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SOME IDENTITIES INVOLVING RATIONAL SUMS

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In this note a procedure to deduce new identities involving elementary rational sums is presented and applying this technique some sums including binomial coefficients and harmonic numbers are obtained.

1. INTRODUCTION

Sums involving products of binomial coefficients, rational functions and occasionally harmonic numbers are usually called combinatorial identities though this term ought to include identities involving sums, double sums and other equations too. Many of these sums and techniques to obtain them can be found in the works of Gould [**1**], Egorichev [**2**] and Larsen [**3**] (see also [**4**, **5**, **6**, **7**]). Recently, in [**8**] one of these identities was given. Namely,

Theorem 1. *Let n be a positive integer. Then, the following identity holds*:

$$\sum_{k=1}^{n}\left\{(-1)^{k+1}\binom{n}{k}\sum_{1\le i\le j\le k}\frac{1}{ij}\right\}=\frac{1}{n^2} \tag{1}$$

The goal of this paper is to generalize the preceding result and to present a general procedure to derive identities of this type.

2. MAIN RESULTS

In what follows we state and prove a general result from which (1) immediately follows as a particular case.

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Theorem 2. *Let n be a positive integer. Then, for all $x \in \mathbb{R}$, holds*

$$\sum_{k=1}^{n} \left\{ (-1)^{k+1} \binom{n}{k} \sum_{1 \le i \le j \le k} \frac{1}{x^2 + (i+j)x + ij} \binom{x+k}{k}^{-1} \right\} = \frac{n}{(x+n)^3}$$

Proof. First, we observe that for all $x \in \mathbb{R}$ and $n > 0$, we have

$$\binom{x+n}{n} = \prod_{k=1}^{n} \frac{x+k}{k}.$$

Differentiating both sides of the preceding identity, we get

$$\frac{\mathrm{d}}{\mathrm{d}x} \binom{x+n}{n} = \binom{x+n}{n} \sum_{j=1}^{n} \frac{1}{x+j} \tag{2}$$

Next we will use the following well known identity

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{x+k}{k}^{-1} = \frac{x}{x+n} \tag{3}$$

It corresponds to formula (5.33) in Graham, Knuth and Patasnik [**5**] with $r = -x-1$:

$$\sum_{j=0}^{m} \binom{m}{j} \binom{r}{j}^{-1} = \frac{r+1}{r+1-m}$$

For ease of reference we give here the proof of (3). We start out with Vandermonde's identity, namely

$$\binom{\alpha}{n} \binom{n}{j} = \binom{\alpha}{j} \binom{\alpha-j}{n-j}$$

and we have

$$\sum_{j=0}^{n} (-1)^j \binom{n}{j} \binom{\beta}{j} \binom{\alpha}{n} \binom{\alpha}{j}^{-1} = \sum_{j=0}^{n} (-1)^j \binom{\beta}{j} \binom{\alpha-j}{n-j} \tag{4}$$

Taking into account Vandermonde's convolution, that is,

$$\sum_{k=0}^{n} \binom{a}{j} \binom{b}{n-j} = \binom{a+b}{n}$$

and the fact that

$$\binom{\alpha}{n} = (-1)^n \binom{n-\alpha-1}{n}$$

then (4) becomes

$$\sum_{j=0}^{n}(-1)^j\binom{\beta}{j}\binom{\alpha-j}{n-j}=(-1)^n\sum_{j=0}^{n}\binom{\beta}{j}\binom{n-\alpha-1}{n-j}$$
$$=(-1)^n\binom{\beta-\alpha+n-1}{n}=\binom{\alpha-\beta}{n}.$$

If we set $\alpha=-(x+1)$ into the preceding expression, we get

$$\sum_{j=0}^{n}\binom{n}{j}\binom{\beta}{j}\binom{x+j}{j}^{-1}=\binom{x+\beta+n}{n}\binom{x+n}{n}^{-1}$$

and setting $\beta=-1$ yields

$$\sum_{j=0}^{n}\binom{n}{j}\binom{-1}{j}\binom{x+j}{j}^{-1}=\binom{x+n-1}{n}\binom{x+n}{n}^{-1}$$

or equivalently

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{x+k}{k}^{-1}=\frac{x}{x+n}$$

as claimed.

Now differentiating two times (3) and taking into account (2) yields

$$\sum_{k=1}^{n}\left\{(-1)^{k+1}\binom{n}{k}\left[\left(\sum_{j=1}^{k}\frac{1}{x+j}\right)^2+\sum_{j=1}^{k}\frac{1}{(x+j)^2}\right]\binom{x+k}{k}^{-1}\right\}$$
$$=\sum_{k=1}^{n}\left\{(-1)^{k+1}\binom{n}{k}\sum_{1\le i\le j\le k}\frac{2}{(x+i)(x+j)}\binom{x+k}{k}^{-1}\right\}$$

$$(5)\qquad =\sum_{k=1}^{n}\left\{(-1)^{k+1}\binom{n}{k}\sum_{1\le i\le j\le k}\frac{2}{x^2+(i+j)x+ij}\binom{x+k}{k}^{-1}\right\}=\frac{2n}{(x+n)^3}$$

from which the statement immediately follows and the theorem is proven. □

Setting $x=0$ into the preceding result, Theorem 1 immediately follows. Applying the same procedure, new identities involving finite sums can be obtained. For instance, setting $x=1$ into (5), we have

Corollary 1. *Let n be a positive integer. Then, holds*:

$$\sum_{k=1}^{n}\left\{\frac{(-1)^{k+1}}{k+1}\binom{n}{k}\sum_{1\le i\le j\le k}\frac{1}{(i+1)(j+1)}\right\}=\frac{n}{(n+1)^3}.$$

Setting $x = n$ into (5), we obtain

Corollary 2. *Let n be a positive integer. Then, holds*:

$$\sum_{k=1}^{n}\left\{(-1)^{k+1}\binom{n}{k}\left[\left(\sum_{j=1}^{k}\frac{1}{n+j}\right)^2+\sum_{j=1}^{k}\frac{1}{(n+j)^2}\right]\binom{n+k}{k}^{-1}\right\}=\frac{1}{4n^2}.$$

Now, using Theorem 2, we get the following identity.

Theorem 3. *Let n be a positive integer. Then, for all $x \in \mathbb{R}$, holds*

$$\sum_{k=1}^{n}\left\{(-1)^{k+1}\binom{n}{k}\left[\sum_{j=1}^{k}\frac{1}{(x+j)^3}+\sum_{1\le i<j\le k}\frac{1}{(x+i)(x+j)(2x+i+j)}\right.\right.$$

$$\left.\left.+\sum_{1\le i<j<\ell\le k}\frac{1}{(x+i)(x+j)(x+\ell)}\right]\binom{x+k}{k}^{-1}\right\}=\frac{n}{(n+x)^4}. \tag{6}$$

Proof. Differentiating three times (3) and taking into account (2) we have

$$\sum_{k=1}^{n}\left\{(-1)^{k+2}\binom{n}{k}\left[\left(\sum_{j=1}^{k}\frac{1}{x+j}\right)^3+3\left(\sum_{j=1}^{k}\frac{1}{x+j}\right)\left(\sum_{j=1}^{k}\frac{1}{(x+j)^2}\right)\right.\right.$$

$$\left.\left.+2\left(\sum_{j=1}^{k}\frac{1}{(x+j)^3}\right)\right]\binom{x+k}{k}^{-1}\right\}=\frac{-6n}{(n+x)^4}.$$

Now we observe that

$$\left(\sum_{j=1}^{k}\frac{1}{x+j}\right)^3+3\left(\sum_{j=1}^{k}\frac{1}{x+j}\right)\left(\sum_{j=1}^{k}\frac{1}{(x+j)^2}\right)+2\left(\sum_{j=1}^{k}\frac{1}{(x+j)^3}\right)$$

$$=6\left(\sum_{j=1}^{k}\frac{1}{(x+j)^3}+\sum_{1\le i<j\le k}\frac{1}{(x+i)(x+j)(2x+i+j)}\right.$$

$$\left.+\sum_{1\le i<j<\ell\le k}\frac{1}{(x+i)(x+j)(x+\ell)}\right)$$

and substituting into the preceding expression the proof follows. □

Proceeding as we have done previously and setting, for example $x = 0$ into identity (6), we have

Corollary 3. *Let n be a positive integer. Then, holds*

$$\sum_{k=1}^{n}(-1)^{k+1}\binom{n}{k}\left[\sum_{j=1}^{k}\frac{1}{j^3}+\sum_{1\le i<j\le k}\frac{1}{ij(i+j)}+\sum_{1\le i<j<\ell\le k}\frac{1}{ij\ell}\right]=\frac{1}{n^3}.$$

Likewise, setting $x = 1$ into (6), we get

Corollary 4. *Let n be a positive integer. Then, holds:*

$$\sum_{k=1}^{n} \frac{(-1)^{k+1}}{k+1} \binom{n}{k} \Bigg[\sum_{j=1}^{k} \frac{1}{(j+1)^3} + \sum_{1 \le i < j \le k} \frac{1}{(i+1)(j+1)(i+j+2)} + \sum_{1 \le i < j < \ell \le k} \frac{1}{(i+1)(j+1)(\ell+1)} \Bigg] = \frac{n}{(n+1)^4}.$$

Finally, if we set $x = n$ into (6), we obtain

Corollary 5. *Let n be a positive integer. Then, holds:*

$$\sum_{k=1}^{n} \Bigg\{ (-1)^{k+1} \binom{n}{k} \Bigg[\sum_{j=1}^{k} \frac{1}{(n+j)^3} + \sum_{1 \le i < j \le k} \frac{1}{(n+i)(n+j)(2n+i+j)} + \sum_{1 \le i < j < \ell \le k} \frac{1}{(n+i)(n+j)(n+\ell)} \Bigg] \binom{n+k}{k}^{-1} \Bigg\} = \frac{1}{16n^3}.$$

Notice that applying the preceding procedure to other guess values of x new nice identities can be derived.

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