maps $\mathcal{M}$ into itself and $f_n(\mathcal{M}) \subseteq g(\mathcal{M})$ for all n. Moreover,

$$\begin{aligned}\|k_n b(gu - gv) + f_n u - f_n v\| &= k_n \|b(gu - gv) + fu - fv\| \\ &\leq k_n(b+1)\|gu - gv\|, \forall u, v \in \mathcal{M}.\end{aligned}$$

Thus, f_n is an enriched g-contraction mapping of the second kind on $\mathcal{M}$. Since f and g are ultraoccasionally weakly compatible, then f_n and g are occasionally weakly compatible for all n. Hence, by Theorem 6, for all n, there exists $u_n \in \mathcal{M}$ such that $u_n \in Fix(f_n, g)$. Since $f(\mathcal{M})$ is bounded and

$$\|u_n - fu_n\| = \|(g - f)u_n\| \leq (1 - k_n)(\|q\| + \|fu_n\|), \tag{9}$$

for all n, then $\|u_n - fu_n\| \to 0$ as $n \to \infty$. Since $\{u_n\}$ is bounded and f is hemicompact, it follows that there exists a subsequence $\{u_{n_m}\}$ of $\{u_n\}$ in $\mathcal{M}$ such that $u_{n_m} \to a$ for some $a \in \mathcal{M}$ as $m \to \infty$. Since $fu_{n_m} = \frac{1}{k_{n_m}} u_{n_m} + (1 - \frac{1}{k_{n_m}})q$, then $fu_{n_m} \to a$ as $m \to \infty$. By the continuity of f, we have $fa = a$. Also, by the continuity of g, we have $u_{n_m} = gu_{n_m} \to ga$ and hence $a \in Fix(f, g)$. □

Corollary 3. *Let $\mathcal{M}$ be a convex subset of a normed space $\mathcal{X}$, $f, g : \mathcal{M} \to \mathcal{M}$, $f(\mathcal{M}) \subseteq g(\mathcal{M})$, and $g(\mathcal{M})$ complete. Assume that $q \in Fix(g)$, $g(\mathcal{M})$ is convex (or, in particular, g is affine), $f(\mathcal{M})$ is bounded and f is a demicompact and b-enriched g-nonexpansive mapping of the second kind, and f and g are continuous and ultraoccasionally weakly compatible, then $Fix(f, g)$ is nonempty.*

4. APPLICATIONS

4.1 Best approximations for convex sets

In 1998, Shahzad [**22**] initiated the study of best approximations for noncommuting mappings. Since then, several important results have been obtained. In the following theorem, we apply Theorem 7 to obtain a best approximation result for ultraoccasionally weakly compatible mappings.

Theorem 8. *Let $\mathcal{X}$ be a normed space, $f, g : \mathcal{X} \to \mathcal{X}$ with $p \in \mathcal{X}$ and $\mathcal{M} \subseteq \mathcal{X}$ with $p \in CAP(f, g)$, $f(\partial\mathcal{M} \cap \mathcal{M}) \subseteq \mathcal{M}$, and $q \in Fix(g)$. Assume that $B_{\mathcal{M}}(p) \subseteq \mathcal{X}$ is convex, $g(B_{\mathcal{M}}(p)) = B_{\mathcal{M}}(p)$, $g(B_{\mathcal{M}}(p))$ is complete, $f(B_{\mathcal{M}}(p))$ is bounded, f and g are continuous and ultraoccasionally weakly compatible mappings on $B_{\mathcal{M}}(p)$ and f is a hemicompact and b-enriched g-nonexpansive mapping of the second kind on $B_{\mathcal{M}}(p)$, then $Fix(f, g) \cap B_{\mathcal{M}}(p)$ is nonempty.*

Proof. Let $u \in \mathscr{B}_{\mathcal{M}}(p)$. Then

$$\|(1-k)u + kp - p\| = (1-k)\|u - p\| = (1-k)\operatorname{dist}(p, \mathcal{M}) < \operatorname{dist}(p, \mathcal{M}), \forall k \in (0, 1).$$

Thus, $\{(1-k)u + kp : k \in (0,1)\} \cap \mathcal{M} = \emptyset$ and so $u \in \partial\mathcal{M} \cap \mathcal{M}$. As $f(\partial\mathcal{M} \cap \mathcal{M}) \subseteq \mathcal{M}$, we obtain $fu \in \mathcal{M}$. Since $u \in \mathscr{B}_{\mathcal{M}}(p)$ and $p \in CAP(f, g)$, it follows that

$$\max\{\|fu - p\|, \|gu - p\|\} \leq \|u - p\| = \operatorname{dist}(p, \mathcal{M}).$$

In particular,

$$\|fu - p\| \leq \|u - p\| = \operatorname{dist}(p, \mathcal{M}).$$

Hence, $fu \in B_{\mathcal{M}}(p)$. Therefore, $f(B_{\mathcal{M}}(p)) \subseteq g(B_{\mathcal{M}}(p))$. Applying Theorem 7 to f and g on $B_{\mathcal{M}}(p)$, we conclude that $B_{\mathcal{M}}(p) \cap Fix(f, g)$ is nonempty. □

The following example demonstrates that the condition '$p \in CAP(f, g)$' is not a necessary condition in Theorem 8.

Example 2. *Consider $\mathcal{X} = \mathbb{R}$ equipped with the usual norm. For all $u \in \mathcal{X}$, define self-mappings f and g on $\mathcal{X}$ by*

$$f(u) = \frac{u^2}{2} \text{ and } g(u) = 4u,$$

with $q = 0 \in Fix(g)$.

Consider $\mathcal{M} = [0, 2]$. *For* $p = \frac{-1}{4} \in \mathcal{X}$, *we have* $p \notin CAP(f, g)$, *since for* $u = 2 \in \mathcal{M}$, *we have*

$$\max\{|fu - p|, |gu - p|\} = \max\{\frac{9}{4}, \frac{33}{4}\} > |u - p| = \frac{9}{4}.$$

Notice that the set $B_{\mathcal{M}}(p) = \{0\}$ *is convex. These conditions, along with the other hypotheses of Theorem 8, are satisfied. However, we observe that*

$$B_{\mathcal{M}}(p) \cap Fix(f, g) = \{0\}.$$

Example 3. *Consider* $\mathcal{X} = \mathbb{R}^2$ *equipped with the norm*

$$\|(u_1, u_2)\| = \max\{|u_1|, |u_2|\}, (u_1, u_2) \in \mathcal{X}.$$

For each $(u_1, u_2) \in \mathcal{X}$, *define self-mappings* f *and* g *on* $\mathcal{X}$ *by*

$$f(u_1, u_2) = \begin{cases} \left(\frac{1}{u_1}, 0\right), & \text{if } (u_1, u_2) \in \mathcal{X},\ u_1 \neq 0 \\ (0, 0), & \text{if } u_1 = 0, \end{cases}$$

and

$$g(u_1, u_2) = (u_1, 0), \forall (u_1, u_2) \in \mathcal{X},$$

with $q = (1, 0) \in Fix(g)$.

Consider $\mathcal{M} = \left[\frac{1}{2}, 2\right] \times [0, 2]$. *For* $p = (1, -1) \in \mathcal{X}$, $p \in CAP(f, g)$. *Furthermore, we have* $f(\partial\mathcal{M} \cap \mathcal{M}) \subseteq \mathcal{M}$. *Also, the set*

$$B_{\mathcal{M}}(p) = \{(\alpha, 0) : \frac{1}{2} \leq \alpha \leq 2\},$$

is convex, $g(B_{\mathcal{M}}(p)) = B_{\mathcal{M}}(p)$, $g(B_{\mathcal{M}}(p))$ *is complete, and* g *is continuous on* $B_{\mathcal{M}}(p)$. *Since* $f(B_{\mathcal{M}}(p)) = \left\{\left(\frac{1}{\alpha}, 0\right) : \frac{1}{2} \leq \alpha \leq 2\right\}$, *then* $f(B_{\mathcal{M}}(p))$ *is bounded. Also,* f *is continuous and hemicompact on* $B_{\mathcal{M}}(p)$. *Additionally,* f *and* g *are ultraoccasionally weakly compatible on* $B_{\mathcal{M}}(p)$. *To see this, we have*

$$f_k(u_1, u_2) = kf(u_1, u_2) + (1 - k)(1, 0) = \left(k\frac{1}{u_1} + 1 - k, 0\right),$$

for every $(u_1, u_2) \in B_{\mathcal{M}}(p)$ *and* $k \in [0, 1]$. *Thus,* $C(f_k, g) = \{(1, 0)\}$ *and since* $gf_k(1, 0) = f_k g(1, 0) = (1, 0)$, *we have* f_k *and* g *are occasionally weakly compatible mappings on* $B_{\mathcal{M}}(p)$. *Hence,* f *and* g *are ultraoccasionally weakly compatible on* $B_{\mathcal{M}}(p)$.

Notice that f is an enriched g-nonexpansive mapping on $B_{\mathcal{M}}(p)$ of the second kind. To verify this, consider condition (8). This implies

$$\|b(g(u_1,u_2)-g(v_1,v_2))+f(u_1,u_2)-f(v_1,v_2)\| = |(u_1-v_1)(b-\frac{1}{u_1v_1})|,$$

for all $(u_1,u_2),(v_1,v_2)\in B_{\mathcal{M}}(p)$. Since, for any $b\geq\frac{3}{2}$, $\left|b-\frac{1}{u_1v_1}\right|\leq b+1$ holds, we conclude that

$$\begin{aligned}\|b(g(u_1,u_2)-g(v_1,v_2))+f(u_1,u_2)-f(v_1,v_2)\| &\leq (b+1)\,|u_1-v_1|\\ &= (b+1)\,\|g(u_1,u_2)-g(v_1,v_2)\|,\end{aligned}$$

for all $(u_1,u_2),(v_1,v_2)\in B_{\mathcal{M}}(p)$. Therefore, the hypotheses of Theorem 8 are satisfied and so $B_{\mathcal{M}}(p)\cap Fix(f,g)=\{(1,0)\}$.

4.2 Volterra integral equations of the second kind

Consider $C([0,a],\mathbb{R})$, the space of all real continuous functions on $[0,a]$, equipped with the Bielecki's norm

$$\|\psi\|_\beta = \max_{t\in[0,a]}\{e^{-\beta t}\,|\psi(t)|\},\ \psi\in C([0,a],\mathbb{R}),\ \beta>0.$$

Lemma 1. *[16] If $\beta>0$ is a constant, then $(C([0,a],\mathbb{R}),\|.\|_\beta)$ is a Banach space.*

In this subsection, we study the existence and uniqueness of a common solution to the following Volterra integral equations of the second kind,

$$\begin{aligned}\psi(t) &= \int_0^t A_1(t,s,\psi(s))ds+h(t),\ t\in[0,a],\\ \psi(t) &= \int_0^t A_2(t,s,\psi(s))ds+h(t),\ t\in[0,a],\end{aligned}\tag{10}$$

where $a>0$, $A_1,A_2\in C([0,a]\times[0,a]\times I,\mathbb{R}),(I\subset\mathbb{R})$ are bounded on the domain, $h\in C([0,a],\mathbb{R})$, and $\psi\in C([0,a],\mathbb{R})$ is the unknown function, $\psi(t)\in I$. For more details, see [**3**].

Theorem 9. *Let the operators f and g on $C([0,a],\mathbb{R})$ be defined by*

$$\begin{aligned}(f\psi)(t) &= \int_0^t A_1(t,s,\psi(s))ds+h(t),\ t\in[0,a],\\ (g\psi)(t) &= \int_0^t A_2(t,s,\psi(s))ds+h(t),\ t\in[0,a],\end{aligned}$$

where $a>0$, $h\in C([0,a],\mathbb{R})$, and $\psi\in C([0,a],\mathbb{R}),\psi(t)\in I\subset\mathbb{R}$. Assume that $A_1,A_2\in C([0,a]\times[0,a]\times I,\mathbb{R})$ are bounded on the domain, and the following conditions hold:

(i) The functions A_1 and A_2 satisfy

$$|A_1(t,s,\psi_1(t)) - A_1(t,s,\psi_2(t))| \leq \beta e^{-\beta} |(g\psi_1)(t) - (g\psi_2)(t)|,$$

$$|A_1(t,s,(g\psi)(t)) - A_2(t,s,(f\psi)(t))| \leq \beta e^{-\beta} |(g\psi)(t) - (f\psi)(t)|,$$

where $\beta > 0$;

(ii) $g(C([0,a],\mathbb{R})) = C([0,a],\mathbb{R})$;

(iii) A_2 is affine with respect to the third variable.

Then the Volterra integral equations (10) have a unique common solution, say $\psi^* \in C([0,a],\mathbb{R})$. Moreover, the iteration $\{g\psi_{n+1}\}$ of a sequence of functions, given by

$$(g\psi_{n+1})(t) = \int_0^t A_1(t,s,\psi_n(s))ds + h(t),\ n \geq 0, \tag{11}$$

converges strongly to the unique common solution of the Volterra integral equations, for any $\psi_0 \in C([0,a],\mathbb{R})$.

Proof. Let $\psi_1, \psi_2 \in C([0,a],\mathbb{R})$. The operator f is a g-contraction. To see this, we have

$$\begin{aligned}
|(f\psi_1)(t) - (f\psi_2)(t)| &\leq \int_0^t |A_1(t,s,\psi_1(s)) - A_1(t,s,\psi_2(s))|ds \\
&\leq \int_0^t \beta e^{-\beta} |(g\psi_1)(s) - (g\psi_2)(s)|\, ds \\
&\leq \beta e^{-\beta} \|g\psi_1 - g\psi_2\|_\beta \int_0^t e^{\beta s} ds \\
&\leq e^{-\beta} \|g\psi_1 - g\psi_2\|_\beta e^{\beta t},
\end{aligned}$$

where $t \in [0,a]$ and $\beta > 0$. Thus,

$$|(f\psi_1)(t) - (f\psi_2)(t)|e^{-\beta t} \leq e^{-\beta} \|g\psi_1 - g\psi_2\|_\beta,$$

where $t \in [0,a]$ and $\beta > 0$. Hence,

$$\|f\psi_1 - f\psi_2\|_\beta \leq e^{-\beta} \|g\psi_1 - g\psi_2\|_\beta,$$

where $\beta > 0$. Since $\theta = e^{-\beta} < 1$, the operator f is a g-contraction. Also, since A_2 is affine in the third coordinate, it follows that

$$\begin{aligned}
(g((1-\alpha)\psi_1 + \alpha\psi_2))(t) &= (1-\alpha) \int_0^t A_2(t,s,\psi_1(s))ds + (1-\alpha)h(t) \\
&\quad + \alpha \int_0^t A_2(t,s,\psi_2(s))ds + \alpha h(t) \\
&= (1-\alpha)(g\psi_1)(t) + \alpha(g\psi_2)(t),
\end{aligned}$$

for all $\alpha \in [0,1]$ and $t \in [0,a]$. Thus, g is affine.

Now, let $\psi \in C([0,a],\mathbb{R})$, then we have

$$\begin{aligned}|(fg\psi)(t)-(gf\psi)(t)| &\leq \int_0^t |A_1(t,s,(g\psi)(s))-A_2(t,s,(f\psi)(s))|ds\\ &\leq \int_0^t \beta e^{-\beta}\,|(g\psi)(s)-(f\psi)(s)|\,ds\\ &\leq \beta e^{-\beta}\,\|g\psi-f\psi\|_\beta \int_0^t e^{\beta s}ds\\ &\leq e^{-\beta}\,\|g\psi-f\psi\|_\beta\, e^{\beta t},\end{aligned}$$

where $t \in [0,a]$ and $\beta > 0$. Thus,

$$|(fg\psi)(t)-(gf\psi)(t)|e^{-\beta t} \leq e^{-\beta}\,\|g\psi-f\psi\|_\beta \leq \|g\psi-f\psi\|_\beta\,,$$

where $t \in [0,a]$ and $\beta > 0$. Hence,

$$\|fg\psi-gf\psi\|_\beta \leq \|g\psi-f\psi\|_\beta\,,$$

where $\beta > 0$. Therefore, the operators f and g are weakly commuting and hence, they are also occasionally weakly compatible. Furthermore, we have

$$f(C([0,a],\mathbb{R})) \subseteq C([0,a],\mathbb{R}) = g(C([0,a],\mathbb{R})).$$

Thus, by Theorem 5, f and g have a unique common fixed point in $C([0,a],\mathbb{R})$, which is the unique common solution of the Volterra integral equations (10). Moreover, the iteration $\{g\psi_{n+1}\}$ of a sequence of functions, given by (11), converges strongly to the unique common solution of the Volterra integral equations, for any $\psi_0 \in C([0,a],\mathbb{R})$. □

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