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ACCELERATING FIXED POINT ITERATIONS WITH MULTI-STEP INERTIALS FOR NONEXPANSIVE MAPPINGS

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We present a novel method that combines multi-step inertial approximations with SP algorithm to solve fixed-point problems for a family of nonexpansive mappings. The approach aims to accelerate the convergence speed. We have established a weak convergence theorem under mild conditions to validate the proposed method. Additionally, we have connected the fixed point problem to the variational inclusion problem, the split feasibility problem, and signal recovery models, broadening the algorithm's range of potential applications.

1. INTRODUCTION

Fixed point theory plays a pivotal role in many optimization branches and has garnered significant recognition due to its broad applicability across diverse fields. This mathematical framework provides foundational tools and methods for solving problems where certain functions map elements back to themselves, known as fixed points. Its use extends far beyond pure mathematics, finding relevance in various disciplines. In science and applied science, fixed point theory is often employed to model and analyze complex systems, enabling researchers to identify equilibrium states and predict system behavior. In machine learning, these methods underpin algorithms for clustering, dimensionality reduction, and iterative optimization tasks, forming the basis of many learning models.

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Consider a real Hilbert space $\mathcal{H}$ and a nonempty subset $\mathcal{C} \subseteq \mathcal{H}$ with $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$ as the inner product and norm, respectively. A mapping $\mathcal{T} : \mathcal{C} \to \mathcal{C}$ is said to be $\mathcal{L}$-Lipschtiz if there exists $\mathcal{L} > 0$ such that $\|\mathcal{T}u - \mathcal{T}w\| \leq \mathcal{L}\|u - w\|$ for any $u, w \in \mathcal{C}$. If $\mathcal{L} = 1$, the mapping $\mathcal{T}$ is called nonexpansive. The set of all fixed points of $\mathcal{T}$ is denoted by $\mathcal{F}(\mathcal{T})$ with $\mathcal{F}(\mathcal{T}) = \{ u \in \mathcal{C} : u = \mathcal{T}u\}$.

For a family of nonexpansive mappings $\{\mathcal{S}_k\}$, a point u is called a common fixed point of $\mathcal{S}_k$ if it belongs to the intersection of the fixed-point sets of all mappings in the family, i.e., $u \in \bigcap_{k=1}^{\infty} \mathcal{F}(\mathcal{S}_k)$. In 1965, Browder [**5**], Göhde [**12**], and Kirk [**20**] established fixed point theorems for nonexpansive mappings under specific geometric conditions, such as standard structure or uniform convexity. Following their work, many researchers have developed various extensions and generalizations of nonexpansive mappings and their associated results.

A traditional iterative approach to solve the fixed point problem is Picard algorithm, which is defined as follows:

$$u_{k+1} = \mathcal{T}u_k, \qquad k \geq 1,$$

where the initial point u_1 is chosen randomly.

Mann algorithm [**22**] is a classical iterative method to solve fixed point problems. The algorithm is defined as follows:

$$u_{k+1} = (1 - \theta_k)u_k + \theta_k \mathcal{T}u_k, \qquad k \geq 1,$$

where u_1 is chosen randomly and $\{\theta_k\}$ is a sequence in $(0, 1]$.

Ishikawa algorithm [**18**] is an extension of Mann algorithm. It incorporates a two-step method, making it more flexible and faster than Mann algorithm under certain conditions. Ishikawa algorithm is defined as follows:

$$\begin{cases} w_k = (1 - \alpha_k)u_k + \alpha_k \mathcal{T}u_k, \\ u_{k+1} = (1 - \theta_k)u_k + \theta_k \mathcal{T}w_k, \quad k \geq 1, \end{cases}$$

where u_1 is chosen randomly and $\{\alpha_k\}$, $\{\theta_k\}$ are sequences in $(0, 1]$.

Noor algorithm [**26**] further generalizes Mann and Ishikawa algorithms and has a three-step method. The Noor algorithm is defined as follows:

$$\begin{cases} z_k = (1 - \beta_k)u_k + \beta_k \mathcal{T}u_k, \\ w_k = (1 - \alpha_k)u_k + \alpha_k \mathcal{T}z_k, \\ u_{k+1} = (1 - \theta_k)u_k + \theta_k \mathcal{T}w_k, \quad k \geq 1, \end{cases}$$

where u_1 is chosen randomly and $\{\beta_k\}$, $\{\alpha_k\}$, $\{\theta_k\}$ are sequences in $(0, 1]$.

SP algorithm [**30**] is a three-step method, which is a modified version of Noor algorithm. It is defined as follows:

$$\begin{cases} z_k = (1 - \beta_k)u_k + \beta_k \mathcal{T}u_k, \\ w_k = (1 - \alpha_k)z_k + \alpha_k \mathcal{T}z_k, \\ u_{k+1} = (1 - \theta_k)w_k + \theta_k \mathcal{T}w_k, \quad k \geq 1, \end{cases}$$

where u_1 is chosen randomly and $\{\beta_k\}$, $\{\alpha_k\}$, $\{\theta_k\}$ are sequences in $(0, 1]$.

In recent years, many researchers have investigated fixed point problems in Banach spaces. In 2020, Ali and Ali [**1**] proposed the F^* algorithm as a novel iterative process to estimate fixed points of weakly contractive mappings in Banach spaces. They further employed this procedure to obtain an approximate solution of a nonlinear quadratic Volterra integral equation. Moreover, Zaheer et al. [**37**] introduced the Z algorithm as a three-step iteration for estimating fixed points of weakly contractive operators. They demonstrated its effectiveness by applying it to approximate the solution of a delay Caputo fractional differential equation.

The inertial extrapolation technique, introduced by Polyak [**31**], is designed to accelerate the convergence of iterative algorithms. It is commonly known as the heavy ball method of the second-order time dynamical system. Later, Nesterov [**25**] applied the inertial process to solve minimization problems, widely known as the inertial iteration. Recently, numerous iterative algorithms incorporating inertial extrapolation techniques have been proposed. These include inertial extragradient methods [**11**], inertial forward-backward algorithms [**2, 4, 21**], and the fast iterative shrinkage-thresholding algorithm (FISTA) [**3, 8**]. A key characteristic of inertial extrapolation algorithms is their two-step iterative framework, where each subsequent iterate is computed using information from the two most recent iterates.

In 1970, Ortega and Rheinboldt [**28**] proposed a generalized iterative process: for $s \in \mathbb{N}$,

$$u_{k+1} = g(u_k, u_{k-1}, ..., u_{k-s+1}), \qquad k \geq 1,$$

where g is a function that performs extrapolation based on the points $u_k, u_{k-1}, ...,$ u_{k-s+1}. This iterative process is referred to as the k-step method. Polyak [**32**] demonstrated that multi-step inertial algorithms can enhance the speed of optimization processes, even though their convergence or convergence rate has yet to be rigorously established. As a result, the inertial extrapolation step is widely regarded as an effective technique for improving the convergence performance of various algorithms presented in the literature (see also [**9, 13, 15, 17, 19**]).

DHLCR Algorithm [**10**] is a modified multi-step inertial and Mann iterative scheme which is determined by:

$$\begin{cases} z_k = u_k + \sum\limits_{k \in \Gamma_k} \delta_{s,k}(u_{k-s+1} - u_{k-s}), \\ w_k = u_k + \sum\limits_{k \in \Gamma_k} \vartheta_{s,k}(u_{k-s+1} - u_{k-s}), \\ u_{k+1} = (1 - \beta_k)w_k + \beta_k \mathcal{T} w_k, \end{cases}$$

where $\Gamma_k \subseteq \{0, 1, 2, ..., k-1\}$, $k \geq 1$ and $\{\delta_{s,k}\}, \{\vartheta_{s,k}\} \subseteq (-1, 2]^{|\Gamma_k|}$ for each $s \geq 2$. They proved it converges weakly to a fixed point of $\mathcal{T}$.

ZDC Algorithm [**38**] is a multi-step inertial algorithm for variational inclusion

problem which is determined by:

$$\begin{cases} z_k = u_k + \sum_{k\in\Gamma} \delta_{s,k}(u_{k-s+1} - u_{k-s}), \\ w_k = J^{\mathcal{A}}_{\rho_k}(z_k - \varsigma_k f(z_k)), \\ d(z_k, w_k) = (z_k - w_k) - \varsigma_k(f(z_k) - f(w_k)), \\ u_{k+1} = w_k + \gamma\beta_k d(z_k, w_k), \end{cases}$$

where $\gamma \in (0, 2)$,

$$\beta_k = \begin{cases} \dfrac{\langle z_k - w_k, d(z_k, w_k)\rangle}{\|d(z_k, w_k)\|^2}, & \|d(z_k, w_k)\|^2 \neq 0, \\ c, & \|d(z_k, w_k)\|^2 = 0, \end{cases}$$

where $\Gamma \subseteq \{0, 1, 2, ..., k-1\}$, $k \geq 1$ and $\{\delta_{s,k}\} \subseteq (-1, 1)^{\Gamma}$ for each $s \geq 2$.

The contributions of this work can be summarized as follows:

- We develop a novel iterative method that combines multi-inertial techniques with the SP algorithm to address common fixed point problems for a family of nonexpansive mappings.
- We establish a fundamental lemma to formulate the multi-inertial conditions and rigorously prove the weak convergence of the proposed scheme under mild assumptions.
- The proposed method is designed with simple parameter settings, avoiding the need for complicated tuning strategies often required in inertial-type algorithms.
- We demonstrate the broader applicability of our approach by connecting the fixed point framework to variational inclusion problems and split feasibility problems.
- Numerical experiments are presented to demonstrate the efficiency and practicality of the algorithm, with a particular emphasis on its successful application to signal recovery problems.

The structure of this work is outlined as follows: Section 1 provides a review of relevant previous works to highlight the background and motivation of our study. Section 2 introduces the theoretical preliminaries and essential mathematical tools employed in our analysis. Section 3 contains the main results, where we introduce the multi-inertial SP algorithm and establish its weak convergence properties. Section 4 concerns variational inclusion problems and split feasibility problems. Section 5 focuses on the application of our algorithm to the signal recovery problem, demonstrating its practical relevance. The final section provides concluding remarks.

2. PRELIMINARIES

Definition 2.1. *[24, 36] Let $\{\mathcal{S}_k\}$ and $\mathcal{S}$ be a family of nonexpansive mappings such that $\emptyset \neq \mathcal{F}(\mathcal{S}) \subset \bigcap_{k=1}^{\infty} \mathcal{F}(\mathcal{S}_k)$. Then $\{\mathcal{S}_k\}$ is said to satisfy NST-condition(I) with $\mathcal{S}$ if for each bounded sequence $\{u_k\}$,*

$$\lim_{k\to\infty} \|u_k - \mathcal{S}_k u_k\| = 0 \text{ implies } \lim_{k\to\infty} \|u_k - \mathcal{T} u_k\| = 0 \text{ for all } \mathcal{T} \in \mathcal{S}.$$

Lemma 2.2. *[29] Let $\{s_k\}, \{t_k\}$ and $\{\delta_k\}$ be nonnegative sequences such that*

$$s_{k+1} \leq (1+\delta_k)s_k + t_k, \qquad k \geq 1.$$

If $\sum_{k=1}^{\infty} \delta_k < +\infty$ and $\sum_{k=1}^{\infty} t_k < +\infty$, then $\lim_{k\to\infty} s_k$ exists.

Lemma 2.3. *[34] In a Hilbert space, we have*

(i) $\|hu + (1-h)z\|^2 = h\|u\|^2 + (1-h)\|z\|^2 - h(1-h)\|u-z\|^2$, for all $h \in \mathbb{R}$ and for all $u, z \in \mathcal{H}$;

(ii) $\|u \pm z\|^2 = \|u\|^2 \pm 2\langle u, z\rangle + \|z\|^2$, for all $u, z \in \mathcal{H}$.

Lemma 2.4. *[27] Let Ψ be a subset of $\mathcal{H}$ and $\{u_k\}$ be a sequence in $\mathcal{H}$ that satisfy:*

(i) for every $\bar{z} \in \Psi$, $\lim_{k\to\infty} \|u_k - \bar{z}\|$ exists;

(ii) each weak-cluster point of the sequence $\{u_k\}$ is in Ψ.

Then $\{u_k\}$ converges weakly to an element in Ψ.

3. ALGORITHM AND CONVERGENCE ANALYSIS

In this section, we introduce a multi-inertial SP algorithm aimed at finding a common fixed point for a family of nonexpansive mappings. Let $\{\mathcal{S}_k\}$ be a family of nonexpansive mappings on $\mathcal{H}$. The focus of this method is to use a combination of inertial techniques and fixed point iteration. The following lemma, crucial for our analysis, can be established using mathematical induction. For further details, we refer to [**16**].

Lemma 3.5. *[16] Let $k \in \{1, 2, ..., b\}$ for some $b \in \mathbb{N}$. Let $a_{1-b}, a_{2-b}, ..., a_0$ be non-negative real numbers. Let $\{a_k\}_{k=1}^{\infty}$ and $\{\delta_{s,k}\}_{k=1}^{\infty}$ be non-negative real sequences. If*

$$a_{k+1} \leq a_k + \sum_{s=1}^{b} (a_{k-s+1} + a_{k-s})\delta_{s,k}, \quad k \in \mathbb{N},$$

then

$$a_{k+1} \leq K \cdot \prod_{j=1}^{k} (1 + 2\delta_{1,j} + 2\delta_{2,j} + ... + 2\delta_{b-1,j} + 2\delta_{b,j}), \quad k \in \mathbb{N},$$

where $K = \max\{a_{1-b}, a_{2-b}, ..., a_0, a_1\}$. *Moreover, if for each* s, $\sum_{k=1}^{\infty} \delta_{s,k} < +\infty$, *then* $\{a_k\}$ *is bounded.*

Algorithm 3.6. *SPMI (SP algorithm with multi-step inertial extrapolations) Let* $\gamma_k \in (0,1)$, $\beta_k \in (0,1)$ *and* $\alpha_k \in (0,1)$. *Fix* $b \in \mathbb{N}$ *and let* $u_{1-b}, u_{2-b}, ..., u_0, u_1 \in \mathcal{H}$ *be arbitrary. For each* $s \in \{1, 2, ..., b\}$, *let* $\{\delta_{s,k}\}_{k=1}^{\infty}$ *be a real non-negative sequence. Compute:*

$$\begin{aligned} z_k &= \gamma_k u_k + (1-\gamma_k)\mathcal{S}_k\left(u_k + \sum_{s=1}^{b} \delta_{s,k}(u_{k-s+1} - u_{k-s})\right), \\ y_k &= \beta_k z_k + (1-\beta_k)\mathcal{S}_k z_k, \\ u_{k+1} &= \alpha_k y_k + (1-\alpha_k)\mathcal{S}_k y_k, \quad k \geq 1. \end{aligned}$$

Theorem 3.7. *Let* $\{u_k\}$ *be defined by Algorithm 3.6. Let* $\{\mathcal{S}_k\}$ *and* $\mathcal{S}$ *be a family of nonexpansive mappings such that* $\{\mathcal{S}_k\}$ *satisfies NST-condition(I) with* $\mathcal{S}$. *Suppose that* $\emptyset \neq \mathcal{F}(\mathcal{S}) \subset \bigcap_{k=1}^{\infty} \mathcal{F}(\mathcal{S}_k)$. *If* $\sum_{k=1}^{\infty} \delta_{s,k} < +\infty$, $0 < \liminf_{k\to\infty} \gamma_k \leq \limsup_{k\to\infty} \gamma_k < 1$, $0 < \liminf_{k\to\infty} \beta_k \leq \limsup_{k\to\infty} \beta_k < 1$ *and* $0 < \alpha_k < 1$, *then*

(i) for each $\bar{z} \in \mathcal{F}(\mathcal{S})$, *we have* $\|u_{k+1} - \bar{z}\| \leq K \cdot \prod_{j=1}^{n}(1 + 2\delta_{1,j} + 2\delta_{2,j} + \ldots + 2\delta_{b-1,j} + 2\delta_{b,j})$, *where* $K = \max\{\|u_{1-b} - \bar{z}\|, \|u_{2-b} - \bar{z}\|, \ldots, \|u_0 - \bar{z}\|, \|u_1 - \bar{z}\|\}$;

(ii) the sequence $\{u_k\}$ *converges weakly to an element in* $\mathcal{F}(\mathcal{S})$.

Proof. Let $\bar{z} \in \mathcal{F}(\mathcal{S})$. Then,

$$\begin{aligned} \|z_k - \bar{z}\| &= \left\|\gamma_k u_k + (1-\gamma_k)\mathcal{S}_k\left(u_k + \sum_{s=1}^{b} \delta_{s,k}(u_{k-s+1} - u_{k-s})\right) - \bar{z}\right\| \\ &\leq \gamma_k\|u_k - \bar{z}\| + (1-\gamma_k)\left\|\mathcal{S}_k\left(u_k + \sum_{s=1}^{b} \delta_{s,k}(u_{k-s+1} - u_{k-s})\right) - \bar{z}\right\| \\ &\leq \gamma_k\|u_k - \bar{z}\| + (1-\gamma_k)\left\|u_k + \sum_{s=1}^{b} \delta_{s,k}(u_{k-s+1} - u_{k-s}) - \bar{z}\right\| \\ &\leq \gamma_k\|u_k - \bar{z}\| + (1-\gamma_k)\|u_k - \bar{z}\| \\ &\leq +(1-\gamma_k)\sum_{s=1}^{b} \delta_{s,k}\|u_{k-s+1} - u_{k-s}\| \\ &= \|u_k - \bar{z}\| + (1-\gamma_k)\sum_{s=1}^{b} \delta_{s,k}\|u_{k-s+1} - u_{k-s}\|. \end{aligned} \tag{3.1}$$

Consider

$$\begin{aligned} \|y_k - \bar{z}\| &= \|\beta_k z_k + (1-\beta_k)\mathcal{S}_k z_k - \bar{z}\| \\ &\leq \beta_k \|z_k - \bar{z}\| + (1-\beta_k)\|\mathcal{S}_k z_k - \bar{z}\| \\ &\leq \beta_k \|z_k - \bar{z}\| + (1-\beta_k)\|z_k - \bar{z}\| \\ &= \|z_k - \bar{z}\|. \end{aligned} \tag{3.2}$$

From (3.1) and (3.2), it follows that

$$\begin{aligned} \|u_{k+1} - \bar{z}\| &= \|\alpha_k y_k + (1-\alpha_k)\mathcal{S}_k y_k - \bar{z}\| \\ &\leq \alpha_k \|y_k - \bar{z}\| + (1-\alpha_k)\|\mathcal{S}_k y_k - \bar{z}\| \\ &\leq \alpha_k \|y_k - \bar{z}\| + (1-\alpha_k)\|y_k - \bar{z}\| \\ &= \|y_k - \bar{z}\| \\ &\leq \|z_k - \bar{z}\| \\ &\leq \|u_k - \bar{z}\| + (1-\gamma_k)\sum_{s=1}^{b} \delta_{s,k} \|u_{k-s+1} - u_{k-s}\|. \end{aligned} \tag{3.3}$$

Using Lemma 3.5, we obtain

$$\|u_{k+1} - \bar{z}\| \leq K \cdot \prod_{j=1}^{k} (1 + 2\delta_{1,j} + 2\delta_{2,j} + ... + 2\delta_{b-1,j} + 2\delta_{b,j}), \qquad k \geq 1,$$

where $K = \max\{\|u_{1-b} - \bar{z}\|, \|u_{2-b} - \bar{z}\|, ..., \|u_0 - \bar{z}\|, \|u_1 - \bar{z}\|\}$. Thus, $\{u_k\}$ is bounded because $\sum_{k=1}^{\infty} \delta_{s,k} < +\infty$. Then $\sum_{k=1}^{\infty} \delta_{1,k}\|u_k - u_{k-1}\| < +\infty$, $\sum_{k=1}^{\infty} \delta_{2,k}\|u_{k-1} - u_{k-2}\| < +\infty$, ..., $\sum_{k=1}^{\infty} \delta_{b-1,k}\|u_{k-b+2} - u_{k-b+1}\| < +\infty$, and $\sum_{k=1}^{\infty} \delta_{b,k}\|u_{k-b+1} - u_{k-b}\| < +\infty$. Applying Lemma 2.2 in (3.3), we conclude that $\lim_{k\to\infty} \|u_k - \bar{z}\|$ exists.

Now, by Lemma 2.3 (i), we see that

$$
\begin{aligned}
\|z_k-\bar{z}\|^2 &= \left\|\gamma_k u_k+(1-\gamma_k)\mathcal{S}_k\left(u_k+\sum_{s=1}^{b}\delta_{s,k}(u_{k-s+1}-u_{k-s})\right)-\bar{z}\right\|^2\\
&= \gamma_k\|u_k-\bar{z}\|^2\\
&\quad +(1-\gamma_k)\left\|\mathcal{S}_k\left(u_k+\sum_{s=1}^{b}\delta_{s,k}(u_{k-s+1}-u_{k-s})\right)-\bar{z}\right\|^2\\
&\quad -\gamma_k(1-\gamma_k)\left\|\mathcal{S}_k\left(u_k+\sum_{s=1}^{b}\delta_{s,k}(u_{k-s+1}-u_{k-s})\right)-u_k\right\|^2\\
&\leq \gamma_k\|u_k-\bar{z}\|^2\\
&\quad +(1-\gamma_k)\left\|u_k+\sum_{s=1}^{b}\delta_{s,k}(u_{k-s+1}-u_{k-s})-\bar{z}\right\|^2\\
&\quad -\gamma_k(1-\gamma_k)\left\|\mathcal{S}_k\left(u_k+\sum_{s=1}^{b}\delta_{s,k}(u_{k-s+1}-u_{k-s})\right)-u_k\right\|^2\\
&= \gamma_k\|u_k-\bar{z}\|^2+(1-\gamma_k)\|u_k-\bar{z}\|^2\\
&\quad +(1-\gamma_k)\left\|\sum_{s=1}^{b}\delta_{s,k}(u_{k-s+1}-u_{k-s})\right\|^2\\
&\quad +2(1-\gamma_k)\left\langle u_k-\bar{z},\sum_{s=1}^{b}\delta_{s,k}(u_{k-s+1}-u_{k-s})\right\rangle\\
&\quad -\gamma_k(1-\gamma_k)\left\|\mathcal{S}_k\left(u_k+\sum_{s=1}^{b}\delta_{s,k}(u_{k-s+1}-u_{k-s})\right)-u_k\right\|^2\\
&\leq \|u_k-\bar{z}\|^2+(1-\gamma_k)\left(\sum_{s=1}^{b}\delta_{s,k}\|u_{k-s+1}-u_{k-s}\|\right)^2\\
&\quad +2(1-\gamma_k)\sum_{s=1}^{b}\delta_{s,k}\|u_k-\bar{z}\|\|u_{k-s+1}-u_{k-s}\|\\
&\quad -\gamma_k(1-\gamma_k)\left\|\mathcal{S}_k\left(u_k+\sum_{s=1}^{b}\delta_{s,k}(u_{k-s+1}-u_{k-s})\right)-u_k\right\|^2. \qquad (3.4)
\end{aligned}
$$

Using (3.4), we get

$$\begin{aligned}
\|y_k-\bar{z}\|^2 &= \|\beta_k z_k+(1-\beta_k)\mathcal{S}_k z_k-\bar{z}\|^2\\
&= \beta_k\|z_k-\bar{z}\|^2+(1-\beta_k)\|\mathcal{S}_k z_k-\bar{z}\|^2-\beta_k(1-\beta_k)\|\mathcal{S}_k z_k-z_k\|^2\\
&\leq \beta_k\|z_k-\bar{z}\|^2+(1-\beta_k)\|z_k-\bar{z}\|^2-\beta_k(1-\beta_k)\|\mathcal{S}_k z_k-z_k\|^2\\
&= \|z_k-\bar{z}\|^2-\beta_k(1-\beta_k)\|\mathcal{S}_k z_k-z_k\|^2\\
&\leq \|u_k-\bar{z}\|^2+(1-\gamma_k)\left(\sum_{s=1}^{b}\delta_{s,k}\|u_{k-s+1}-u_{k-s}\|\right)^2\\
&\quad +2(1-\gamma_k)\sum_{s=1}^{b}\delta_{s,k}\|u_k-\bar{z}\|\|u_{k-s+1}-u_{k-s}\|\\
&\quad -\gamma_k(1-\gamma_k)\left\|\mathcal{S}_k\left(u_k+\sum_{s=1}^{b}\delta_{s,k}(u_{k-s+1}-u_{k-s})\right)-u_k\right\|^2\\
&\quad -\beta_k(1-\beta_k)\|\mathcal{S}_k z_k-z_k\|^2. \qquad (3.5)
\end{aligned}$$

By the definition of u_{k+1}, (3.5) and Lemma 2.3 (i), we have

$$\begin{aligned}
\|u_{k+1}-\bar{z}\|^2 &= \|\alpha_k y_k+(1-\alpha_k)\mathcal{S}_k y_k-\bar{z}\|^2\\
&= \alpha_k\|y_k-\bar{z}\|^2+(1-\alpha_k)\|\mathcal{S}_k y_k-\bar{z}\|^2-\alpha_k(1-\alpha_k)\|\mathcal{S}_k y_k-y_k\|^2\\
&\leq \alpha_k\|y_k-\bar{z}\|^2+(1-\alpha_k)\|y_k-\bar{z}\|^2-\alpha_k(1-\alpha_k)\|\mathcal{S}_k y_k-y_k\|^2\\
&= \|y_k-\bar{z}\|^2-\alpha_k(1-\alpha_k)\|\mathcal{S}_k y_k-y_k\|^2\\
&\leq \|u_k-\bar{z}\|^2+(1-\gamma_k)\left(\sum_{s=1}^{b}\delta_{s,k}\|u_{k-s+1}-u_{k-s}\|\right)^2\\
&\quad +2(1-\gamma_k)\sum_{s=1}^{b}\delta_{s,k}\|u_k-\bar{z}\|\|u_{k-s+1}-u_{k-s}\|\\
&\quad -\gamma_k(1-\gamma_k)\left\|\mathcal{S}_k\left(u_k+\sum_{s=1}^{b}\delta_{s,k}(u_{k-s+1}-u_{k-s})\right)-u_k\right\|^2\\
&\quad -\beta_k(1-\beta_k)\|\mathcal{S}_k z_k-z_k\|^2-\alpha_k(1-\alpha_k)\|\mathcal{S}_k y_k-y_k\|^2.
\end{aligned}$$

since $\sum_{k=1}^{\infty}\delta_{s,k}<+\infty$ and $0<\liminf_{k\to\infty}\gamma_k\leq\limsup_{k\to\infty}\gamma_k<1$, $0<\liminf_{k\to\infty}\beta_k\leq\limsup_{k\to\infty}\beta_k<1$ and $0<\alpha_k<1$, we have

$$\lim_{k\to\infty}\|\mathcal{S}_k z_k-z_k\|=0 \text{ and } \lim_{k\to\infty}\left\|\mathcal{S}_k\left(u_k+\sum_{s=1}^{b}\delta_{s,k}(u_{k-s+1}-u_{k-s})\right)-u_k\right\|=0.$$

Consequently

$$\begin{aligned}\lim_{k\to\infty}\|z_k-u_k\| &= \lim_{k\to\infty}\left\|\gamma_k u_k+(1-\gamma_k)\mathcal{S}_k\left(u_k+\sum_{s=1}^{b}\delta_{s,k}(u_{k-s+1}-u_{k-s})\right)-u_k\right\| \\ &= \lim_{k\to\infty}(1-\gamma_k)\left\|\mathcal{S}_k\left(u_k+\sum_{s=1}^{b}\delta_{s,k}(u_{k-s+1}-u_{k-s})\right)-u_k\right\| \\ &= 0.\end{aligned}$$

Since $\{\mathcal{S}_k\}$ satisfies NST-conditon(I) with $\mathcal{S}$, we get that $\lim_{k\to\infty}\|\mathcal{T}z_k-z_k\|=0$ for all $\mathcal{T}\in\mathcal{S}$. Let z^* be a weak cluster point of $\{z_k\}$. Then there is a subsequence $\{z_{k_s}\}$ of $\{z_k\}$ such that $\{z_{k_s}\}$ converges weakly to $z^*\in\mathcal{H}$. By the demiclosedness of nonexpansive mappings, we have $z^*\in\mathcal{F}(\mathcal{S})$. So by Lemma 2.4, it is concluded $\{z_k\}$ converge weakly to a point in $\mathcal{F}(\mathcal{S})$. Furthermore, since $\|z_k-u_k\|\to 0$, then $\{u_k\}$ converges weakly to a point in $\mathcal{F}(\mathcal{S})$. □

Remark 3.8. *Algorithm 3.6, which converges in the sense of weak convergence, is sufficient for the illustrative example considered in the finite-dimensional spaces. However, in infinite-dimensional spaces, it is often necessary to establish strong convergence results. In this work, our main algorithm can be suitably modified to obtain strong convergence by employing classical techniques such as the Halpern iteration method [**14**], the viscosity approximation method [**23**], and the shrinking projection method [**35**].*

4. APPLICATIONS

4.1 Variational Inclusion Problems

The variational inclusion problem is to find a point $u\in\mathcal{H}$ such that

$$0\in\mathcal{A}u+\mathcal{B}u, \tag{4.6}$$

where $\mathcal{A}:\mathcal{H}\to\mathcal{H}$ is a single-valued operator and $\mathcal{B}:\mathcal{H}\to 2^{\mathcal{H}}$ is a set-valued operator. The set of solutions to this problem, referred to as the zero set of $\mathcal{A}+\mathcal{B}$, is denoted by

$$zer(\mathcal{A}+\mathcal{B})=\{u\in\mathcal{H}:0\in\mathcal{A}u+\mathcal{B}u\}.$$

The resolvent operator of $\mathcal{B}$, denoted by $J_\zeta^{\mathcal{B}}$, is defined as

$$J_\zeta^{\mathcal{B}}=(I+\zeta\mathcal{B})^{-1},$$

where $\zeta>0$ and I is the identity operator on $\mathcal{H}$.

An operator $\mathcal{B} : \mathcal{H} \to 2^{\mathcal{H}}$ is called monotone if for all $(a, u), (b, v) \in \text{Graph}(\mathcal{B})$, the following condition holds:

$$\langle u - v, a - b\rangle \geq 0,$$

where the graph of $\mathcal{B}$, denoted $\text{Graph}(\mathcal{B})$, is given by

$$\text{Graph}(\mathcal{B}) = \{(u, v) \in \mathcal{H} \times \mathcal{H} : u \in \text{dom}(\mathcal{B}),\ v \in \mathcal{B}(u)\}.$$

An operator $\mathcal{B}$ is called maximally monotone if its graph is not strictly contained within the graph of any other monotone operator. In other words, $\mathcal{B}$ is maximally monotone if there is no other monotone operator $\mathcal{A}$ whose graph strictly contains $\text{Graph}(\mathcal{B})$.

If $\mathcal{A}$ is $\mathcal{L}-$Lipschitz and monotone operator and $\mathcal{B}$ is maximally monotone, then by setting $\mathcal{S}_k = J^{\mathcal{B}}_{\zeta_k}(I - \zeta_k\mathcal{A})$, where $\zeta_k \in (c, \frac{2}{\mathcal{L}})$ for some $c > 0$, it follows from [**19**] that $\mathcal{S}_k$ satisfies NST-condition(I) with $\mathcal{S}_\zeta = J^{\mathcal{B}}_{\zeta}(I - \zeta\mathcal{A})$ where $\zeta \in (0, \frac{2}{\mathcal{L}})$. Consequently, we obtain the following result:

Corollary 4.9. *Let $\mathcal{H}$ be a Hilbert space. Let $\mathcal{A} : \mathcal{H} \to \mathcal{H}$ be $\mathcal{L}$ − Lipschitz and monotone and $\mathcal{B} : \mathcal{H} \to 2^{\mathcal{H}}$ be maximally monotone. Let $\zeta \in (0, \frac{2}{\mathcal{L}})$ and $\{\zeta_k\} \subset (c, \frac{2}{\mathcal{L}})$ for some $c > 0$. Suppose that zer$(\mathcal{A}+\mathcal{B}) \neq \emptyset$. Let $\{u_k\}$ be a sequence in $\mathcal{H}$ defined by: $u_{1-b}, u_{2-b}, ..., u_0, u_1 \in \mathcal{H}$ and*

$$\begin{aligned} w_k &= u_k + \sum_{s=1}^{b} \delta_{s,k}(u_{k-s+1} - u_{k-s}), \\ z_k &= \gamma_k u_k + (1 - \gamma_k) J^{\mathcal{B}}_{\zeta_k}(I - \zeta_k\mathcal{A})w_k, \\ y_k &= \beta_k z_k + (1 - \beta_k) J^{\mathcal{B}}_{\zeta_k}(I - \zeta_k\mathcal{A})z_k, \\ u_{k+1} &= \alpha_k y_k + (1 - \alpha_k) J^{\mathcal{B}}_{\zeta_k}(I - \zeta_k\mathcal{A})y_k, \end{aligned}$$

where $\delta_{s,k} \geq 0$, $\sum_{k=1}^{\infty} \delta_{s,k} < +\infty$, $0 < \liminf_{k\to\infty} \gamma_k \leq \limsup_{k\to\infty} \gamma_k < 1$, $0 < \liminf_{k\to\infty} \beta_k \leq \limsup_{k\to\infty} \beta_k < 1$ and $0 < \alpha_k < 1$. Then $\{u_k\}$ converges weakly to a point in zer$(\mathcal{A} + \mathcal{B})$.

Next, we present a numerical example in $\mathbb{R}^3$, focusing primarily on studying the variational inclusion problem as discussed in Collorary 4.9. We will begin by comparing the results from various choices of inertial methods, followed by a performance comparison of our algorithm with Noor Algorithm, SP Algorithm, DHLCR Algorithm, and ZDC Algorithm, which yield the following results.

Example 4.10. *Let $\mathcal{A} : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by $\mathcal{A}v = 4v + (1, 1, 1)$ and let $\mathcal{B} : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by $\mathcal{B}v = 3v$ where $v = (v_1, v_2, v_3) \in \mathbb{R}^3$.*

We see that $\mathcal{A}$ is $4-$Lipschitz and monotone operator and $\mathcal{B}$ is a maximally monotone operator. Set starting points $u_{-5} = (-4, -7, -1.3)$, $u_{-4} = (0.4, 2, 2.4)$, $u_{-3} = (3, -1, 1)$, $u_{-2} = (2, 4, 3.5)$, $u_{-1} = (-7, 8, 9)$, $u_0 = (0, 0, 0)$, $u_1 = (2, 2, 2)$ and the stopping criterion is $D_k = \frac{1}{2}\|u_k - J^{\mathcal{B}}_{\zeta_k}(I - \zeta_k\mathcal{A})u_k\| < 10^{-5}$.

We choose $\zeta_k = 0.2,\ \gamma_k = 0.4,\ \beta_k = 0.1,\ \alpha_k = 0.1,\ \delta_{1,k} = \frac{1}{(5k+1)^3},\ \delta_{2,k} = \frac{1}{(6k+1)^4},\ \delta_{3,k} = \frac{1}{(2k+1)^{10}},\ \delta_{4,k} = \frac{1}{(7k+1)^5}$ and $\delta_{5,k} = \frac{1}{(8k+1)^5}$. For Noor Algorithm, we set $\beta_k = \frac{1}{(4k^2+1)^3},\ \alpha_k = \frac{1}{(3k^2+1)^4}$ and $\theta_k = 0.7$. For SP Algorithm, we use $\beta_k = \frac{1}{(5k^2+1)^3},\ \alpha_k = \frac{1}{(2k^2+1)^4}$ and $\theta_k = 0.8$. For DHLCR Algorithm, we choose $\delta_1 = 1.1, \delta_2 = -0.3, \delta_3 = 0.6, \vartheta_1 = 1.8, \vartheta_2 = -0.5, \vartheta_3 = 1, a = 0.01, b = 0.01, \delta = 0.9, \sigma = 0.001, \xi = 0.01$ and $\lambda = 0.98$. For ZDC Algorithm, we choose $\delta_1 = 0.2, \delta_2 = 0.3, \delta_3 = -0.6$ and $\gamma = 1.2$.

Table 1: Numerical comparison of different inertial with number of iterations and CPU time in Example 4.10

	1 inertial	2 inertial	3 inertial	4 inertial	5 inertial
Number of iterations	5	4	4	4	4
CPU time(s)	0.0126	0.0014	0.0013	0.0017	0.0017

Table 2: Numerical comparison of SPMI Algorithm and other algorithm in Example 4.10

	SPMI	Noor	SP	DHLCR	ZDC
Number of iterations	4	14	11	23	18
CPU time(s)	0.0013	0.0017	0.0017	0.0018	0.0044

Table 1 shows that using only a single inertial step requires more iterations and a higher CPU time compared to the multi-step cases. In contrast, when two or more inertial steps are employed, the number of iterations decreases to four and the CPU time is significantly reduced, with the three-step inertial scheme achieving the shortest execution time. This suggests that multi-step inertial techniques can significantly improve computational efficiency. Table 2 presents a comparison between our main algorithm and several other iterative schemes. The results reveal that our method attains solutions with fewer iterations and shorter computational time, confirming its efficiency and potential.

Figure 1 shows the numerical error plots in Example 4.10 for different inertial steps. It can be seen that the one-step inertial case decreases more slowly than the multi-step cases, which highlights the advantage of using multiple inertial terms. Figure 2 compares SPMI algorithm with Noor, SP, DHLCR, and ZDC methods. SPMI algorithm converges most rapidly, reaching low error levels in only a few iterations. Other algorithms converge more gradually, each exhibiting distinct convergence patterns.

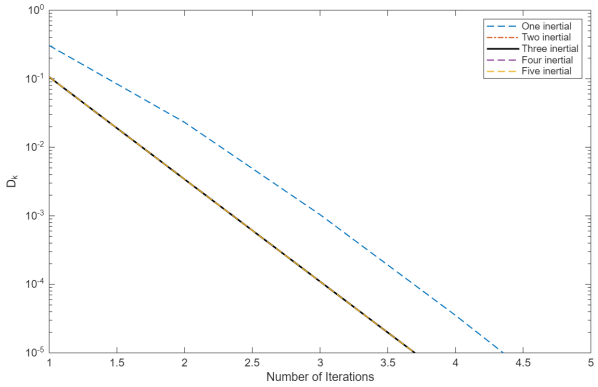

Figure 1: Numerical error plots in Example 4.10 for different values of $\delta_{s,k}$

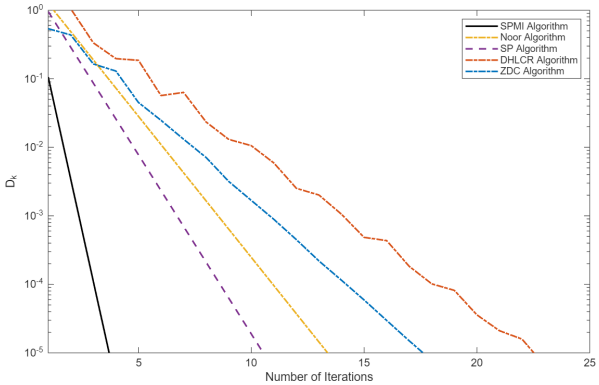

Figure 2: Numerical error plots of SPMI Algorithm and other algorithms in Example 4.10

4.2 Split Feasibility Problems

Let $\mathcal{K}$ and $\mathcal{Q}$ be nonempty, closed, and convex subsets of the real Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively, and let $\mathcal{D} : \mathcal{H}_1 \to \mathcal{H}_2$ be a bounded linear operator, the adjoint of which is denoted by $\mathcal{D}^T : \mathcal{H}_2 \to \mathcal{H}_1$. We now recall the split feasibility problems (SFP):

$$\text{find } u \in \mathcal{K} \text{ such that } \mathcal{D}u \in \mathcal{Q}. \tag{4.7}$$

The set $\mathcal{X} := \{u \in \mathcal{K} : \mathcal{D}u \in \mathcal{Q}\}$ is denoted for the solution set of (4.7). Let f be convex, differentiable and weakly lower semicontinuous on $\mathcal{H}_1$ and $f : \mathcal{H}_1 \to \mathbb{R}$ be a function defined by $f(u) = \frac{1}{2}\|\mathcal{D}u - \mathcal{P}_{\mathcal{Q}}\mathcal{D}u\|^2$ for all $u \in \mathcal{H}_1$. Hence, we have $\nabla f(u) = \mathcal{D}^T(I - \mathcal{P}_{\mathcal{Q}})\mathcal{D}u$ for all $u \in \mathcal{H}_1$. We know that ∇f is $\frac{1}{\|\mathcal{D}\|^2}$-inverse strongly

monotone [**7**]. In 2002, Byrne [**6**] introduced iterative method for solving the (4.7) by:

$$u_{k+1} = \mathcal{P}_{\mathcal{K}}(u_k - \zeta_k \nabla f(u_k)),$$

where $0 < \zeta_k < \dfrac{2}{\|\mathcal{D}\|^2}$. So we have the following result:

Corollary 4.11. *Let $\mathcal{K}$ and $\mathcal{Q}$ be nonempty, closed, and convex subsets of the real Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively, and let $\mathcal{D} : \mathcal{H}_1 \to \mathcal{H}_2$ be a bounded linear operator. Assume that $\mathcal{X} \neq \emptyset$. Let $\{u_k\}$ be a sequence in $\mathcal{H}$ defined by*

$$\begin{aligned} w_k &= u_k + \sum_{s=1}^{b} \delta_{s,k}(u_{k-s+1} - u_{k-s}), \\ z_k &= \gamma_k u_k + (1-\gamma_k)\mathcal{P}_{\mathcal{K}}(w_k - \zeta_k \nabla f(w_k)), \\ y_k &= \beta_k z_k + (1-\beta_k)\mathcal{P}_{\mathcal{K}}(z_k - \zeta_k \nabla f(z_k)), \\ u_{k+1} &= \alpha_k y_k + (1-\alpha_k)\mathcal{P}_{\mathcal{K}}(y_k - \zeta_k \nabla f(y_k)), \end{aligned}$$

where $\delta_{s,k} \geq 0$, $\sum_{k=1}^{\infty} \delta_{s,k} < +\infty$, $0 < \liminf_{k\to\infty} \gamma_k \leq \limsup_{k\to\infty} \gamma_k < 1$, $0 < \liminf_{k\to\infty} \beta_k \leq \limsup_{k\to\infty} \beta_k < 1$ and $0 < \alpha_k < 1$. Then $\{u_k\}$ converges weakly to a point in $\mathcal{X}$.

Next, we present a numerical example in $\mathbb{R}^2$, focusing primarily on studying the split feasibility problem as discussed in Collorary 4.11. We will begin by comparing the results from various choices of inertial methods, followed by a performance comparison of our algorithm with Noor Algorithm, SP Algorithm, DHLCR Algorithm, and ZDC Algorithm, which yield the following results.

Example 4.12. *Let $f(x_1, x_2) = x_1^3 + 2x_2^2 + 1$, and $\mathcal{C} = \{u \in \mathbb{R}^2 : \|u - (2,3)\| \leq 5\}$. Then*

$$\mathcal{P}_{\mathcal{C}}(u) = \begin{cases} (2,3) + \dfrac{5}{\|u-(2,3)\|}(u-(2,3)) \ , & \text{if } \|u-(2,3)\| > 5; \\ u, & \text{if } \|u-(2,3)\| \leq 5. \end{cases}$$

The starting point $u_{-5} = (-2.8, -7)$, $u_{-4} = (0.5, 7)$, $u_{-3} = (3, -2)$, $u_{-2} = (4, 5)$, $u_{-1} = (-2, 9)$, $u_0 = (-1, -9)$, $u_1 = (1, 2)$ and the stopping criterion is $D_k = \dfrac{1}{2}\|u_k - \mathcal{P}_{\mathcal{C}}(u_k - \zeta_k \nabla f(u_k))\| < 10^{-5}$.

We also choose Lipschitz constant $L = \sqrt{\max\{36M^2, 16\}}$ where $M = \max\{|x_1|, |x_2|\}$ and $\zeta_k = \frac{1.99k}{(k+1)L}$, $\gamma_k = 0.45$, $\beta_k = 0.7$, $\alpha_k = 0.7$, $\delta_{1,k} = \frac{1}{(10k+1)^3}$ if $1 \leq k \leq 300$, we set $s_1 = 1$, $\delta_{2,k} = \frac{s_k - 1}{s_{k+1}}$ where $s_{k+1} = \dfrac{1 + \sqrt{1 + 4s_k^2}}{2}$ and set $\delta_{2,k} = \frac{1}{(5k+1)^2}$, if $k > 300$, $\delta_{3,k} = \frac{1}{(7k+1)^8}$, $\delta_{4,k} = \frac{1}{(2k+1)^5}$ and $\delta_{5,k} = \frac{1}{(6k+1)^{10}}$. For Noor Algorithm, we set $\beta_k = \frac{1}{(4k^2+1)^3}$, $\alpha_k = \frac{1}{(3k^2+1)^4}$ and $\theta_k = \frac{1}{k^{0.2}+1}$. For

SP Algorithm, we use $\beta_k = \frac{1}{(5k^2+1)^3}$, $\alpha_k = \frac{1}{(2k^2+1)^4}$ and $\theta_k = \frac{1}{k^{0.2}+1}$. For DHLCR Algorithm, we choose $\delta_1 = 1.1, \delta_2 = -0.3, \delta_3 = 0.6, \vartheta_1 = 1.8, \vartheta_2 = -0.5, \vartheta_3 = 1, a = 0.01, b = 0.01, \delta = 0.9, \sigma = 0.001, \xi = 0.01$ and $\lambda = 0.98$. For ZDC Algorithm, we choose $\delta_1 = 0.1, \delta_2 = 0.1, \delta_3 = 0.1, \delta = 10, \alpha = 0.02, \sigma = 0.95$ and $\gamma = 0.9$.

Table 3: Numerical comparison of different inertial with number of iterations and CPU time in Example 4.12

	1 inertial	2 inertial	3 inertial	4 inertial	5 inertial
Number of iterations	13	9	6	6	6
CPU time(s)	0.0124	0.0045	0.0040	0.0047	0.0060

Table 4: Numerical comparison of SPMI Algorithm and other algorithms in Example 4.12

	SPMI	Noor	SP	DHLCR	ZDC
Number of iterations	6	12	10	40	142
CPU time(s)	0.0040	0.0043	0.0042	0.0044	0.0044

Table 3 presents the comparison of the number of iterations and computation times for our main algorithm, considering inertial values ranging from 1 to 5. The results indicate that the choice of 3 inertial is the most suitable, as it yields the lowest number of iterations and requires less computation time than 4 inertial and 5 inertial.

Subsequently, Table 4 presents a comparison of the results between our main algorithm (SPMI Algorithm), which uses 3 inertial, and other algorithms, including Noor Algorithm, SP Algorithm, DHLCR Algorithm, and ZDC Algorithm. The comparison is made in terms of computation times and the number of iterations. The results demonstrate that the SPMI Algorithm requires less computation times and fewer iterations compared to the other algorithms.

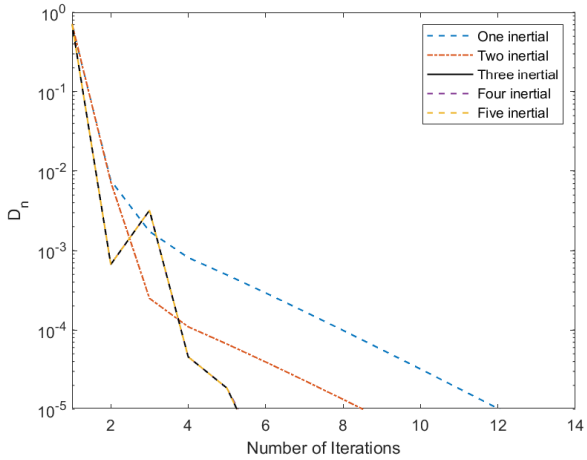

Figure 3: Numerical error plots in Example 4.12 for different values of $\delta_{s,k}$

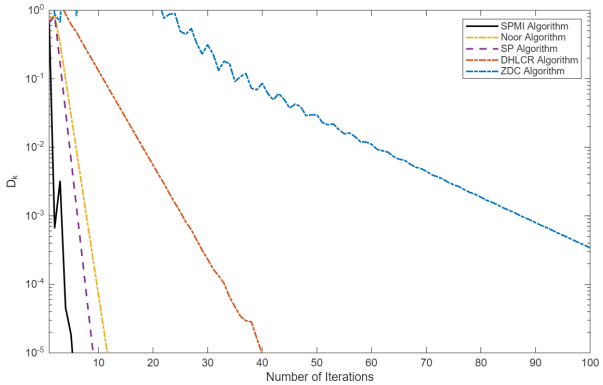

Figure 4: Numerical error plots of SPMI Algorithm and other algorithms in Example 4.12

Figure 3 illustrates the graph of the fixed point error against the number of iterations, focusing primarily on analysing our main algorithm with inertial values ranging from 1 to 5. The results indicate that when using 3 inertial, tolerance 10^{-5} is achieved at the 6th iteration, which is fewer iterations than 1 inertial and 2 inertial but equal to 4 inertial and 5 inertial.

Figure 4 presents a graph illustrating the relationship between the fixed point error and the number of iterations, comparing SPMI Algorithm (with 3 inertial) with Noor Algorithm, SP Algorithm, DHLCR Algorithm, and ZDC Algorithm. The results demonstrate that at 10^{-5} tolerance, the SPMI Algorithm achieves fewer iterations than the other algorithms.

5. NUMERICAL EXPERIMENT IN SIGNAL RECOVERY PROBLEM

In signal recovery problem, compressed sensing in the frequency domain can be modeled by the following linear equation system, where $u \in \mathbb{R}^M$ is the original signal and $y \in \mathbb{R}^N$ is the observed data:

$$y = \mathcal{W}u + \epsilon, \tag{5.8}$$

where $\mathcal{W} \in \mathbb{R}^{N\times M}$ (with $M > N$), and $\epsilon \in \mathbb{R}^N$ is noisy. To solve this equation (5.8), we can formulate the problem as a LASSO (Least Absolute Shrinkage and Selection Operator) optimization problem:

$$\min_{u\in\mathbb{R}^M} \frac{1}{2}\|\mathcal{W}u - y\|_2^2 + \rho\|u\|_1, \tag{5.9}$$

where $\rho > 0$ is a regularization parameter.

In this connection, (5.9) can be transformed to the following fixed point problem:

$$\mathcal{S}_k u = \text{prox}_{\zeta_k g}(u - \zeta_k \nabla f(u)), \qquad k \geq 1,$$

where $f(u) = \frac{1}{2}\|\mathcal{W}u - y\|_2^2$, $g(u) = \rho\|u\|_1$, $\nabla f(u) = \mathcal{W}^T(y - \mathcal{W}u)$ and $\zeta_k \in \left(0, \frac{2}{\|\mathcal{W}\|_2^2}\right)$.

In this experiment, y is generated by Gaussian noise with SNR = 40, The matrix $\mathcal{W}$ is sampled from a normal distribution with a mean of zero and a variance of one. The vector $u \in \mathbb{R}^M$, containing m nonzero components for recovery, is generated from a uniform distribution over the interval $[-2, 2]$.

We now present a numerical solution to (5.9) and analyze the performance of SPMI Algorithm. Additionally, we compare its behavior with three other algorithmic schemes: SP, DHLCR, and ZDC.

Let M and N be the size of signal. The inertial points $u_{-5}, u_{-4}, u_{-3}, u_{-2}, u_{-1}$, u_0, u_1 are *randn* $(M,\ 1)$. We selected $\zeta_k = \dfrac{1.9k}{(k+1)\|\mathcal{W}\|_2^2}$, $\gamma_k = 0.04$, $\beta_k = 0.07$, $\alpha_k = 0.07$, $\delta_{1,k} = \dfrac{1}{(10k+1)^3}$ if $1 \leq k \leq 300$, we set $s_1 = 1$, $\delta_{2,k} = \dfrac{s_k - 1}{s_{k+1}}$ where $s_{k+1} = \dfrac{1 + \sqrt{1 + 4s_k^2}}{2}$ and set $\delta_{2,k} = \dfrac{1}{(5k+1)^3}$, if $k > 300$, $\delta_{3,k} = \dfrac{1}{(7k+1)^100}$, $\delta_{4,k} = \dfrac{1}{(2k+1)^5}$ and $\delta_{5,k} = \dfrac{1}{(6k+1)^{10}}$. For SP Algorithm, we use $\zeta_k = \dfrac{1.9}{\|\mathcal{W}\|_2^2}$, $\beta_k = \dfrac{k+1}{(7k+2)}$, $\alpha_k = \dfrac{k+1}{(5k+2)^4}$ and $\theta_k = \dfrac{1}{(k+1)^{0.1}}$. For DHLCR Algorithm, we choose $\zeta_k = \dfrac{1}{\|\mathcal{W}\|_2^2}, \delta_1 = 0.01, \delta_2 = -0.01, \delta_3 = 0.02, \vartheta_1 = 0.2, \vartheta_2 = -0.03, \vartheta_3 = 0.01, a = 0.01, b = 0.01, \delta = 0.9, \sigma = 0.001, \xi = 0.01$ and $\lambda = 0.98$.

For ZDC Algorithm, we choose $\zeta_k = \dfrac{1}{\|\mathcal{W}\|_2^2}, \delta_1 = 0.01, \delta_2 = -0.01, \delta_3 = 0.02, \delta = 10, \alpha = 0.02, \sigma = 0.98$ and $\gamma = 0.9$.

The performance is measured by mean-squared error (MSE), which is used to evaluate the performance of a model or algorithm by measuring the difference between predicted values and actual values. If MSE values are low, this indicates high prediction accuracy. The MSE is calculated as:

$$MSE = \frac{1}{N} \|u - u_k\|^2 < 10^{-4},$$

where u_k is the approximated signal of u.

Moreover, we used the signal-to-noise ratio (SNR) to evaluate the quality of a signal by comparing the level of the desired signal to the level of background noise. If a high SNR this mean that indicates that the desired signal has much higher power than the noise, signifying good signal quality, the SNR can be calculated as:

$$SNR = 10 \log_{10} \frac{\|u\|}{\|u - u_k\|},$$

where u_k is the approximated signal of u.

Next, we analyze the convergence behavior of the algorithms used. We begin by displaying the original signal and the observed data distorted by noise. Then, we present the results of the recovered signal obtained from the processing. By comparing the recovered signal with the original signal, we evaluate the performance of the algorithms using MSE and SNR. These results are presented alongside the number of iterations for each algorithm to demonstrate the convergence speed and accuracy in signal recovery.

Table 5 provides the performance of our main algorithm (considering inertial values ranging from 1 to 5) and other algorithms. The results are analyzed in terms of the number of iterations and computation time. The table shown signals of sizes M=512, N=256, and M=1024, N=512, with the number of signal bars set to m=30 and m=50, respectively. The results indicate that our algorithm use fewer iterations and less computation time than the other algorithms in all cases considered. Furthermore, it was observed that when the inertial value is set to 3 or higher, the number of iterations remains the same, but the computation time steadily increases.

Table 5: Comparison of CPU Time and number of iterations

sparse signal	method	M=512, N=256		M=1024, N=512	
		No. iter	CPU(s)	No. iter	CPU(s)
m = 30	SPMI				
	- 1 inertial	404	2.9182	701	30.0389
	- 2 inertial	145	1.0437	206	10.2555
	- 3 inertial	72	0.5291	99	3.9943
	- 4 inertial	72	0.5316	99	4.1901
	- 5 inertial	72	0.5366	99	4.2598
	SP	1265	9.8535	2288	116.8671
	DHLCR	2163	16.3615	3789	195.0759
	ZDC	2205	18.9536	3862	194.1483
m = 50	SPMI				
	- 1 inertial	639	4.5945	829	42.0769
	- 2 inertial	155	1.1298	210	11.9705
	- 3 inertial	89	0.6279	105	5.8026
	- 4 inertial	89	0.6653	105	6.0328
	- 5 inertial	89	0.6811	105	6.3367
	SP	2071	15.3556	2737	128.4702
	DHLCR	3419	25.1762	4462	209.4379
	ZDC	3489	27.7235	4547	224.3200

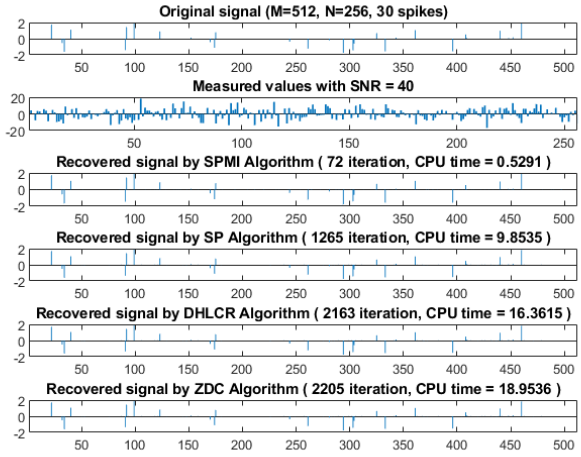

Figure 5: From top to bottom: original signal, observed data, recovered signal by SPMI, SP, DHLCR and ZDC with M = 512, N = 256 and m = 30

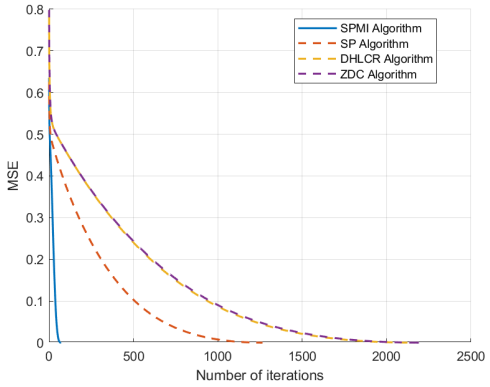

Figure 6: The graph of MSE and number of iterations with M = 512, N = 256 and m = 30

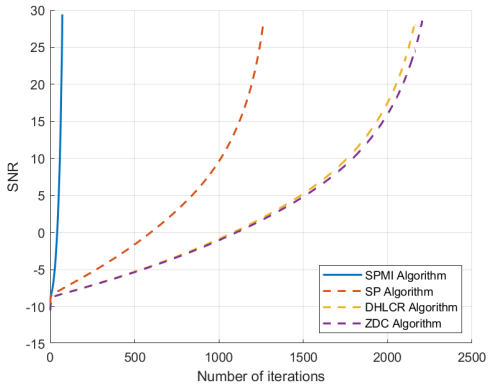

Figure 7: The graph of SNR and number of iterations with M = 512, N = 256 and m = 30

The graph in Figure 5 shows the original signal, observed data, and the recovered signal using various algorithms, with M=512, N=256, and m=30. Each row of the graph shows the results from different algorithms. The recovery by the SPMI Algorithm (with 3 inertial) yields the closest match to the original signal and takes the least time, requiring only 72 iterations (CPU time = 0.5291). In contrast, the other algorithms require significantly more iterations and computation time to recover a signal.

Figure 6 and Figure 9 show the relationship between MSE value and number of iterations. The result demonstrates that SPMI Algorithm outperforms other algorithms in terms of iterations and CPU time.

The graph in Figure 7 and Figure 10 shows the relationship between SNR value and number of iterations. The results indicate that our algorithm is the most

efficient in increasing the SNR compared to the others.

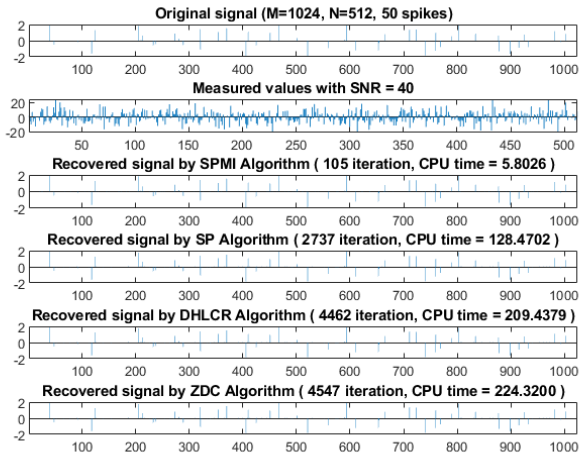

Figure 8: From top to bottom: original signal, observed data, recovered signal by SPMI, SP, DHLCR and ZDC with M = 1024, N = 512 and m = 50

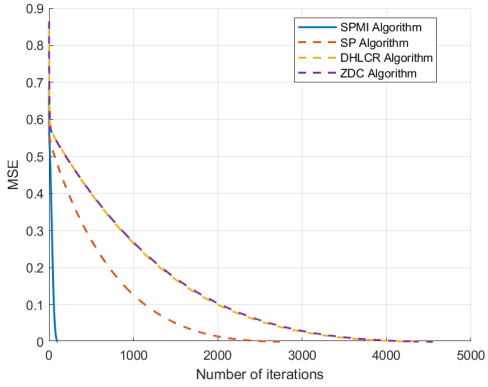

Figure 9: The graph of MSE and number of iterations with M = 1024, N = 512 and m = 50

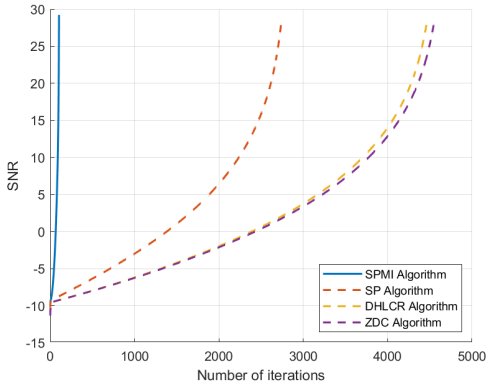

Figure 10: The graph of SNR and number of iterations with M = 1024, N = 512 and m = 50

Figure 8 presents results similar to that in Figure 5, with SPMI Algorithm continuing to demonstrate superior performance by requiring less time and fewer iterations than other algorithms. However, since the signal in Figure 8 has a size of M=1024, N=512 and m=50, which is more complex and contains more data than the signal in Figure 5, it results in higher time consumption and a more significant number of iterations. In conclusion, the increased complexity and number of data points in the signal significantly impact the performance and recovery time of algorithms. Nonetheless, the SPMI Algorithm consistently stands out for its speed and efficiency across all cases.

6. CONCLUSION

In this research, we have proposed a recent fixed point method for a family of nonexpansive mappings. Our approach modified multi-step inertial extrapolations with SP iteration to accelerate the algorithm's convergence speed. This strategy makes finding the fixed point faster and more efficient. Additionally, we have established a weak convergence theorem under mild assumptions. Furthermore, we have connected the fixed point problem with the variational inclusion problem and the split feasibility problem. In terms of numerical testing for signal recovery, the results obtained from the experiments demonstrate the superiority of our algorithm over others in the literature, particularly in terms of convergence speed and the quality of the results. The experiments show that the choice of inertial parameters significantly impacts the algorithm's performance. The best results were obtained by selecting a 3 inertial, which shows that substantial inertial values do not necessarily lead to better outcomes. Choosing a suitable inertial value requires a manageable amount of values to achieve the best results.

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