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A DOUBLE INERTIAL PARALLEL MANN ALGORITHM FOR FINITE NONEXPANSIVE MAPPINGS IN BLIND IMAGE RESTORATION

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This research introduces a novel approach for blind image restoration using a fixed point algorithm enhanced by inertial and parallel computation for a family of nonexpansive mappings. A weak convergence theorem is established in Hilbert spaces under suitable inertial conditions. Numerical results support the theoretical findings and demonstrate that the proposed method achieves superior convergence compared to existing techniques.

1. INTRODUCTION AND PRELIMINARIES

In this paper, we use the following notation: $\mathcal{H}$ denotes a real Hilbert space with $\|\cdot\|$ and $\langle\cdot,\cdot\rangle$ as norm and inner product, respectively, and let $\mathcal{C}$ be a nonempty subset of $\mathcal{H}$. A mapping $\mathcal{T}:\mathcal{C}\to\mathcal{C}$ is nonexpansive, if

$$\|\mathcal{T}u-\mathcal{T}v\|\leq\|u-v\| \quad \text{for all } u,v\in\mathcal{C}.$$

The fixed point problem is to find $u\in\mathcal{C}$ such that

$$u=\mathcal{T}u.$$

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Here u is called a fixed point of $\mathcal{T}$ and the set of all fixed points of $\mathcal{T}$ is denoted by $F(\mathcal{T})$. Fixed point theory is widely recognized in various branches of optimization. Its usefulness extends to numerous fields, including science, applied science, machine learning, engineering, medicine, economics, data classification and image/signal processing, see [**15, 7, 9**]. The approximation of fixed points for nonexpansive mappings was initiated by Browder [**6**], Göhde [**13**], and Kirk [**17**]. Original iterative for solving fixed point problems is the Picard iteration scheme [**24**]. Given $u_1 \in \mathcal{C}$ and

$$u_{k+1} = \mathcal{T}u_k, \qquad k \geq 1.$$

A classical one is Mann iteration [**21**], which is defined by $u_1 \in \mathcal{C}$ and

$$u_{k+1} = \alpha_k u_k + (1-\alpha_k)\mathcal{T}u_k, \quad k \geq 1.$$

It was proved that $\{u_k\}$ converges weakly to a fixed point of $\mathcal{T}$ when $\{\alpha_k\}$ is a sequence in $(0,1)$ and $\sum_{k=1}^{\infty} \alpha_k(1-\alpha_k) = \infty$.

Acceleration of the convergence of algorithms has been studied by many researchers. Polyak [**26**] introduced a new algorithm by adding inertial (momentum) terms to the gradient algorithm. The iterative process known as the heavy ball method is defined as follows: let $u_0 = u_{-1} \in \mathbb{R}^m$ and

$$\begin{aligned} v_k &= u_k + \alpha_k(u_k - u_{k-1}), \\ u_{k+1} &= v_k - \beta_k \nabla g(u_k), \quad k \geq 1, \end{aligned}$$

where $\alpha_k \in [0,1]$, $\beta_k > 0$ is a step-size parameter and $g : \mathbb{R}^m \to \mathbb{R}$ is a convex and differentiable function. Iyiola and Shehu [**14**] introduced a two-point inertial proximal algorithm to find the zeros of a maximal monotone operator. Given a set-valued operator $A : \mathcal{H} \to 2^{\mathcal{H}}$, and initial points $u_{-1}, u_0, v_1 \in \mathcal{H}$ and

$$\begin{aligned} v_{k+1} &= u_k + \theta(u_k - u_{k-1}) + \beta(u_{k-1} - u_{k-2}), \\ u_{k+1} &= J_\lambda^A(v_k), \quad k \geq 1, \end{aligned}$$

where the parameters β and θ satisfy certain conditions in [**14**], and $\lambda > 0$. It was shown that $\{u_k\}$ converges to a zero of A under mild conditions.

In recent years, the adoption of inertial techniques as acceleration methods has garnered considerable attention in research. Numerous authors have devised various rapid iterative algorithms with inertial techniques. Such that inertial extragradient [**10, 11**], inertial Mann algorithm [**29**], inertial forward-backward algorithm [**2**], and the fast iterative shrinkage thresholding algorithm (FISTA) [**5**]. These algorithms have been shown to be useful both in theoretical and numerical assessments. It has been developed for applications in areas such as image processing, signal recovery and machine learning. For further insights, we encourage readers to explore [**3, 9, 18, 19**], and the associated references therein.

Recent research has advanced iterative frameworks for approximating common fixed points. In particular, Ali and Ali [**1**] introduced the F^* algorithm, a

novel iterative scheme for approximating fixed points of weakly contractive mappings in Banach spaces, and further employed it to obtain approximate solutions to a nonlinear quadratic Volterra integral equation. Moreover, Zaheer et al. [**32**] proposed the Z algorithm, a three-step iteration for approximating fixed points of weakly contractive operators, and demonstrated its effectiveness by addressing a Caputo fractional delay differential equation.

Mainge [**20**] proposed the inertial Mann algorithm by the following manner : $u_{-1},\ u_0 \in \mathcal{H}$ and

$$\begin{aligned} w_k &= u_k + \alpha_k(u_k - u_{k-1}), \\ u_{k+1} &= \beta_k w_k + (1-\beta_k)\mathcal{T}w_k, \quad k \geq 1, \end{aligned}$$

where $\alpha_k \geq 0$ and $\beta_k \in (0,1)$. It was shown that $\{u_k\}$ converges weakly to a fixed point of $\mathcal{T}$ under mild assumptions.

Recently, Jun-on et al. [**15**] employed the inertial technique with a parallel approach for a family of nonexpansive mappings and applied it to signal recovery. Let $\{\mathcal{T}_i\}_{i=1}^N$ be a family of nonexpansive mappings on a nonempty set $\mathcal{C}$ such that $\bigcap_{i=1}^N F(\mathcal{T}_i) \neq \emptyset$. The iterative sequences $\{u_k\}$ and $\{w_k\}$ are generated by $u_{-1}, u_0 \in \mathcal{C}$, and

$$\begin{aligned} w_k &= u_k + \theta_k(u_k - u_{k-1}), \\ y_{i,k} &= (1-\beta_{i,k})w_k + \beta_{i,k}\mathcal{T}_i w_k, \\ z_{i,k} &= (1-\alpha_{i,k})\mathcal{T}_i w_k + \alpha_{i,k}\mathcal{T}_i y_{i,k}, \\ u_{k+1} &= \arg\max\{\|z_{i,k} - w_k\| : i = 1,2,\ldots,N\}, \quad k \geq 1, \end{aligned} \tag{1.1}$$

where $\{\theta_k\} \subset [0,\theta]$ for $\theta \in (0,1]$ and $\{\alpha_{i,k}\}$ and $\{\beta_{i,k}\}$ are sequences in $[0,1]$ for all $i = 1,2,...,N$.

Our main contribution is to design a new fixed point method that integrates Mann iteration with double inertial extrapolations to solve blind image debluring. The resulting algorithm is highly parallelizable and can be efficiently implemented on a GPU. We establish the weak convergence of this method for finding a common fixed point for a finite family of nonexpansive mappings.

We denote the weak convergence by $\rightharpoonup$ and strong convergence by $\to$.

Let $\{\mathcal{T}_i\}_{i=1}^N : \mathcal{H} \to \mathcal{H}$ be a finite family of nonexpansive mappings. A point u is called a common fixed point of $\mathcal{T}_i$ if $u \in \bigcap_{i=1}^N \mathcal{F}(\mathcal{T}_i)$.

A space $\mathcal{H}$ is said to satisfy the Opial property [**22**] if for each weakly convergent sequence $\{u_k\}$ in $\mathcal{H}$ with a weak limit z,

$$\liminf_{k\to\infty} \|u_k - z\| < \liminf_{k\to\infty} \|u_k - y\|$$

for all $y \in \mathcal{C}$ with $y \neq z$.

Lemma 1. [**12**] *Let $\mathcal{T}$ be a nonexpansive mapping on a closed and convex subset $\mathcal{C}$ of a Hilbert space $\mathcal{H}$. If $\mathcal{T}$ has a fixed point, then $I - \mathcal{T}$ is demiclosed; that is, whenever a sequence $\{u_k\}$ in $\mathcal{C}$ weakly converges to $u \in \mathcal{C}$ and the sequence $\{u_k - \mathcal{T}u_k\}$ strongly converges to y, it follows that $u - \mathcal{T}u = y$.*

In convergence analysis, we need the following lemma.

Lemma 2. [**23**] *Let $\{s_k\}, \{t_k\}$ and $\{\delta_k\}$ be nonnegative sequences such that*

$$s_{k+1} \leq (1+\delta_k)s_k + t_k, \qquad k \geq 1.$$

If $\sum_{k=1}^{\infty} \delta_k < +\infty$ and $\sum_{k=1}^{\infty} t_k < +\infty$, then $\lim_{k\to\infty} s_k$ exists.

2. MAIN RESULTS

In this section, we present a double inertial parallel Mann algorithm, and prove its weak convergence for finding a common fixed point of nonexpansive mappings.

Algorithm 1. *Double Inertial Parallel Mann Algorithm*

Initialization : *Let $u_{-1}, u_0, u_1 \in \mathcal{H}$ be arbitrary, $\theta_k,\ \sigma_k \in (-\infty, \infty)$, $\alpha_{i,k} \in (0,1)$ for all $i = 0, 1, ..., N$, with $\sum_{i=0}^{N} \alpha_i = 1$.*
Step 1 *For $k = 1$, compute*

$$v_k = u_k + \theta_k(u_k - u_{k-1}) + \sigma_k(u_{k-1} - u_{k-2}).$$

Step 2

$$u_{k+1} = \alpha_{0,k} v_k + \sum_{i=1}^{N} \alpha_{i,k} \mathcal{T}_i v_k.$$

Then, update $k = k + 1$ in ***Step 1.***

Theorem 3. *Let $\mathcal{H}$ be a real Hilbert space and let $\{\mathcal{T}_i\}_{i=1}^{N} : \mathcal{H} \to \mathcal{H}$ be nonexpansive for all $i = 1, 2, \ldots, N$ such that $\bigcap_{i=1}^{N} \mathcal{F}(\mathcal{T}_i)$ is nonempty. Let $\{u_k\}$ be defined by Algorithm 1. Assume that the following conditions are satisfied:*

(i) $\sum_{k=1}^{\infty} |\theta_k| \|u_k - u_{k-1}\| < \infty$;
(ii) $\sum_{k=1}^{\infty} |\sigma_k| \|u_{k-1} - u_{k-2}\| < \infty$;
(iii) $\liminf_{k\to\infty} \alpha_{0,k}\alpha_{i,k} > 0$ for all $i = 1, 2, ..., N$.

Then, $\{u_k\}$ converges weakly to a point in $\bigcap_{i=1}^{N} \mathcal{F}(\mathcal{T}_i)$.

Proof. Let $p \in \bigcap_{i=1}^{N} \mathcal{F}(\mathcal{T}_i)$. Then,

$$\begin{aligned} \|v_k - p\| &= \|u_k + \theta_k(u_k - u_{k-1}) + \sigma_k(u_{k-1} - u_{k-2}) - p\| \\ &\leq \|u_k - p\| + \|\theta_k(u_k - u_{k-1})\| + \|\sigma_k(u_{k-1} - u_{k-2})\| \\ &\leq \|u_k - p\| + |\theta_k|\|u_k - u_{k-1}\| + |\sigma_k|\|u_{k-1} - u_{k-2}\|. \end{aligned} \tag{2.2}$$

So we have

$$\begin{aligned}\|u_{k+1}-p\| &= \|\alpha_{0,k}v_k+\sum_{i=1}^{N}\alpha_{i,k}\mathcal{T}_iv_k-p\|\\ &\le \alpha_{0,k}\|v_k-p\|+\sum_{i=1}^{N}\alpha_{i,k}\|\mathcal{T}_iv_k-p\|\\ &\le \alpha_{0,k}\|v_k-p\|+\sum_{i=1}^{N}\alpha_{i,k}\|v_k-p\|\\ &= \|v_k-p\|. \qquad (2.3)\end{aligned}$$

Using (2.2) and (2.3), we obtain

$$\|u_{k+1}-p\| \le \|u_k-p\|+|\theta_k|\|u_k-u_{k-1}\|+|\sigma_k|\|u_{k-1}-u_{k-2}\|.$$

By Lemma 2 we have $\lim_{k\to\infty}\|u_k-p\|$ exists. This implies that $\{u_k\}$ is bounded. Moreover $\{v_k\}$ is also bounded.

Next, we show that $\{u_k\}$ converges weakly to a point in $\bigcap_{i=1}^{N}\mathcal{F}(\mathcal{T})$. We see that

$$\begin{aligned}\|v_k-p\|^2 &= \|u_k+\theta_k(u_k-u_{k-1})+\sigma_k(u_{k-1}-u_{k-2})-p\|^2\\ &\le \|u_k-p\|^2+2\langle\theta_k(u_k-u_{k-1})+\sigma_k(u_{k-1}-u_{k-2}),v_k-p\rangle. \qquad (2.4)\end{aligned}$$

Hence it follows that

$$\begin{aligned}\|u_{k+1}-p\|^2 &= \|\alpha_{0,k}v_k+\sum_{i=1}^{N}\alpha_{i,k}\mathcal{T}_iv_k-p\|^2\\ &= \alpha_{0,k}\|v_k-p\|^2+\sum_{i=1}^{N}\alpha_{i,k}\|\mathcal{T}_iv_k-p\|^2-\alpha_{0,k}\sum_{i=1}^{N}\alpha_{i,k}\|\mathcal{T}_iv_k-v_k\|^2\\ &\le \|v_k-p\|^2-\alpha_{0,k}\sum_{i=1}^{N}\alpha_{i,k}\|\mathcal{T}_iv_k-v_k\|^2. \qquad (2.5)\end{aligned}$$

Combining (2.4) and (2.5), we get

$$\begin{aligned}\|u_{k+1}-p\|^2 \le\ & \|u_k-p\|^2+2\langle\theta_k(u_k-u_{k-1})+\sigma_k(u_{k-1}-u_{k-2}),v_k-p\rangle\\ & -\alpha_{0,k}\sum_{i=1}^{N}\alpha_{i,k}\|\mathcal{T}_iv_k-v_k\|^2.\end{aligned}$$

So we have

$$\begin{aligned}\alpha_{0,k}\sum_{i=1}^{N}\alpha_{i,k}\|\mathcal{T}_iv_k-v_k\|^2 \le\ & \|u_k-p\|^2+2\langle\theta_k(u_k-u_{k-1})+\sigma_k(u_{k-1}-u_{k-2}),v_k-p\rangle\\ & -\|u_{k+1}-p\|^2.\end{aligned}$$

By assumptions (i)-(iii) and since $\lim_{k\to\infty}\|u_k - p\|$ exists, we have

$$\lim_{k\to\infty}\|\mathcal{T}_i v_k - v_k\| = 0, \qquad \text{for all} \quad i = 1, 2, ..., N. \tag{2.6}$$

Next, we also have

$$\begin{aligned}\lim_{k\to\infty}\|v_k - u_k\| &= \lim_{k\to\infty}\|u_k + \theta_k(u_k - u_{k-1}) + \sigma_k(u_{k-1} - u_{k-2}) - u_k\| \\ &\leq \lim_{k\to\infty}|\theta_k|\|u_k - u_{k-1}\| + \lim_{k\to\infty}|\sigma_k|\|u_{k-1} - u_{k-2}\| \\ &= 0.\end{aligned}$$

Since $\{u_k\}$ is bounded, there exists a subsequence $\{u_{k_i}\}$ of $\{u_k\}$ such that $\{u_{k_i}\}$ converges weakly to $z \in \mathcal{H}$. Arguing by contradiction, assume that $\{u_{k_j}\}$ is a subsequence of $\{u_k\}$ converging weakly to $q \in \mathcal{H}$. We will show that $q \neq z$. By Opial's property,

$$\begin{aligned}\lim_{k\to\infty}\|u_k - z\| &= \lim_{i\to\infty}\|u_{k_i} - z\| < \lim_{i\to\infty}\|u_{k_i} - q\| = \lim_{k\to\infty}\|u_k - q\| \\ &= \lim_{j\to\infty}\|u_{k_j} - q\| < \lim_{j\to\infty}\|u_{k_j} - z\| = \lim_{k\to\infty}\|u_k - z\|\end{aligned} \tag{2.7}$$

which is a contradiction. Hence $\{u_k\}$ converges weakly to z. So $\{v_k\}$ also converges weakly to z. By Lemma 1 and (2.6), we can conclude that $z \in \bigcap_{i=1}^{N}\mathcal{F}(\mathcal{T}_i)$. This completes the proof □

Remark 4. *(I) For solving the parallel minimization of one function as follow:*

$$\min_{x\in\mathcal{H}} f_i(u),$$

where $f_i : \mathcal{H} \to \mathbb{R} \cup \{\infty\}$ is a proper and convex function that is differentiable for all $i = 1, 2, ..., N$, we can set [**25**]

$$\mathcal{T}_i u = u - \lambda_i \nabla f_i(u),$$

with $\lambda_i \in (0, \frac{2}{L_i})$ [**4**]*, L_i is a Lipschitz constant of ∇f_i, and*

$$\nabla f_i(u) := A^t(Au - v).$$

where A is bounded linear operator and v is the observed or measured. It is known that if $\lambda_i \in \left(0, \frac{2}{L_i}\right)$, then $\mathcal{T}_i$ is nonexpansive.

(II) For applying the metric projection $P_{\mathcal{C}}$ to a constrained problem, we can set $\mathcal{T}_1 = P_{\mathcal{C}}$. In this case, we have $\mathcal{C} = \mathcal{F}(\mathcal{T}_1)$.

From the above remarks, we obtain two different algorithms to solve a constrained convex optimization problem. From Theorem 3, we directly get the following results.

Algorithm 2. *Double Inertial Parallel Mann Algorithm with a Projection*

Initialization : *Let* $u_{-1}, u_0, u_1 \in \mathcal{H}$ *be arbitrary,* $\theta_k,\ \sigma_k \in (-\infty, \infty),\ \alpha_{i,k} \in (0,1)$ *for all* $i = 0, 1, ..., N$, *with* $\sum_{i=0}^{N} \alpha_i = 1$.
Step 1 *For* $k = 1$, *we compute*

$$v_k = u_k + \theta_k(u_k - u_{k-1}) + \sigma_k(u_{k-1} - u_{k-2}).$$

Step 2

$$u_{k+1} = \alpha_{0,k} v_k + \alpha_{1,k} P_{\mathcal{C}} v_k + \sum_{i=2}^{N} \alpha_{i,k} \mathcal{T}_i v_k.$$

Then, set $k = k + 1$ *in* ***Step 1.***

Algorithm 3. *Double Inertial Parallel Gradient Algorithm*

Initialization : *Let* $u_{-1}, u_0, u_1 \in \mathcal{H}$ *be arbitrary,* $\theta_k,\ \sigma_k \in (-\infty, \infty),\ \alpha_{i,k} \in (0,1)$ *for all* $i = 0, 1, ..., N$, *with* $\sum_{i=0}^{N} \alpha_i = 1$.
Step 1 *For* $k = 1$, *we compute*

$$v_k = u_k + \theta_k(u_k - u_{k-1}) + \sigma_k(u_{k-1} - u_{k-2}).$$

Step 2

$$u_{k+1} = \alpha_{0,k} v_k + \alpha_{1,k} P_{\mathcal{C}} v_k + \sum_{i=2}^{N} \alpha_{i,k} \left(v_k - \lambda_i \nabla f_i(v_k)\right).$$

Then, set $k = k + 1$ *in* ***Step 1.***

To support our main theorem, we now present an example in $L_2[0,1]$, where the norm is defined by $\|u\| = \sqrt{\int_0^1 |u(t)|^2\, dt}$. The inner product is given by $\langle a, b\rangle = \int_0^1 a(t)b(t)\, dt$. The parameters θ_k and σ_k are defined by

$$\theta_k = \begin{cases} \frac{1}{k^2\|u_{k+1} - u_k\|}, & \text{if } u_{k+1} \neq u_k,\ k > M \\ \theta & \text{otherwise} \end{cases} \tag{2.8}$$

and

$$\sigma_k = \begin{cases} \frac{1}{k^2\|u_{k+1} - u_k\|}, & \text{if } u_{k+1} \neq u_k,\ k > M \\ \sigma & \text{otherwise.} \end{cases} \tag{2.9}$$

Next, we investigate the behavior of our algorithm and compare it with Algorithm (1.1), PISPMHA [**30**] and ISPA [**28**].

Example 5. *Let* $\mathcal{H} = L_2[0,1]$ *and define* $\mathcal{T}_1$ *and* $\mathcal{T}_2$ *by:* $\mathcal{T}_1 u(t) = \frac{2}{3}u(t)$ *and* $\mathcal{T}_2 u(t) = \frac{4}{7}u(t),\ \forall u(t) \in L_2[0,1]$.

We select $u_0(t) = \cos(t)$, $u_1(t) = e^t - 2$, $u_2(t) = \sin(t)$, and $\alpha_0 = 0.1,\ \alpha_1 = 0.45$ and $\alpha_2 = 0.45$. We consider the following cases:

- Case 1 : $\theta = 0.3$ and $\sigma = 0.05$.
- Case 2 : $\theta = 0.3$ and $\sigma = 0$.
- Case 3 : $\theta = 0$ and $\sigma = 0$.

We set $\beta_{i,k} = 0.1$, $\alpha_{i,k} = \frac{k}{k+1}$, and $\theta_k = 0.7$ in Algorithm (1.1). We set $\gamma_{i,k} = 0.7$, $\beta_{i,k} = 0.5$, $\alpha_{i,k} = 0.7$, and $\theta_k = 0.7$ in PISPMHA. We set $\gamma_{i,k} = 0.7, \rho_{i,k} = 0.5$, $\tau_{i,k} = 0.1$, and $\alpha_k = 0.7$ in ISPA. The algorithm terminates when the Cauchy error $E_k = \max(\|u_k - \mathcal{T}_i u_k\|) \leq 10^{-4}$.

Table 1: Computational results in Example 5.

		Number of iterations	CPU time
Algorithm 1	Case 1	16	1.3439
	Case 2	19	1.3584
	Case 3	32	1.5652
Algorithm (1.1)		20	1.4729
PISPMHA		27	1.9777
ISPA		27	1.9275

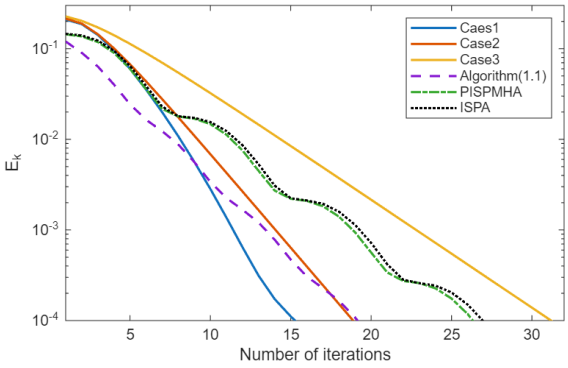


Figure 1: Errors and number of iterations.

From Table 1 and Figure 1, it can be seen that our algorithm in Case 1 converges faster than other methods in term of number of iterations and CPU time.

Example 6. *Let* $\mathcal{H} = L_2[0,1]$ *and* $\mathcal{C} = \{u(t) \in L_2[0,1] : \int_0^1 |u(t) - t^2|\, dt \leq 1\}$. *Define* $\mathcal{T}_2$ *and* $\mathcal{T}_3$ *by* $\mathcal{T}_2 u(t) = \frac{3}{7}u(t)$ *and* $\mathcal{T}_3(t) = \frac{1}{2}u(t)$ *for all* $u(t) \in L_2[0,1]$.

We select $u_0(t) = t^2$, $u_1(t) = \cos(t) - t$, $u_2(t) = e^{2t}$, and $\alpha_0 = 0.1$, $\alpha_1 = 0.3$, $\alpha_2 = 0.3$ and $\alpha_3 = 0.3$. We consider the following cases:

- Case 1 : $\theta = 0.21$ and $\sigma = 0.06$.
- Case 2 : $\theta = 0.21$ and $\sigma = 0$.
- Case 3 : $\theta = 0$ and $\sigma = 0$.

For Algorithm (1.1), we set $\beta_{i,k} = 0.1$, $\alpha_{i,k} = \frac{k}{k+1}$, and $\theta_k = 0.5$. For PISPMHA, we set $\gamma_{i,k} = 0.7$, $\beta_{i,k} = 0.5$, $\alpha_{i,k} = 0.7$, and $\theta_k = 0.5$. For ISPA, we set $\gamma_{i,k} = 0.7$, $\rho_{i,k} = 0.5$, $\tau_{i,k} = 0.1$, and $\alpha_k = 0.5$. The algorithm terminates when $E_k = \max(\|u_k - \mathcal{T}_i u_k\|) \leq 10^{-4}$.

Table 2: Computational results in Example 6.

		Number of iterations	CPU time
Algorithm 2	Case 1	11	1.2082
	Case 2	18	1.1631
	Case 3	25	1.4219
Algorithm (1.1)		13	5.3444
PISPMHA		12	1.1852
ISPA		13	1.1956

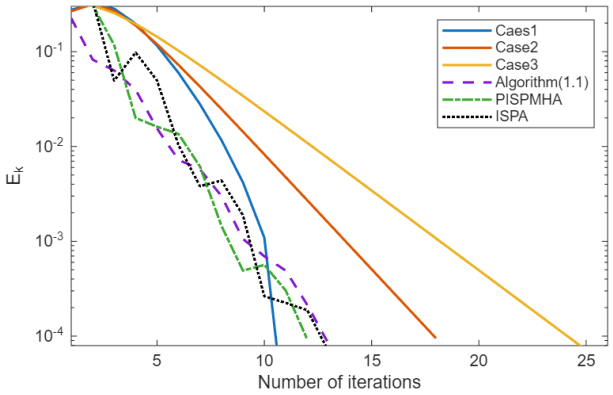


Figure 2: Errors and number of iterations.

From Table 2 and Figure 2, it can be observed that Case 1 attains the lowest number of iterations. In Case 3, with the choice $\theta_k = 0.2$ and $\sigma_k = 0$, our method achieves the fastest CPU time. However, the overall CPU times across all method are quite close to each other.

Example 7. *Let* $\mathcal{H} = L_2[0,1]$, $\mathcal{C} = \{u(t) \in L_2[0,1] : \int_0^1 |u(t) - e^{2t}|\, dt \leq 3\}$, *and define* $f_1(u(t)) = 2\langle u, x(t)\rangle$, $f_2(u(t)) = 3\langle u, x(t)\rangle$, $g_1(u(t)) = 4\|u(t)\|$, *and* $g_2(u(t)) = 5\|u(t)\|$.

We choose $u_0(t) = \cos(t)$, $u_1(t) = e^{t^2}$, $u_2(t) = \cos(t) + t^2$, and $\alpha_0 = 0.1$, $\alpha_1 = 0.1$, $\alpha_2 = 0.4$, and $\alpha_3 = 0.4$. The performance of Algorithm 3 is evaluated with the following cases:

- Case 1 : $\theta = 0.5$ and $\sigma = 0.03$.
- Case 2 : $\theta = 0.5$ and $\sigma = 0$.
- Case 3 : $\theta = 0$ and $\sigma = 0$.

For Algorithm (1.1), we set $\beta_{i,k} = 0.1$, $\alpha_{i,k} = \frac{k}{k+1}$, and $\theta_k = 0.4$. For PISPMHA, we set $\gamma_{i,k} = 0.7$, $\beta_{i,k} = 0.5$, $\alpha_{i,k} = 0.7$, and $\theta_k = 0.4$. For ISPA, we set $\gamma_{i,k} = 0.7$, $\rho_{i,k} = 0.5$, $\tau_{i,k} = 0.1$, and $\alpha_k = 0.4$. The algorithm terminates when $E_k = \max(\|u_k - \mathcal{T}_i u_k\|) \leq 10^{-4}$.

Table 3: Computational results in Example 7.

		Number of iterations	CPU time
Algorithm 3	Case 1	8	88.6997
	Case 2	8	71.5714
	Case 3	6	29.5233
Algorithm (1.1)		9	91.5449
PISPMHA		7	21.2415
ISPA		8	22.9977

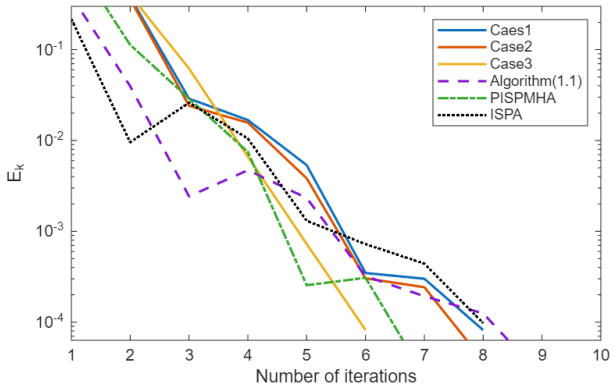


Figure 3: Errors and number of iterations.

From Table 3 and Figure 3, It can be observed that our algorithm in Case 3 requires the fewest number of iterations. However, in terms of CPU time, the

PISPMHA method performs best, requiring only 21.2415 seconds with a comparable number of iterations.

3. APPLICATIONS

In this section, we present a recent technique to restore image for blind case. The experiments were conducted by MATLAB R2022A on a computer with Intel(R) Core(TM) i9-9820X processor, graphics rtx2080 and 64.00 GB RAM.

Restoring blurred images requires techniques that can handle ambiguities in the degradation process. Optimization methods are commonly used to find the best approximation of the original image to address uncertainties in the data. Blurring typically refers to the loss of sharpness in specific image regions. This degradation arises due to various factors, including camera settings, subject movement, improper focus, or a narrow aperture, leading to a shallow depth of field.

Numerous studies have sought to resolve the challenge of image restoration without prior knowledge of the blurring operator. The four most common types of blur in digital imaging are average, Gaussian, out-of-focus, and motion. In this work, we concentrate specifically on motion blur, which occurs when there is relative movement between the camera and the subject, resulting in stretching effects along the motion direction.

The image recovery problem can be thought of as a linear equation given by:

$$b = Dx + \upsilon, \tag{3.10}$$

where an original image is denoted as $x \in \mathbb{R}^N$, a degraded image is $b \in \mathbb{R}^M$, a noise term is given by $\upsilon \in \mathbb{R}^M$, and $D \in \mathbb{R}^{M\times N}$ is a blurring matrix. It is worth noting that the problem (3.10) can be formulated equivalently as the subsequent convex minimization model:

$$\min_{x\in\mathbb{R}^N} \frac{1}{2}\|b - Dx\|_2^2$$

Recently, several researchers [**8, 16, 27, 31**] have studied in image restoration without knowing the specifics of the blurring matrix. This investigation has resulted in the following minimization problem:

$$\min_{x\in\mathbb{R}^N} \frac{1}{2}\|D_1x - b_1\|_2^2, \min_{x\in\mathbb{R}^N} \frac{1}{2}\|D_2x - b_2\|_2^2, ..., \min_{x\in\mathbb{R}^N} \frac{1}{2}\|D_Nx - b_N\|_2^2 \tag{3.11}$$

where x represent the original image, D_i is the blurring matrix, and b_i is the blurred image. However, this technique (3.11) has not yet been applied in practice. Therefore, we consider the problem as follows:

$$\min_{x\in\mathbb{R}^N} \frac{1}{2}\|D_1x - b\|_2^2, \min_{x\in\mathbb{R}^N} \frac{1}{2}\|D_2x - b\|_2^2, ..., \min_{x\in\mathbb{R}^N} \frac{1}{2}\|D_Nx - b\|_2^2. \tag{3.12}$$

In this case, D_i is randomly selected to solve the problem without prior knowledge of the true blurring operator. This approach aims to discover a more effective solution for recovering images in the blind case.

By Remark 4(I), we can use our Algorithm 3 to solve (3.12) by setting $f_i(x) = \frac{1}{2}\|D_i x - b\|_2^2$ with $\lambda_i \in \left(0, \frac{2}{\|D_i\|_2^2}\right)$.

We set D_i for a group of motion blur as follows:

$$\begin{aligned} D_1 &= (L, \Theta), & D_6 &= (L+1, \Theta-1),\\ D_2 &= (L+1, \Theta), & D_7 &= (L+2, \Theta-1),\\ D_3 &= (L+2, \Theta), & D_8 &= (L, \Theta+1),\\ D_4 &= (L, \Theta), & D_9 &= (L+1, \Theta+1),\\ D_5 &= (L, \Theta-1), & D_{10} &= (L+2, \Theta+1), \end{aligned}$$

where L is a len and Θ is an angle of motion blurs.

The parameters of Algorithm 3 are defined by $\alpha_{0,k} = 0.1$, $\alpha_{1,k} = \alpha_{2,k} = ... = \alpha_{10,k} = 0.09$, $\lambda_i = \dfrac{1.99}{\|D_i\|^2}$ for $i = 1, 2, ..., 10$, and θ_k, σ_k are defined as in (2.8) and (2.9) with $\theta = 0.95$ and $\sigma = -0.01$. The initial points u_0, u_1, u_2 are blurred images.

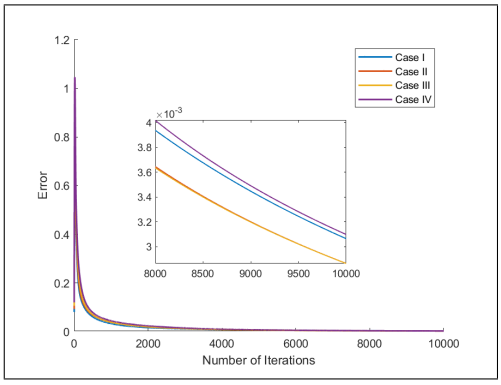
We consider 4 cases with $\Theta = 159$, 162, 165, 168, 171.

- Case I : $L = 57$;
- Case II : $L = 60$;
- Case III : $L = 63$;
- Case IV : $L = 66$.

First, we have the degraded image (license plate), which is a blurred image. We then apply our Algorithm 3 using groups of motion blurring matrix with different lens (L) and angles (Θ) to recover the image. Next, we present the blurred images with 10,000 iterations for our algorithm shown in Figures 4-8. Moreover, we show the graph of Cauchy error $\|u_{k+1} - u_k\| < 10^{-5}$ for confirming the convergence of our algorithm.

Figure 4 shows the results with $\Theta = 159$ and different values of L. Despite the algorithm's convergence, as indicated by the Cauchy error, the recovered image still fails to reveal the numbers and letters on the license plate clearly. Among the cases, case III suggests a marginally better performance in this figure.

(a) Degraded image



(b) Cauchy error

(c) Case I

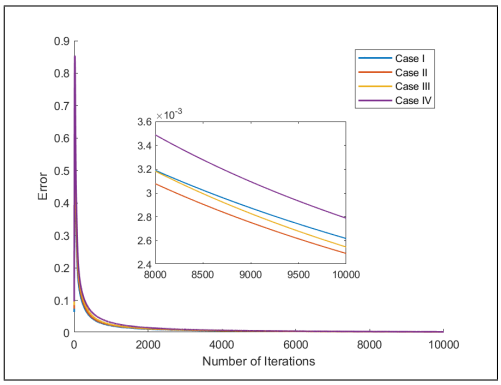
(d) Case II

(e) Case III

(f) Case IV

Figure 4: Restored image and Cauchy error for Θ =159

(a) Degraded image

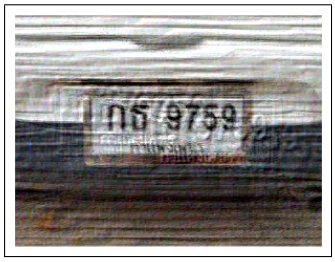


(b) Cauchy error

(c) Case I

(d) Case II

(e) Case III

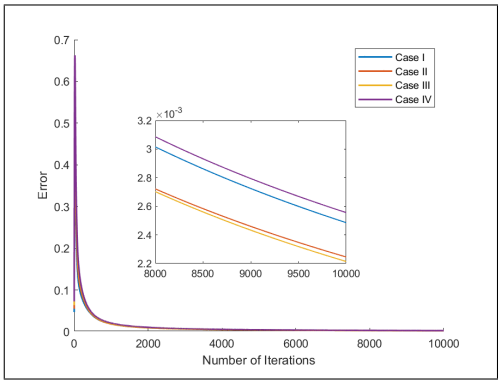


(f) Case IV

Figure 5: Restored image and Cauchy error for Θ =162

In Figure 5, with $\Theta = 162$, the algorithm shows convergence across all cases. The visibility of numbers and letters on the license plate improves slightly, indicating that this combination of blurring parameters and algorithm improvements is more effective than those in Figure 4.

(a) Degraded image



(b) Cauchy error

(c) Case I

(d) Case II

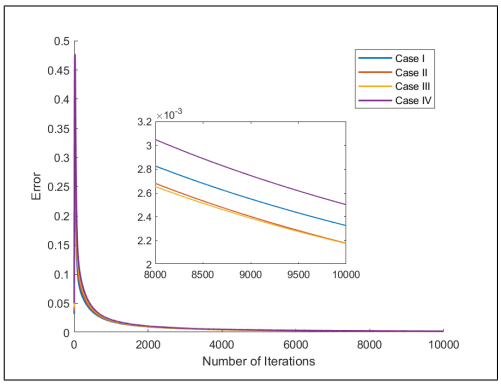
(e) Case III

(f) Case IV

Figure 6: Restored image and Cauchy error for Θ =165

Figure 6 presents the results when the image is blurred with $\Theta = 165$. The recovered images in cases II and III show significant recovery, with the numbers and letters becoming the clearest among all the figures.

(a) Degraded image



(b) Cauchy error

(c) Case I

(d) Case II

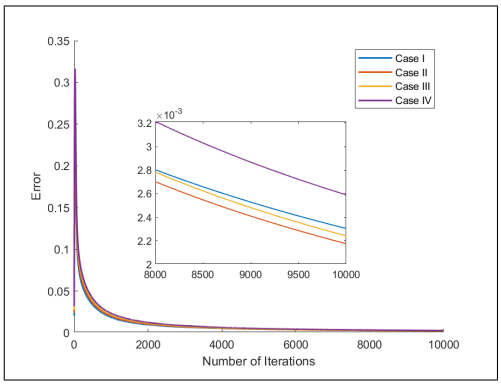
(e) Case III

(f) Case IV

Figure 7: Restored image and Cauchy error for Θ =168

When $\Theta = 168$, as shown in Figure 7, the algorithm successfully recovers the license plate details. Case III demonstrates a recovery that is not as distinct as in previous figures. The recovery quality is improved (see Figure 4 and 6).

(a) Degraded image



(b) Cauchy error

(c) Case I

(d) Case II

(e) Case III

(f) Case IV

Figure 8: Restored image and Cauchy error for Θ =171

Figure 8 shows the performance of the algorithm with $\Theta = 171$. The image clarity is not as high as in Figures 5 and 6. The numbers and letters are visible, but the recovery is less effective than cases II and III in previous figures. Based on Figures 4 through 8, it can be concluded that the Double Inertial Parallel Mann Algorithm performs most effectively when the image is blurred with $\Theta = 165$

(Figure 6). In this case, the algorithm not only achieves the lowest Cauchy error, indicating better convergence, but also produces the clearest recovery of the license plate details, particularly in cases II and III. These results demonstrate that the algorithm is effective in image deblurring under these specific conditions.

4. CONCLUSION

We have studied the fixed point problem using double inertial parallel Mann algorithm, establishing its weak convergence under appropriate conditions. To assess the practical performance of our approach, we applied it to the challenging task of blind image deblurring. The results demonstrate that the proposed algorithm is highly effective in recovering images, particularly under motion blur conditions, offering significant improvements over existing methods. Additionally, the parallel nature of the algorithm makes it well-suited for GPU implementation, enabling faster and more efficient deblurring in real-world scenarios.

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