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RESULTS ON COMPARISON AND SUB/SUPER-STABILIZABILITY OF SOME NEW MEANS

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We present analysis of some new means introduced by M. Raïssouli and A. Rezgui. We establish comparison relations and results on (K, N)-sub/super-stabilizability. Assuming that means involved have asymptotic expansions, we present the complete asymptotic expansion of the resultant mean-map. As an application of the obtained asymptotic expansions and the asymptotic inequality between M and $\mathcal{R}(B_p, M, B_q)$, we show how to find the optimal parameters p and q for which M is (B_p, B_q)-sub/super-stabilizable.

1. INTRODUCTION

Through this paper we consider *bivariate mean*, i.e. a function $M\colon \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ such that $\min(s,t) \le M(s,t) \le \max(s,t)$. We say that mean M is *symmetric* if $M(s,t) = M(t,s)$ for all $s,t > 0$, and *homogeneous* (of degree 1) if $M(\lambda s, \lambda t) = \lambda M(s,t)$ for all $\lambda, s, t, > 0$. For three homogeneous symmetric bivariate means K, M and N, we define the so-called resultant mean-map

$$\mathcal{R}(K, M, N)(s,t) = K\Big(M\big(s, N(s,t)\big), M\big(N(s,t), t\big)\Big).$$

A symmetric mean M is said to be *stable* (*balanced*), if $\mathcal{R}(M, M, M) = M$.

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Definition 1 ([**9, 11**])**.** Let K, N be two nontrivial stable means. Mean M is called

1. (K,N)-*stabilizable*, if the following relation is satisfied:
$$M(s,t) = \mathcal{R}(K,M,N)(s,t) = K\Big(M\big(s,N(s,t)\big), M\big(N(s,t),t\big)\Big).$$
2. (K,N)-*sub-stabilizable*, if $\mathcal{R}(K,M,N) \leq M$ and M is between K and N,
3. (K,N)-*super-stabilizable*, if $M \leq \mathcal{R}(K,M,N)$ and M is between K and N.

Some interesting results regarding these notions can also be found in papers [**1, 12**]. Motivated by the results on sub/super-stabilizability for bivariate means studied in the paper [**11**] in combination with the general algorithms for the coefficients in the asymptotic expansion of the resultant mean-map and consequently of the stabilizable mean, obtained in the paper [**8**], we present the results on sub/super-stabilizability in the context of the asymptotic expansions. Since the previously obtained asymptotic expansions were suitable for a specific class of means, which in their asymptotic expansions can only have odd powers x^{-2n+1}, and did not cover all types of means introduced in [**10**], we extended the algorithms to be applicable to means whose asymptotic expansion consists of all powers x^{-n+1}, $n \in \mathbb{N}_0$.

Definition 2. For an asymptotic sequence of functions $(\varphi_n)_{n\in\mathbb{N}_0}$ the (formal) series $\sum_{n=0}^{\infty} a_n\varphi_n(x)$ is said to be an *asymptotic expansion* of a function $f(x)$ as $x \to x_0$, if for each $N \in \mathbb{N}_0$
$$f(x) = \sum_{n=0}^{N} a_n\varphi_n(x) + o(\varphi_N(x)).$$

Theoretical background from theory of asymptotic expansions can be found in [**7**]. Through this paper, we mainly use asymptotic expansion with respect to the asymptotic sequence $\varphi_n(x) = x^{1-n}$, $n \in \mathbb{N}_0$, as $x \to \infty$. Based on the sign of the first term in such asymptotic expansion, we introduce the notion of the asymptotic inequality.

Definition 3 ([**13**])**.** Let $F(s,t)$ be any homogeneous bivariate function such that
$$F(x+s, x+t) = c_k(t,s)x^{-k+1} + \mathcal{O}(x^{-k}).$$
If $c_k(s,t) > 0$ for all s and t, then we say F is *asymptotically* greater than zero, and write
$$F \succ 0.$$

Asymptotic inequality is the necessary condition for the proper inequality, i.e. if $F \geq 0$, then $F \succ 0$.

The subject of study in this paper are means from [**10**] alone for themselves and also in combinations with power means. Recall the definition of the r-th power mean

$$B_r(a,b)=\begin{cases}\left(\dfrac{a^r+b^r}{2}\right)^{1/r}, & r\neq 0,\\ \sqrt{ab}, & r=0.\end{cases}$$

This class of means covers some well known classical means such as arithmetic mean $A=B_1$, geometric mean $G=B_0$ and harmonic mean $H=B_{-1}$. It has been proved ([**9**]) that power means B_p are stable for all real values of parameter p. Regarding its asymptotic expansion which has been studied in [**6**], one may find that

$$B_p(x-t,x+t)\sim x+\tfrac{1}{2}(p-1)t^2x^{-1}+\tfrac{1}{24}(p-1)(3+p-2p^2)t^4x^{-3}+\mathcal{O}(x^{-5}). \tag{1}$$

Remark 4. This so-called one variable asymptotic expansion, i.e. asymptotic expansion of a mean in variables $(x-t,x+t)$, is sufficient to determine completely the two variable asymptotic expansion, i.e. asymptotic expansion in variables $(x+s,x+t)$ as it was proved in [**2**, Lemma 2.1.]. Following the same procedure as in the mentioned Lemma, we may conclude that the first non-zero coefficient in both of those expansions is the same.

For the convenience of the reader, let us list all the means from [**10**] which will be involved in analysis in this paper. Means from the mentioned paper can be written in a form

$$m_f(a,b)=\frac{2(a-b)}{f(\frac{a}{b})-f(\frac{b}{a})},\ a\neq b,\qquad m_f(a,a)=a.$$

We are interested in some of the special cases. Let

$$g(x)=\begin{cases}\mu(\ln x), & x\geq 1,\\ -\mu(\ln\frac{1}{x}), & 0<x<1,\end{cases}\qquad \mu(x)=\int_0^x u(t)\,\mathrm{d}t \text{ odd function,}$$

and

$$m_g(a,b)=\frac{|a-b|}{\mu(|\ln\frac{a}{b}|)}=M_u(a,b). \tag{2}$$

With $u(x)=\cosh(\alpha x)$, and for $|\alpha|\leq 1$, we have

$$L_\alpha(a,b)\coloneqq M_u(a,b)=\frac{2\alpha a^\alpha b^\alpha(b-a)}{b^{2\alpha}-a^{2\alpha}}. \tag{3}$$

When $u(x)=\frac{1}{\cosh(\alpha x)}$, and for $|\alpha|\leq 1$, we have

$$S_\alpha(a,b)\coloneqq M_u(a,b)=\frac{2\alpha(b-a)}{4\arctan(\frac{b}{a})^\alpha-\pi}=\frac{\alpha(b-a)}{2\arctan\frac{b^\alpha-a^\alpha}{b^\alpha+a^\alpha}}. \tag{4}$$

Remark that $L_{-\alpha} = L_\alpha$ and $S_{-\alpha} = S_\alpha$. We can then assume that $0 \le \alpha \le 1$. The means L_α and S_α cover some well-known standard means and their properties are embodied in the following result. The first part can be seen by a simple verification and for the second part see [**10**].

Proposition 5. *The following statements are valid:*

(i) $L_0 = S_0 = L$, $L_{1/2} = G$, $L_1 = H$, $S_{1/2} = P$, $S_1 = T$ *and* $L_{1/4} = HZ_{1/4}$, *where* L, G, H *denote logarithmic, geometric and harmonic mean,* P *and* T *refer to the first and the second Seiffert mean respectively, and* $HZ_{1/4}$ *is the Heinz mean.*

(ii) For fixed $a \neq b$, *the map* $\alpha \longmapsto L_\alpha(a,b)$ *is strictly decreasing in* $\alpha \in [0,1]$ *while* $\alpha \longmapsto S_\alpha(a,b)$ *is strictly increasing in* $\alpha \in [0,1]$.

We also recall the definitions of the following means which were introduced in [**10**, Examples 4.4–4.7], where $a, b > 0, a \neq b$, $r > 0$, $|\alpha| \le 1$:

$$
\begin{aligned}
M_1(a,b) &\coloneqq \frac{|b-a|}{\ln(1+|\ln b - \ln a|)},\\
M_2(a,b) &\coloneqq \frac{b-a}{\sqrt{2}\arctan\frac{\ln b - \ln a}{\sqrt{2}}},\\
M_3(a,b) &\coloneqq \frac{|b-a|}{2\arctan\big(|\ln b - \ln a| + 1\big) - \frac{\pi}{2}},\\
M_4(a,b) &\coloneqq \frac{b-a}{\sqrt{2}\,\mathrm{arcsinh}\frac{\ln b - \ln a}{\sqrt{2}}},\\
M_5(a,b) &\coloneqq \frac{|b-a|}{\sqrt{2}\Big(\mathrm{arcsinh}\big(1+|\ln b - \ln a|\big) - \mathrm{arcsinh}\,1\Big)},\\
M_{\alpha,r}(a,b) &\coloneqq \frac{(r+\alpha)|b-a|}{\big(1+r|\ln b - \ln a|\big)^{\frac{r+\alpha}{r}} - 1}.
\end{aligned} \tag{5}
$$

The (double) parameterized mean $M_{\alpha,r}$ includes some of the known classical means as well as the other means that appear to be new. The following result, which is a simple exercise of Real Analysis, clarifies this claim.

Proposition 6. *Let* $r > 0$ *and* $|\alpha| \le 1$. *Then the following statements hold:*

(i) $M_{\alpha,0}(a,b) = \frac{\alpha|b-a|}{\exp\big(\alpha|\ln b - \ln a|\big) - 1}$, *with* $M_{0,0} = L$.

(ii) $M_{0,r} = L$ *for* $r > 0$, *and* $M_{\alpha,\infty} = L$ *for* $|\alpha| \le 1$.

(iii) $M_{-r,r}(a,b) = \frac{r|b-a|}{\ln\big(1+r|\ln b - \ln a|\big)}$ *for* $0 < r \le 1$. *In particular,* $M_{-1,1} = M_1$.

Let M be a (bivariate) mean. For $a, b > 0$ we can set $a = e^{-x}G$ and $b = e^{x}G$ for some $x \in \mathbb{R}$, where $G := G(a,b) = \sqrt{ab}$. If M is homogeneous, then

$$M(a,b) = G\, M(e^{-x}, e^{x}) =: G\, f_M(x). \tag{6}$$

If moreover M is symmetric we can assume that $x \geq 0$. Relying on the associated functions f_{m_i} we may express the characterization of the comparability condition $m_1 < m_2$ for any two symmetric homogeneous means m_1 and m_2. Namely, it is obvious that

$$m_1(a,b) < m_2(a,b),\ \forall a \neq b \iff f_{m_1}(x) < f_{m_2}(x),\ \forall x > 0. \tag{7}$$

For example, the associated functions of the standard means, are

$$\begin{gathered} f_G(x) = 1, \quad f_A(x) = \cosh x, \quad f_H(x) = \frac{1}{\cosh x}, \quad f_L(x) = \frac{\sinh x}{x}, \\ f_P(x) = \frac{\sinh x}{2\arctan\big(\tanh(x/2)\big)}, \quad f_T(x) = \frac{\sinh x}{\arctan(\tanh x)}, \end{gathered} \tag{8}$$

from which we easily deduce the well-known chain of inequalities $H < G < L < P < A < T$.

Regarding the means L_α and S_α, defined by (3) and (4), for $x \neq 0$ we have

$$f_{L_\alpha}(x) = 4\alpha \frac{\sinh x}{\sinh(2\alpha x)}, \quad f_{S_\alpha}(x) = \alpha \frac{\sinh x}{\arctan(\tanh \alpha x)}, \tag{9}$$

which implies that $L_\alpha < S_\alpha$ for all α, $0 < \alpha \leq 1$.

For the means (5), we can easily find that (for $x > 0$)

$$\begin{aligned} f_{M_1}(x) &= \frac{2\sinh x}{\ln(1+2x)}, & f_{M_2}(x) &= \frac{\sqrt{2}\sinh x}{\arctan(x\sqrt{2})}, \\ f_{M_3}(x) &= \frac{\sinh x}{\arctan(1+2x) - \pi/4}, & f_{M_4}(x) &= \frac{\sqrt{2}\sinh x}{\operatorname{arcsinh}(x\sqrt{2})}, \\ f_{M_5}(x) &= \frac{\sqrt{2}\sinh x}{\operatorname{arcsinh}(1+2x) - \operatorname{arcsinh} 1}, & f_{M_{\alpha,r}}(x) &= \frac{2(r+\alpha)\sinh x}{\big(1+2rx\big)^{\frac{r+\alpha}{r}} - 1}. \end{aligned} \tag{10}$$

The rest of the paper is organized as follows. In Section 2, we establish comparison relations involving means from [**10**] and some other well-known means. We examine the possibility of being (A,G)- or (G,A)-sub/super-stabilizable for means defined in (3), (4) and (5). In Section 3, we present complete asymptotic expansions of the above mentioned means. We also extend the result from [**8**] in order to find the complete asymptotic expansion of the resultant mean-map of means whose asymptotic expansion may include all terms x^{1-n}, $n \in \mathbb{N}_0$, which then could be applied on means which are subject of study in this paper. As a consequence of the coefficient comparison, we find parameters for which means L_α

are stable and disprove the stability for other means. With use of the coefficients in the asymptotic expansion of power means ([**6**]) we present the coefficients in the asymptotic expansion of the resultant mean-map $\mathcal{R}(B_p, M, B_q)(x-t, x+t)$ as $x \to \infty$. In Section 4, we show some of the applications of the obtained results. We analyze the behaviour of the difference $M - \mathcal{R}(B_p, M, B_q)$, for each of the means defined in (3), (4) and (5). We examine when each of these means can be (B_p, B_q)-sub/super-stabilizable and, when possible, how to find such optimal parameters p and q.

2. COMPARISON OF MEANS AND SUB/SUPER-STABILIZABILITY

We start this section by stating some results about comparison between the bivariate means mentioned in the previous section.

Proposition 7. *Let $\alpha \in [0, 1]$. The following statements hold:*

(i) If $0 < \alpha < 1/2$ then $G < L_\alpha < L < S_\alpha < A$.

(ii) If $1/2 < \alpha < 1$ then $H < L_\alpha < G < L < S_\alpha < T$.

(iii) If $1/2 \le \alpha \le \sqrt{2}/2$ then $S_\alpha < A$.

Proof. For proving ((i)) and ((ii)) we use the statement ((ii)) of Proposition 5 with the help of ((i)). The details are straightforward and therefore are omitted here.

To show ((iii)) we use (7) with f_A defined in (8) and f_{S_α} defined in the second formula in (9). Thus, we have to establish that

$$\alpha \frac{\sinh x}{\arctan(\tanh \alpha x)} < \cosh x, \quad \forall x > 0,$$

or equivalently,

$$g(x) \coloneqq \alpha \tanh x - \arctan(\tanh \alpha x) < 0, \quad \forall x > 0.$$

Simple computation leads to

$$\begin{aligned} g'(x) &= \frac{\alpha}{(\cosh x)^2} - \frac{1}{1 + (\tanh \alpha x)^2} \frac{\alpha}{(\cosh \alpha x)^2} \\ &= \frac{\alpha}{(\cosh x)^2} - \frac{\alpha}{(\cosh \alpha x)^2 + (\sinh \alpha x)^2}. \end{aligned}$$

We need to study the sign of

$$h(x) \coloneqq (\cosh \alpha x)^2 + (\sinh \alpha x)^2 - (\cosh x)^2,$$

for which we have

$$h'(x) = 2\alpha(\sinh 2\alpha x) - \sinh 2x \ \text{ and } \ h''(x) = 2\big(2\alpha^2 \cosh 2\alpha x - \cosh 2x\big).$$

If $\alpha \leq \sqrt{2}/2$, it is easy to see that $h''(x) < 0$ for all $x > 0$. Thus, h' is strictly decreasing for $x > 0$ and so, $h'(x) < h'(0) = 0$ for all $x > 0$. By the same arguments, we deduce that $h(x) < h(0) = 0$ and therefore $g(x) < g(0) = 0$, for all $x > 0$. The proof is finished. □

Remark 8. Numerical computations show that if $\sqrt{2}/2 < \alpha < 1$ then S_α and A are not comparable.

The following Proposition is an extension of the result from [**10**].

Proposition 9. *We have the following assertions:*

(i) $M_4 < M_5 < M_1 < M_3$ *and* $L < M_4 < A$*. The mean* A *is not comparable to either one of* M_1, M_2, M_3 *and* M_5*.*

(ii) $L < M_2 < M_3$*. The mean* M_2 *is not comparable to either one of* M_1, M_4 *and* M_5*.*

Proof. To prove $M_4 < M_5 < M_1 < M_3$ in ((i)), relying on (7) and (10), it is equivalent to show that for all $x > 0$ the following chain of inequalities holds

$$\frac{\sqrt{2}\sinh x}{\operatorname{arcsinh}(x\sqrt{2})} < \frac{\sqrt{2}\sinh x}{\operatorname{arcsinh}(1+2x) - \operatorname{arcsinh} 1} < \frac{2\sinh x}{\ln(1+2x)} < \frac{\sinh x}{\arctan(1+2x) - \pi/4}. \tag{11}$$

For the first inequality of (11) we consider

$$g(x) := \operatorname{arcsinh}(1+2x) - \operatorname{arcsinh}(x\sqrt{2}) - \operatorname{arcsinh} 1,$$

and then

$$g'(x) = \frac{2}{\sqrt{1+(1+2x)^2}} - \frac{\sqrt{2}}{\sqrt{1+2x^2}},$$

for which it is easy to see that $g'(x) < 0$ and so $g(x) < g(0) = 0$, for all $x > 0$.

The proof of the two other inequalities in (11) as well as the proof of the inequalities $L < M_4 < A$ and also $L < M_2 < M_3$ from part ((ii)) is similar. The details are therefore omitted here.

To show that, for example, A is not comparable with M_1 we proceed as follows: we compare $f_A(x)$ and $f_{M_1}(x)$ for $x > 0$, or equivalently, we study the sign of $g(x) := \ln(1+2x) - 2\tanh x$. It is easy to check that $\lim_{x\uparrow\infty} g(x) = +\infty$ and $g(2) = \ln 5 - 2\tanh 2 < 0$. We then have the conclusion. □

The following result concerns comparison of $M_{\alpha,r}$ with A and G.

Proposition 10. *Let* $r > 0$*. If* $\alpha \leq 0$ *then* $M_{\alpha,r} > G$ *and, if* $\alpha \geq 0$ *then* $M_{\alpha,r} < A$*.*

Proof. Assume that $\alpha \le 0$. We want to show that $f_{M_{\alpha,r}}(x) > f_G(x)$ for all $x > 0$. We consider

$$g(x) := \frac{h(x)}{\big(1+2rx\big)^{\frac{\alpha+r}{r}} - 1},$$

with

$$h(x) = 2(\alpha+r)\sinh x - \big(1+2rx\big)^{\frac{\alpha+r}{r}} + 1.$$

It is clear that $h'(x) = 2(r+\alpha)\Big(\cosh x - \big(1+2rx\big)^{\frac{\alpha}{r}}\Big)$. We have the following situations:

- If $\alpha + r > 0$ then $\big(1+2rx\big)^{\frac{\alpha+r}{r}} - 1 > 0$ and if moreover $\alpha \le 0$ then $\cosh x > 1 \ge (1+2rx)^{\alpha/r}$ for all $x > 0$. In this case, $h'(x) > 0$ and so $h(x) > h(0) = 0$, for all $x > 0$. We then deduce that $g(x) > 0$ for all $x > 0$.

- If $\alpha + r < 0$ then $\big(1+2rx\big)^{\frac{\alpha+r}{r}} - 1 < 0$ and if moreover $\alpha \le 0$ then $h'(x) < 0$ and so $h(x) < h(0) = 0$, for all $x > 0$. We then infer that $g(x) > 0$ for all $x > 0$.

Summarizing, we have shown the desired result.

The inequality $M_{\alpha,r} < A$, for $\alpha \ge 0$, can be established in a similar way, and we leave it to the reader. □

Remark 11. Numerical computations show that, if $\alpha > 0$ (resp. $\alpha < 0$) then $M_{\alpha,r}$ is not comparable with G (resp. A).

We will now study the sub/super-stabilizability of some of the above means. We recall the following result.

Proposition 12 ([**11**])**.** *Let M be a continuous symmetric mean. Then*

(i) If M is (A,G)-sub-stabilizable then $L \le M \le A$.

(ii) If M is (A,G)-super-stabilizable then $G \le M \le L$.

Combining Proposition 12 with Proposition 7 and Proposition 9 we immediately deduce the following corollary.

Corollary 13. *Let $\alpha \in [0,1]$. The following statements hold:*

(i) L_α is not (A,G)-sub-stabilizable and S_α is not (A,G)-super-stabilizable.

(ii) If $1/2 < \alpha \le 1$ then L_α is not (A,G)-super-stabilizable.

(iii) The three means M_1, M_3 and M_5 are neither (A,G)-sub/super-stabilizable nor (G,A)-sub/super-stabilizable.

To giving more results about sub/super-stabilizability we need the following lemma.

Lemma 14 ([9])**.** *Let m be a symmetric homogeneous mean. Then, for all $a, b > 0$, we have*

$$\mathcal{R}(A, m, G)(a, b) = A\big(\sqrt{a}, \sqrt{b}\big)\; m\big(\sqrt{a}, \sqrt{b}\big).$$

Now, we may state the following result.

Theorem 15. *Let $0 < \alpha \le \sqrt{2}/2$. Then, S_α is strictly (A, G)-sub-stabilizable. In particular, the first Seiffert mean $P = S_{1/2}$ is strictly (A, G)-sub-stabilizable.*

Proof. Firstly, by following Proposition 7 we have $G \le S_\alpha \le A$ for $0 < \alpha \le \sqrt{2}/2$. According to Definition 1 we have to prove that the inequality $\mathcal{R}\big(A, S_\alpha, G\big)(a, b) < S_\alpha(a, b)$ holds for $a > b$. By Lemma 14, with substitution (6), it is equivalent to show that the inequality

$$f_A(\tfrac{x}{2}) f_{S_\alpha}(\tfrac{x}{2}) < f_{S_\alpha}(x)$$

holds for all $x > 0$, and in combination with the corresponding relations in (8) and in (9), this is equivalent to

$$\alpha \cosh(x/2) \frac{\sinh(x/2)}{\arctan\big(\tanh(\alpha x/2)\big)} < \alpha \frac{\sinh x}{\arctan\big(\tanh(\alpha x)\big)},$$

for $x > 0$. Using formula $\sinh x = 2\sinh(x/2)\cosh(x/2)$ and setting $g(x) := \arctan\big(\tanh(\alpha x)\big) - 2\arctan\big(\tanh(\alpha x/2)\big)$ we easily verify that $g'(x) < 0$ and so $g(x) < g(0) = 0$, for all $x > 0$. We then deduce the desired result. □

Theorem 16. *Let $0 < \alpha < 1/2$. Then L_α is strictly (A, G)-super-stabilizable. In particular, the Heinz mean $HZ_{1/4} = L_{1/4}$ is strictly (A, G)-super-stabilizable.*

Proof. By similar way and similar arguments as in the proof of the previous theorem, the problem is reduced here on studying the sign of $g(x) := \sinh(2\alpha x) - 2\sinh(\alpha x)$ for $x > 0$. Obviously, $g'(x) := 2\alpha\cosh(2\alpha x) - 2\alpha\cosh(\alpha x) > 0$ and so $g(x) > g(0) = 0$, for any $x > 0$, so concluding the proof. □

Theorem 17. *The means M_2 and M_4 are both strictly (A, G)-sub-stabilizable.*

Proof. It is also similar to the previous proofs. The details are straightforward and therefore omitted here. □

3. ASYMPTOTIC ANALYSIS OF NEW MEANS

3.1 Asymptotic expansions of means from [10]

Let us find asymptotic expansions of means from the Section 1.

The most used result is the expansion for the power of an asymptotic series, which we recall here.

Lemma 18 ([**3**]). *Let*

$$g(x) \sim \sum_{n=0}^{\infty} a_n x^{-n}, \ \textit{as } x \to \infty,$$

be a given asymptotic expansion of $g(x)$ *with* $a_0 \neq 0$. *Then for all real* r *it holds*

$$[g(x)]^r \sim \sum_{n=0}^{\infty} P[n, r, \mathbf{a}] x^{-n},$$

where $P[0, r, \mathbf{a}] = a_0^r$ *and*

$$P[n, r, \mathbf{a}] = \frac{1}{n a_0} \sum_{k=1}^{n} [k(1+r) - n] a_k P[n-k, r, \mathbf{a}], \quad n \in \mathbb{N}.$$

We assume all sequences are enumerated from 0. Here $P[n, r, \mathbf{a}]$ denotes the coefficient by the x^{-n} in the r-th power of series assigned to a sequence $\mathbf{a} = (a_i)_{i \in \mathbb{N}_0}$.

Proposition 19. *Complete asymptotic expansion of mean* L_α, $|\alpha| \leq 1$, *is given by:*

$$L_\alpha(x-t, x+t) \sim x \sum_{n=0}^{\infty} 2\alpha \sum_{k=0}^{n} \binom{\alpha}{n-k} (-1)^{n-k} P[k, -1, (\tbinom{2\alpha}{2i+1})_{i \in \mathbb{N}_0}] t^{2n} x^{-2n}.$$

Proof.

$$\begin{aligned}
L_\alpha(x-t, x+t) &= \frac{4\alpha t (x-t)^\alpha (x+t)^\alpha}{(x+t)^{2\alpha} - (x-t)^{2\alpha}} \\
&= 4\alpha t \left(1 - \frac{t}{x}\right)^\alpha \left(1 + \frac{t}{x}\right)^\alpha \left[\left(1 + \frac{t}{x}\right)^{2\alpha} - \left(1 - \frac{t}{x}\right)^{2\alpha}\right]^{-1} \\
&\sim 4\alpha t \left(1 - \frac{t^2}{x^2}\right)^\alpha \left[\sum_{k=0}^{\infty} \binom{2\alpha}{k} t^k x^{-k} - \sum_{k=0}^{\infty} \binom{2\alpha}{k} (-1)^k t^k x^{-k}\right]^{-1} \\
&\sim 2\alpha t \left(1 - \frac{t^2}{x^2}\right)^\alpha \left[\sum_{k=0}^{\infty} \binom{2\alpha}{2k+1} t^{2k+1} x^{-(2k+1)}\right]^{-1}
\end{aligned}$$

$$\sim 2\alpha t \sum_{j=0}^{\infty} \binom{\alpha}{j}(-1)^j t^{2j} x^{-2j} \cdot \frac{x}{t} \sum_{k=0}^{\infty} P[k,-1,(\tbinom{2\alpha}{2i+1})_{i\in\mathbb{N}_0}] t^{2k} x^{-2k}$$
$$\sim 2\alpha x \sum_{n=0}^{\infty}\sum_{k=0}^{n} \binom{\alpha}{n-k}(-1)^{n-k} P[k,-1,(\tbinom{2\alpha}{2i+1})_{i\in\mathbb{N}_0}] t^{2n} x^{-2n}.$$

□

The beginning of the asymptotic expansion:

$$\begin{aligned} L_\alpha(x-t,x+t) \sim x &- \tfrac{1}{3}(2\alpha^2+1)t^2x^{-1} + \tfrac{2}{45}(\alpha-1)(\alpha+1)(7\alpha^2+2)t^4x^{-3} \\ &- \tfrac{2}{945}(\alpha-1)(\alpha+1)(62\alpha^4-85\alpha^2-22)t^6x^{-5} \\ &+ \tfrac{2}{14175}(\alpha-1)(\alpha+1)(381\alpha^6-1169\alpha^4+889\alpha^2+214)t^8x^{-7}+\dots \end{aligned} \tag{12}$$

For $\alpha = 1$ and $\alpha = \frac{1}{2}$ these coefficients coincide with coefficients obtained in paper [**5**] for harmonic and geometric mean respectively.

Proposition 20. *Complete asymptotic expansion of mean S_α, $|\alpha| \le 1$, is given by:*

$$S_\alpha(x-t,x+t) \sim x \sum_{n=0}^{\infty} \alpha P[n,-1,(D_i)_{i\in\mathbb{N}_0}] t^{2n} x^{-2n},$$

where

$$D_n = \sum_{m=0}^{n} \frac{(-1)^m}{2m+1} P[n-m, 2m+1, (C_i)_{i\in\mathbb{N}_0}],$$

and

$$C_n = \sum_{k=0}^{n} \binom{\alpha}{2k+1} P[n-k,-1,(\tbinom{\alpha}{2i})_{i\in\mathbb{N}_0}].$$

Proof.

$$S_\alpha(x-t,x+t) = \alpha t \left[\arctan \frac{(x+t)^\alpha-(x-t)^\alpha}{(x+t)^\alpha+(x-t)^\alpha}\right]^{-1}$$
$$= \alpha t \left[\arctan \frac{\left(1+\frac{t}{x}\right)^\alpha - \left(1-\frac{t}{x}\right)^\alpha}{\left(1+\frac{t}{x}\right)^\alpha + \left(1-\frac{t}{x}\right)}\right]^{-1}$$
$$\sim \alpha t \left[\arctan \frac{\sum_{k=0}^{\infty}\binom{\alpha}{k} t^k x^{-k} - \sum_{k=0}^{\infty}\binom{\alpha}{k}(-1)^k t^k x^{-k}}{\sum_{k=0}^{\infty}\binom{\alpha}{k} t^k x^{-k} + \sum_{k=0}^{\infty}\binom{\alpha}{k}(-1)^k t^k x^{-k}}\right]^{-1}$$
$$\sim \alpha t \left[\arctan \frac{\sum_{k=0}^{\infty}\binom{\alpha}{2k+1} t^{2k+1} x^{-(2k+1)}}{\sum_{k=0}^{\infty}\binom{\alpha}{2k} t^{2k} x^{-2k}}\right]^{-1}$$
$$\sim \alpha t \left[\arctan \left(\frac{t}{x}\sum_{k=0}^{\infty}\binom{\alpha}{2k+1} t^{2k} x^{-2k} \cdot \sum_{j=0}^{\infty} P[j,-1,(\tbinom{\alpha}{2i})_{i\in\mathbb{N}_0}] t^{2j} x^{-2j}\right)\right]^{-1}$$

$$\sim \alpha t\left[\arctan\left(\frac{t}{x}\sum_{n=0}^{\infty}\underbrace{\sum_{k=0}^{n}\binom{\alpha}{2k+1}P[n-k,-1,(\tbinom{\alpha}{2i})_{i\in\mathbb{N}_0}]}_{C_n}t^{2n}x^{-2n}\right)\right]^{-1}$$

$$\sim \alpha t\left[\sum_{m=0}^{\infty}\frac{(-1)^m}{2m+1}\left(\frac{t}{x}\sum_{n=0}^{\infty}C_n t^{2n}x^{-2n}\right)^{2m+1}\right]^{-1}$$

$$\sim \alpha t\left[\sum_{m=0}^{\infty}\frac{(-1)^m}{2m+1}\left(\frac{t}{x}\right)^{2m+1}\sum_{j=0}^{\infty}P[j,2m+1,(C_i)_{i\in\mathbb{N}_0}]t^{2j}x^{-2j}\right]^{-1}$$

$$\sim \alpha t\left[\frac{t}{x}\sum_{n=0}^{\infty}\underbrace{\sum_{m=0}^{n}\frac{(-1)^m}{2m+1}P[n-m,2m+1,(C_i)_{i\in\mathbb{N}_0}]}_{D_n}t^{2n}x^{-2n}\right]^{-1}$$

$$\sim \alpha x\sum_{n=0}^{\infty}P[n,-1,(D_i)_{i\in\mathbb{N}_0}]t^{2n}x^{-2n}.$$

□

Although computed using several recursively defined sequences, the coefficients have a nice form. The beginning of the asymptotic expansion is given by

$$\begin{aligned}S_\alpha(x-t,x+t) &\sim x+\tfrac{1}{3}(2\alpha^2-1)t^2x^{-1}-\tfrac{2}{45}(5\alpha^4-5\alpha^2+2)t^4x^{-3}\\ &+\tfrac{2}{945}(86\alpha^6-105\alpha^4+63\alpha^2-22)t^6x^{-5}\\ &-\tfrac{2}{14175}\big(214+5\alpha^2(\alpha-1)(\alpha+1)(135-159\alpha^2+271\alpha^4)\big)t^8x^{-7}+\dots\end{aligned}\tag{13}$$

For $\alpha=\frac{1}{2}$ and $\alpha=1$ these coefficients coincide with the coefficients from [**13**] for the first and the second Seiffert mean.

We may state more general result for means of the type (2).

Theorem 21. *Assume that the odd function* $\mu\colon\mathbb{R}\to\mathbb{R}$ *has the following expansion*

$$\mu(x)\sim\sum_{n=0}^{\infty}c_nx^{2n+1},\ \text{as } x\to 0,\tag{14}$$

with $c_0=1$. *Then mean* m_g *defined in* (2) *has the following expansion*

$$m_g(x-t,x+t)\sim x\sum_{m=0}^{\infty}P[m,-1,(E_i)_{i\in\mathbb{N}_0}]t^{2m}x^{-2m},\ \text{as } x\to\infty,\tag{15}$$

where

$$E_m=\sum_{n=0}^{m}c_n2^{2n}P[m-n,2n+1,(\tfrac{1}{2i+1})_{i\in\mathbb{N}_0}].$$

Proof. Observe the expression $|\ln \frac{a}{b}|$ when $a = x - t$ and $b = x + t$, which under assumption $t > 0$ and for x large enough, is equal to

$$\ln \frac{x+t}{x-t} \sim \ln\left(1+\frac{t}{x}\right) - \ln\left(1-\frac{t}{x}\right) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} t^k x^{-k} + \sum_{k=1}^{\infty} \frac{1}{k} t^k x^{-k}$$
$$\sim 2\sum_{k=0}^{\infty} \frac{1}{2k+1} t^{2k+1} x^{-(2k+1)}.$$

Now we have, for any t,

$$m_g(x-t, x+t) \sim 2t\left[\mu\left(2\sum_{k=0}^{\infty}\frac{1}{2k+1}t^{2k+1}x^{-(2k+1)}\right)\right]^{-1}$$
$$\sim 2t\left[\sum_{n=0}^{\infty} c_n \left(2\sum_{k=0}^{\infty}\frac{1}{2k+1}t^{2k+1}x^{-(2k+1)}\right)^{2n+1}\right]^{-1}$$
$$\sim 2t\left[\sum_{n=0}^{\infty} c_n t^{2n+1}x^{-2n-1}2^{2n+1}\sum_{k=0}^{\infty} P[k, 2n+1, (\tfrac{1}{2i+1})_{i\in\mathbb{N}_0}]t^{2k}x^{-2k}\right]^{-1}$$
$$\sim 2t\left[2tx^{-1}\sum_{m=0}^{\infty}\underbrace{\sum_{n=0}^{m} c_n 2^{2n} P[m-n, 2n+1, (\tfrac{1}{2i+1})_{i\in\mathbb{N}_0}]}_{E_m} t^{2m}x^{-2m}\right]^{-1}$$
$$\sim x\sum_{m=0}^{\infty} P[m, -1, (E_i)_{i\in\mathbb{N}_0}]t^{2m}x^{-2m}.$$

□

Example 22. 1. Asymptotic expansion of mean L_α can also be deduced form Theorem 21. Let $\mu(x) = \frac{1}{\alpha}\sinh(\alpha x)$, i.e. let $c_n = \frac{\alpha^{2n}}{(2n+1)!}$ in (14). Then using formula (15) we may also obtain the coefficients from (12).

2. Let $u(y) = \frac{1}{\cosh(\alpha y)}$ whose asymptotic expansion as $y \to 0$ is equal to $u(y) \sim \sum_{n=0}^{\infty} P[n, -1, (\frac{\alpha^{2i}}{(2i)!})_{i\in\mathbb{N}_0}]y^{2n}$. Then, with $c_n = \frac{1}{2n+1}P[n, -1, (\frac{\alpha^{2i}}{(2i)!})_{i\in\mathbb{N}_0}]$ in (14), formula (15) gives the coefficients as in (13).

Regarding the rest of the means from Section 1, similar computations lead to their asymptotic expansions. With prior use of the arctangent addition formula in the denominator of M_3 and the inverse hyperbolic sine addition (subtraction) formula for the denominator of M_5, with application of Lemma 18 and Taylor series expansion of logarithmic, arctangent or inverse hyperbolic sine function in variable t/x, we obtain the following expansions.

Proposition 23. *As $x \to \infty$, for means defined in* (5)*, the following expansions hold:*

$$\begin{aligned}
M_1(x-t,x+t) &\sim x + |t| - \tfrac{2}{3}t^2x^{-1} + \tfrac{1}{3}|t|^3x^{-2} - \tfrac{28}{45}t^4x^{-3} + \tfrac{37}{45}|t|^5x^{-4} \\
&\quad - \tfrac{1369}{945}t^6x^{-5} + \dots \\
M_2(x-t,x+t) &\sim x + \tfrac{1}{3}t^2x^{-1} - \tfrac{2}{9}t^4x^{-3} + \tfrac{14}{135}t^6x^{-5} - \tfrac{122}{945}t^8x^{-7} + \dots \\
M_3(x-t,x+t) &\sim x + |t| - \tfrac{1}{3}|t|^3x^{-2} + \tfrac{4}{15}t^4x^{-3} - \tfrac{13}{45}|t|^5x^{-4} + \tfrac{1}{9}t^6x^{-5} + \dots \\
M_4(x-t,x+t) &\sim x - \tfrac{1}{6}t^4x^{-3} + \tfrac{8}{315}t^6x^{-5} - \tfrac{367}{4536}t^8x^{-7} + \dots \\
M_5(x-t,x+t) &\sim x + \tfrac{1}{2}|t| - \tfrac{1}{4}t^2x^{-1} - \tfrac{1}{6}|t|^3x^{-2} + \tfrac{5}{48}t^4x^{-3} - \tfrac{73}{360}|t|^5x^{-4} \\
&\quad + \tfrac{1033}{10080}t^6x^{-5} + \dots \\
M_{\alpha,r}(x-t,x+t) &\sim x + \alpha|t| + \tfrac{1}{3}(\alpha^2+2r\alpha-1)t^2x^{-1} - \tfrac{1}{3}r\alpha(2r+\alpha)|t|^3x^{-2} \\
&\quad - \tfrac{1}{45}\left(\alpha(\alpha+2r)\left(\alpha^2-18r^2+2\alpha r-5\right)+4\right)t^4x^{-3} \\
&\quad - \tfrac{1}{45}\alpha r(\alpha+2r)\big(3(2r-\alpha)(\alpha+4r)+10\big)|t|^5x^{-4} + \dots
\end{aligned}$$

3.2 Asymptotic expansion of the resultant mean-map

Assuming that all means involved were bivariate, symmetric, homogeneous and had the asymptotic expansions as $x \to \infty$ of the following type

$$\begin{aligned}
K(x-t,x+t) &\sim \sum_{n=0}^{\infty} a_n^K t^{2n}x^{-2n+1}, \\
M(x-t,x+t) &\sim \sum_{n=0}^{\infty} a_n^M t^{2n}x^{-2n+1}, \\
N(x-t,x+t) &\sim \sum_{n=0}^{\infty} a_n^N t^{2n}x^{-2n+1},
\end{aligned} \tag{16}$$

in [**8**] we found the complete asymptotic expansion of the resultant mean–map:

$$R(x-t,x+t) = \mathcal{R}(K,M,N)(x-t,x+t) \sim \sum_{n=0}^{\infty} a_n^R t^{2n}x^{-2n+1}. \tag{17}$$

Using coefficients of the resultant mean–map of the corresponding means, we found formula for calculating the coefficients in the asymptotic power series expansion of type (16) of stable mean, which, up to five terms reads as ([**8**, formula (34)]):

$$\begin{aligned}
M(x-t,x+t) &= x + a_1t^2x^{-1} + \tfrac{1}{6}a_1(1+a_1)(1-4a_1)t^4x^{-3} \\
&\quad + \tfrac{1}{90}a_1(1+a_1)\left(6-31a_1+36a_1^2+64a_1^3\right)t^6x^{-5} \\
&\quad + \tfrac{1}{2520}a_1(1+a_1)\big(90-531a_1+937a_1^2+568a_1^3-3088a_1^4-2176a_1^5\big)t^8x^{-7} \\
&\quad + \mathcal{O}(x^{-9}).
\end{aligned} \tag{18}$$

With $K = B_p$ and $N = B_q$, we obtain coefficients in the asymptotic expansion (17) of the resultant mean–map (consequence of [**8**, formula (26)]):

$$\begin{aligned}(19)\qquad &\mathcal{R}(B_p, M, B_q)(x-t, x+t) = x + \tfrac{1}{8}(2a_1^M + p + 2q - 3)t^2x^{-1} \\ &+ \tfrac{1}{384}\Big(24a_2^M + 12a_1^M(-4pq + p + 2q(q+1) + 1) \\ &- 2p^3 + 3p^2 + 2p(7-6q) + 4q\left(-4q^2 + 6q + 7\right) - 39\Big)t^4x^{-3} + \mathcal{O}(x^{-5}).\end{aligned}$$

As a consequence of the results of the above mentioned paper, specially relying on the form of the stable mean coefficients given in (18), in combination with the asymptotic expansion given in Proposition 19 we obtain the following.

Corollary 24. *Means L_α are stable iff $\alpha \in \{\pm\frac{1}{2}, \pm 1\}$.*

Proof. By comparing coefficients (12) in the asymptotic expansion of mean L_α with the stable mean coefficients (18) by the powers x^{-1} and x^{-3} we obtain the following equations

$$\begin{aligned}-\tfrac{1}{3}(2\alpha^2+1) &= a_1,\\ \tfrac{2}{45}(\alpha-1)(\alpha+1)(7\alpha^2+2) &= \tfrac{1}{6}a_1(1+a_1)(1-4a_1)\\ &= \tfrac{1}{6}(-\tfrac{1}{3}(2\alpha^2+1))(1-\tfrac{1}{3}(2\alpha^2+1))(1+\tfrac{4}{3}(2\alpha^2+1)),\end{aligned}$$

which combined give the following

$$18(\alpha^2-1)(7\alpha^2+2) = 5(2\alpha^2+1)(\alpha^2-1)(8\alpha^2+7).$$

The only real solutions are $\alpha \in \{\pm\frac{1}{2}, \pm 1\}$. For either of those values of α the stable mean is obtained. Namely, $L_{\pm 1} = H$ and $L_{\pm\frac{1}{2}} = G$. ☐

The similar procedure as in Corollary 24, involving coefficients given in (13), gives no solutions for $|\alpha| \le 1$, so we have the following conclusion.

Corollary 25. *Means S_α, $|\alpha| \le 1$, are not stable.*

Not all of the means (3), (4), (5) have asymptotic expansion of the form (16). In order to obtain the asymptotic expansion of the resultant mean–map for all of the means mentioned in the introduction, and afterwards find which of them are stable and study asymptotic expansion of $\mathcal{R}(B_p, M, B_q)$, we need to adjust the algorithm obtained in the paper [**8**].

Theorem 26. *Assume that means K, M and N have expansions*

$$(20)\qquad K(x-t, x+t) \sim \sum_{n=0}^{\infty} a_n^K t^n x^{-n+1},$$

$$(21)\qquad M(x-t, x+t) \sim \sum_{n=0}^{\infty} a_n^M t^n x^{-n+1},$$

$$N(x-t,x+t)\sim\sum_{n=0}^{\infty}a_n^N t^n x^{-n+1}, \tag{22}$$

as $x\to\infty$, such that $a_1^N\neq\pm1$. Then the coefficients in the asymptotic expansion of the resultant mean–map

$$R(x-t,x+t)=\mathcal{R}(K,M,N)(x-t,x+t)\sim\sum_{n=0}^{\infty}a_n^R t^n x^{-n+1}, \tag{23}$$

can be calulated by the recursive formula

$$a_m^R=\frac{1}{4}\sum_{n=0}^{m}\sum_{k=0}^{m-n}a_n^K P[k,n,\mathbf{d}]P[m-n-k,-n+1,\mathbf{s}],\quad m\in\mathbb{N}_0, \tag{24}$$

where

$$\begin{aligned}
d_{m-1}=\sum_{n=0}^{m}a_n^M\sum_{k=0}^{m-n}\big(&P[k,n,\tilde{\mathbf{g}}]P[m-n-k,-n+1,\tilde{\mathbf{h}}]\\
&-P[k,n,\mathbf{g}]P[m-n-k,-n+1,\mathbf{h}]\big),\quad m\in\mathbb{N},\\
s_m=\sum_{n=0}^{m}a_n^M\sum_{k=0}^{m-n}\big(&P[k,n,\tilde{\mathbf{g}}]P[m-n-k,-n+1,\tilde{\mathbf{h}}]\\
&+P[k,n,\mathbf{g}]P[m-n-k,-n+1,\mathbf{h}]\big),\quad m\in\mathbb{N}_0,
\end{aligned}$$

and

$$\begin{aligned}
\mathbf{g}&=(1+a_1^N,a_2^N,a_3^N,\dots), & \mathbf{h}&=(2,a_1^N-1,a_2^N,a_3^N,\dots),\\
\tilde{\mathbf{g}}&=(1-a_1^N,-a_2^N,-a_3^N,\dots), & \tilde{\mathbf{h}}&=(2,1+a_1^N,a_2^N,a_3^N,\dots).
\end{aligned} \tag{25}$$

Proof. The proof goes by the similar procedure as the proof of the somewhat specific analogue form paper [**8**]. With N being the abbreviated version of $N(x-t,x+t)$, the following holds:

$$\begin{aligned}
M(x-t,N(x-t,x+t))&\sim\sum_{n=0}^{\infty}a_n^M\big(\tfrac{1}{2}(N-x+t)\big)^n\big(\tfrac{1}{2}(N+x-t)\big)^{-n+1}\\
&\sim\frac{1}{2}\sum_{n=0}^{\infty}a_n^M\Big(t\big(1+a_1^N\big)+\sum_{k=2}^{\infty}a_k^N t^k x^{-k+1}\Big)^n\Big(2x+\big(a_1^N-1\big)t+\sum_{j=2}^{\infty}a_j^N t^j x^{-j+1}\Big)^{-n+1}\\
&\sim\frac{1}{2}\sum_{n=0}^{\infty}a_n^M t^n\sum_{k=0}^{\infty}P[k,n,\mathbf{g}]t^k x^{-k}\cdot x^{-n+1}\sum_{j=0}^{\infty}P[j,-n+1,\mathbf{h}]t^j x^{-j}\\
&\sim\frac{1}{2}\sum_{m=0}^{\infty}\sum_{n=0}^{m}a_n^M\sum_{k=0}^{m-n}P[k,n,\mathbf{g}]P[m-n-k,-n+1,\mathbf{h}]t^m x^{-m+1}.
\end{aligned}$$

Similarly, the second component to be composed with the mean K has the following expansion:

$$M(N(x-t,x+t),x+t) \sim \sum_{n=0}^{\infty} a_n^M \big(\tfrac{1}{2}(x+t-N)\big)^n \big(\tfrac{1}{2}(x+t+N)\big)^{-n+1}$$
$$\sim \frac{1}{2}\sum_{n=0}^{\infty} a_n^M \Big(t\big(1-a_1^N\big) - \sum_{k=2}^{\infty} a_k^N t^k x^{-k+1}\Big)^n \Big(2x + \big(a_1^N+1\big)t + \sum_{j=2}^{\infty} a_j^N t^j x^{-j+1}\Big)^{-n+1}$$
$$\sim \frac{1}{2}\sum_{n=0}^{\infty} a_n^M t^n \sum_{k=0}^{\infty} P[k,n,\tilde{\mathbf{g}}] t^k x^{-k} \cdot x^{-n+1} \sum_{j=0}^{\infty} P[j,-n+1,\tilde{\mathbf{h}}] t^j x^{-j}$$
$$\sim \frac{1}{2}\sum_{m=0}^{\infty}\sum_{n=0}^{m} a_n^M \sum_{k=0}^{m-n} P[k,n,\tilde{\mathbf{g}}] P[m-n-k,-n+1,\tilde{\mathbf{h}}] t^m x^{-m+1}.$$

In order to obtain the desired formula, we need to calculate one half of the difference of the previous two expressions

$$T = \frac{1}{2}\big(M(N(x-t,x+t),x+t) - M(x-t,N(x-t,x+t))\big) \sim \frac{1}{4}\sum_{m=1}^{\infty} d_{m-1} t^m x^{-m+1},$$

and one half of the sum of those two expressions

$$X = \frac{1}{2}\big(M(N(x-t,x+t),x+t) + M(x-t,N(x-t,x+t))\big) \sim \frac{1}{4}\sum_{m=0}^{\infty} s_m t^m x^{-m+1}.$$

The resultant mean-map can then be written in a following way

$$R = K(X-T,X+T)$$
$$\sim \sum_{n=0}^{\infty} a_n^K \Big(\tfrac{1}{4}\sum_{k=1}^{\infty} d_{k-1} t^k x^{-k+1}\Big)^n \Big(\tfrac{1}{4}\sum_{j=0}^{\infty} s_j t^j x^{-j+1}\Big)^{-n+1}$$
$$\sim \frac{1}{4}\sum_{n=0}^{\infty} a_n^K t^n \sum_{k=0}^{\infty} P[k,n,\mathbf{d}] t^k x^{-k} \cdot x^{-n+1} \sum_{j=0}^{\infty} P[j,-n+1,\mathbf{s}] t^j x^{-j}$$
$$\sim \frac{1}{4}\sum_{m=0}^{\infty}\sum_{n=0}^{m}\sum_{k=0}^{m-n} a_n^K P[k,n,\mathbf{d}] P[m-n-k,-n+1,\mathbf{s}] t^m x^{-m+1}$$
$$\sim \sum_{m=0}^{\infty} a_m^R t^m x^{-m+1}.$$

□

With assumptions from Theorem 26, the first few coefficients in the expansion

(23) of the resultant mean are given by:

$$
\begin{aligned}
a_0^R &= 1,\\
a_1^R &= \tfrac{1}{2}\big(a_1^K + a_1^M + a_1^N - a_1^K a_1^M a_1^N\big),\\
a_2^R &= \tfrac{1}{4}\Big(a_2^N(2-2a_1^K a_1^M) + a_2^K(-1+a_1^M a_1^N)^2 + a_2^M + a_2^M a_1^N(a_1^N - 2a_1^K)\Big),\\
a_3^R &= \tfrac{1}{64}\Bigg(-32a_3^N(a_1^K a_1^M - 1) - 8a_3^K(a_1^M a_1^N - 1)^3 - 8a_2^K(-1+a_1^M a_1^N)\times\\
&\qquad \times\Big((a_1^N)^2 a_1^M - a_1^M(4a_2^N+1) + a_1^N\big((a_1^M)^2 - 4a_2^M - 1\big)\Big)\\
&\qquad + 8a_2^M(a_1^K - a_1^N)\big((a_1^N)^2 - 4a_2^N - 1\big)\\
&\qquad - 8a_3^M(a_1^K a_1^N(3+(a_1^N)^2) - 3(a_1^N)^2 - 1)\Bigg).
\end{aligned}
\tag{26}
$$

Remark 27. In the previous Theorem formula for the Case I: $a_1^N \neq \pm 1$ was presented. In other cases, first few terms in the sequence $\mathbf{g}$ or $\tilde{\mathbf{g}}$ is equal to 0 and therefore the expression $P[\cdot,\cdot,\mathbf{g}]$ or $P[\cdot,\cdot,\tilde{\mathbf{g}}]$ is not well defined. In order to complement the result of Theorem 26, let z be the minimal integer, greater or equal to 2, such that $a_z^N \neq 0$. In Case II: $a_1^N = -1$, let us redefine $\mathbf{g} = (a_z^N, a_{z+1}^N, \ldots)$ and let $\mathbf{h}, \tilde{\mathbf{g}}, \tilde{\mathbf{h}}$ be the same as in (25). Then

$$
\begin{aligned}
d_{m-1} &= \sum_{n=0}^{m} a_n^M \sum_{k=0}^{m-n} P[k,n,\tilde{\mathbf{g}}]P[m-n-k,-n+1,\tilde{\mathbf{h}}]\\
&\qquad - \sum_{n=0}^{\lfloor \frac{m}{z}\rfloor} a_n^M \sum_{k=0}^{m-nz} P[k,n,\mathbf{g}]P[m-nz-k,-n+1,\mathbf{h}], \quad m\in\mathbb{N},\\
s_m &= \sum_{n=0}^{m} a_n^M \sum_{k=0}^{m-n} P[k,n,\tilde{\mathbf{g}}]P[m-n-k,-n+1,\tilde{\mathbf{h}}]\\
&\qquad + \sum_{n=0}^{\lfloor \frac{m}{z}\rfloor} a_n^M \sum_{k=0}^{m-nz} P[k,n,\mathbf{g}]P[m-nz-k,-n+1,\mathbf{h}], \quad m\in\mathbb{N}_0,
\end{aligned}
$$

and the coefficients of the resultant mean-map can be calculated by the formula (24). In Case III: $a_1^N = 1$, the sequence $\tilde{\mathbf{g}}$ needs to be redefined in a similar manner and the rest of the procedure goes analogously.

Remark 28. Once calculated by the procedure given in Theorem 26, the coefficients behave well for the special values mentioned in Cases II and III from Remark 27 where the existence has been shown. As it was proved in [**6**, Lemma 2.2] we may use coefficients from the list (26) with $a_1^N = \pm 1$ and $a_2^N = \ldots = a_{z-1}^N = 0$ to obtain the coefficients in Cases II and III as well.

Remark 29. Regarding the stability examination, when equating the coefficients a_m^R from (26) with $a_m^M = a_m^N = a_m^K$, for $m = 1$ we obtain that $a_1^M(1+a_1^M)(1-a_1^M) =$

0. The first possibility ($a_1^M = 0$) implies (inductively) that also $a_{2m+1} = 0$, for all $m \in \mathbb{N}$, which means that M has the asymptotic expansion of the type (16). Each of other two possibilities ($a_1^M = \pm 1$) implies that $a_m = 0$, for all $m \geq 2$, which correspond to the first and the second projection.

As a consequence of the coefficients comparison, using the stable mean expansion (18), for means whose asymptotic expansion does not contain even powers of x, on the one side and the corresponding expansions given in Proposition 23 on the other side, and employing Remark 29 for those means whose asymptotic expansions contain all powers x^{-n+1}, $n \in \mathbb{N}_0$, we have the following conclusion.

Corollary 30. *None of the means form the list* (5) *is stable.*

We finish this Section with a result based on Theorem 23 which will be used in sequel. With $K = B_p$ and $N = B_q$, by incorporating coefficients (1) into the expansions (20) and (22) with parameters p and q respectively, we obtain coefficients in the asymptotic expansion (23) of the resultant mean–map:

$$\begin{aligned}\mathcal{R}(B_p, M, B_q)(x-t, x+t) &= x + \tfrac{1}{2}a_1^M t + \tfrac{1}{8}\big(2a_2^M + p + 2q - 3\big)t^2x^{-1} \\ &\quad + \tfrac{1}{16}\big(2a_3^M - (p-1)(2q-1)a_1^M\big)t^3x^{-2} + \mathcal{O}(x^{-3}).\end{aligned} \tag{27}$$

4. SUB-STABILIZABILITY AND SUPER-STABILIZABILITY WITH POWER MEANS

In this Section we present some of the possible applications of the previously obtained results. We will show the use of asymptotic expansions when examining the possibility for a given mean to be (B_p, B_q)-sub- or super-stabilizable and when possible, how to find optimal parameters p and q.

Since the asymptotic inequality is the necessary condition for the proper inequality, we analyse when

$$M - \mathcal{R}(B_p, M, B_q) \succ 0. \tag{28}$$

The best approximation is obtained when as many as possible first coefficients are equal to 0. See [**13**] for detailed analysis. Because of the reasoning in Remark 4, it is sufficient to use the expansions in variables $(x-t, x+t)$.

Asymptotic inequality corresponds the proper inequality for means when variables s and t are close enough to each other. In order to complement the information obtained form asymptotic side, it is often useful to observe relation between means about the point $(0,1)$. For more details we refer to paper [**4**].

Example 31. Let $M = L_\alpha$. Observe the difference between (12) and (19). When equating the coefficient by x^{-1} with 0, we have that $q = \frac{1}{2}(1-p) - 2\alpha^2$ and then

$$(L_\alpha - \mathcal{R}(B_p, L_\alpha, B_q))(x-t, x+t) \sim -\tfrac{1}{384}(4\alpha^2-1)(p^2-1-16\alpha^2+16\alpha^4)t^4x^{-3} + \dots$$

and if also $p = \pm\sqrt{1+16\alpha^2-16\alpha^4}$ then

$$(L_\alpha - \mathcal{R}(B_p, L_\alpha, B_q))(x-t, x+t) \sim -\tfrac{1}{720}\alpha^2(\alpha^2-1)(4\alpha^2-1)^3 t^6 x^{-5} + \ldots$$

Hence, for such p and q, (28) holds for $\frac{1}{2} < |\alpha| < 1$, and we have the opposite inequality sign in (28) for $0 < |\alpha| < \frac{1}{2}$.

Additionally, when we equate coefficient by x^{-5} with 0, we obtain the following trivial (and already known) cases: $\alpha = \pm\frac{1}{2}, q = -p$; $\alpha = 0, p = -1, q = 1$; $\alpha = 0, p = 1, q = 0$; $\alpha = \pm 1, p = 1, q = -2$; where the stabilizability is achieved.

On the other side, values of limits as $s \to 0$: $\lim_{s\to 0} L_\alpha(s, 1-s) = 0$ and $\lim_{s\to 0} \mathcal{R}(B_p, L_\alpha, B_q)(s, 1-s) > 0$ for any p and q, imply that L_α could only be (B_p, B_q)-super-stabilizable.

Combining the observations about the sign of the difference $L_\alpha - \mathcal{R}(B_p, L_\alpha, B_q)$ in two limiting points, with the intent to achieve the best possible order of inequality, we see that for $\frac{1}{2} < |\alpha| < 1$ and above mentioned specific values of p and q mean L_α cannot be either sub- or super-stabilizable with the pair of power means (B_p, B_q).

Motivated by the Example 31, numerical experiments indicate that the following statement should be true.

Conjecture 32. Let $q = \frac{1}{2}(1-p) - 2\alpha^2$ and $p = \pm\sqrt{1+16\alpha^2-16\alpha^4}$. Then $L_\alpha - \mathcal{R}(B_p, L_\alpha, B_q) < 0$ for $|\alpha| < \frac{1}{2}$.

Example 33. Let $M = S_\alpha$. Observe the difference between (13) and (19). The best order of approximation as $x \to \infty$ is obtained when $q = \frac{1}{2}(1-p) + 2\alpha^2$ and $p^2 = \frac{1-12\alpha^2+112\alpha^4-64\alpha^6}{1+4\alpha^2}$. For such values p and q we have

$$(S_\alpha - \mathcal{R}(B_p, S_\alpha, B_q))(x-t, x+t) \sim \tfrac{1}{720}\alpha^2(1+\alpha^2)(1-16\alpha^2+16\alpha^4)^2(1+4\alpha^2)^{-1} t^6 x^{-5} + \ldots$$

Then, (28) holds when $|\alpha| \neq \frac{1}{2}\sqrt{2\pm\sqrt{3}}$ and $\alpha \neq 0$. Furthermore, one way to make the coefficient by the x^{-5} equal to zero yields trivial cases: $\alpha = 0,\ p = 1,\ q = 0$ and $\alpha = 0,\ p = -1,\ q = 1$, where the stabilizability is achieved. The remaining option gives the condition $|\alpha| = \frac{1}{2}\sqrt{2\pm\sqrt{3}}$. Notice that $\frac{1}{2}\sqrt{2-\sqrt{3}} \approx 0.2588$ and $\frac{1}{2}\sqrt{2+\sqrt{3}} \approx 0.9659$, which belong to the valid range. For $|\alpha| = \frac{1}{2}\sqrt{2-\sqrt{3}}$, we have

$$(S_\alpha - \mathcal{R}(B_p, S_\alpha, B_q))(x-t, x+t) \sim \tfrac{17(-97+56\sqrt{3})}{967680}\, t^8 x^{-7} + \mathcal{O}(x^{-9}),$$

where the numerical value of the first nonzero coefficient is $\approx -9.0558 \cdot 10^{-8}$. On the other side, the value of the same difference in variables $(s, 1-s)$ when $s = 0$ is approximately equal to $-6.4286 \cdot 10^{-4}$. For $|\alpha| = \frac{1}{2}\sqrt{2+\sqrt{3}}$, we have

$$(S_\alpha - \mathcal{R}(B_p, S_\alpha, B_q))(x-t, x+t) \sim -\tfrac{17(97+56\sqrt{3})}{967680}\, t^8 x^{-7} + \mathcal{O}(x^{-9}),$$

where the numerical value of the first nonzero coefficient is $\approx -3.4081 \cdot 10^{-3}$. On the other side, the value of the same difference in variables $(s, 1-s)$ when $s = 0$

is approximately equal to $-2.39996 \cdot 10^{-3}$. In both cases, sign of the first nonzero coefficient, which is by x^{-7}, implies that the opposite sign in (28) holds. The asymptotic inequality describes the behavior when variables s and t are relatively close to each other. Additionally, on the other side, for $(s,t) = (0,1)$ (or by symmetry for $(1,0)$) we have the same sign of the difference, which motivated the following Conjecture.

Conjecture 34. Let $|\alpha| = \frac{1}{2}\sqrt{2-\sqrt{3}}$, $p = -1+\sqrt{3}$, $q = 2-\sqrt{3}$, or $|\alpha| = \frac{1}{2}\sqrt{2+\sqrt{3}}$, $p = 1+\sqrt{3}$, $q = 1$. Then $S_\alpha - \mathcal{R}(B_p, S_\alpha, B_q) < 0$.

Example 35. Let $M = M_1$. Mean M has the asymptotic expansion of the form (21) and hence from (27) follows that

$$(M_1 - \mathcal{R}(B_p, M_1, B_q))(x-t, x+t) \sim \tfrac{1}{2} a_1^M t + \mathcal{O}(x^{-1}).$$

From Proposition 23 we see that $a_1^M = \frac{|t|}{t}$ so the expansion of the difference starts with coefficient $\frac{1}{2}a_1^M t = \frac{1}{2}|t|$ by x^0, and the asymptotic inequality (28) holds, which means that for s and t close enough and $s \neq t$, the difference $M_1(s,t) - \mathcal{R}(B_p, M_1, B_q)(s,t)$ is greater than 0. On the other side, observing the values of limits $\lim_{s\to 0} M_1(s, 1-s) = 0$ and $\lim_{s\to 0} \mathcal{R}(B_p, M_1, B_q)(s, 1-s) = 2^{-1/p} \cdot |2^{-1/q} - 1| / \ln(1 + |\ln 2^{-1/q}|) > 0$, $\forall p, q \in \mathbb{R}$, implies that there are s and t for which the difference between M_1 and $\mathcal{R}(B_p, M_1, B_q)$ is negative. We may conclude that M_1 cannot be either sub- or super-stabilizable with power means.

Example 36. Let $M = M_2$. Then from Proposition 23 and (19) we have

$$\begin{aligned}(M_2 - &\mathcal{R}(B_p, M_2, B_q))(x-t, x+t) \sim \tfrac{1}{8}(5-p-2q)t^2x^{-1} \\ &+ \tfrac{1}{384}\left(2p^3 - 3p^2 + 2p(14q-9) + 4q(4(q-2)q-9) - 45\right)t^4x^{-3} + \dots\end{aligned}$$

For $p = 5-2q$ and $q = \frac{1}{2}(5 \pm \sqrt{17})$, coefficients by x^{-1} and x^{-3} become equal to 0, and coefficient by x^{-5} is equal to $-\frac{11}{180}t^6$. Therefore,

$$M_2 - \mathcal{R}(B_p, M_2, B_q) \prec 0.$$

Conjecture 37. Let $p = 5-2q$ and $q = \frac{1}{2}(5 \pm \sqrt{17})$. Then $M_2 - \mathcal{R}(B_p, M_2, B_q) < 0$.

Example 38. Let $M = M_3$. Then we have the similar situation as in the Example 35 with the same value of the coefficient a_1^M, and therefore asymptotic inequality (28) holds. On the other side, as $(s,t) \to (0,1)$, values of the difference $M_3 - \mathcal{R}(B_p, M_3, B_q)$ may be positive or negative, depending on p and q. Hence, M_3 could only be sub-stabilizable with power means.

Example 39. Let $M = M_4$. Then

$$\begin{aligned}(M_4 - \mathcal{R}(B_p, M_4, B_q))&(x-t, x+t) \sim \tfrac{1}{8}(3-p-2q)t^2x^{-1} \\ &+ \tfrac{1}{384}\left(p(2+p)(-7+2p) + 12pq - 24q^2 + 16q^3 - 7(3+4q)\right)t^4x^{-3} + \dots\end{aligned}$$

For $p = 3-2q$ and $q = \frac{1}{2}(3 \pm \sqrt{21})$ coefficients by x^{-1} and x^{-3} become equal to 0, the coefficient by x^{-5} is equal to $\frac{13}{120}t^6$ and therefore the asymptotic inequality (28) holds.

Conjecture 40. Let $p = 3-2q$ and $q = \frac{1}{2}(3 \pm \sqrt{21})$. Then $M_4 - \mathcal{R}(B_p, M_4, B_q) > 0$.

Example 41. Mean $M = M_5$, when compared with the resultant mean–map behaves similarly as mean M_1. Namely, asymptotic inequality (28) holds, and further, $\lim_{s\to 0} M_5(s, 1-s) = 0$ and $\lim_{s\to 0} \mathcal{R}(B_p, M_5, B_q)(s, 1-s) > 0$, $\forall p, q \in \mathbb{R}$. Analogously as in the Example 35, we may conclude that mean M_5 cannot be either (B_p, B_q)-sub or super-stabilizable.

Example 42. Let $M = M_{\alpha,r}$, defined in (5) with special and limit cases described in Proposition 6. From (27) we find the asymptotic expansion of the difference

$$(M_{\alpha,r} - \mathcal{R}(B_p, M_{\alpha,r}, B_q))(x-t, x+t) \sim \tfrac{1}{2} a_1^M t + \dots,$$

where from Proposition 23 we see that $a_1^M = \alpha \frac{|t|}{t}$. Therefore, asymptotic inequality (28) holds when $\alpha > 0$, and the opposite asymptotic inequality holds when $\alpha < 0$.

Observing the limit of $M_{\alpha,r}(s, 1-s)$ as $s \to 0$, we see that it is equal to 0 for $r + \alpha \geq 0$ and $-(r+\alpha)$ for $r + \alpha < 0$.

If $\alpha > 0$, then by the same argument as in the Example 35, mean $M_{\alpha,r}$ cannot be either (B_p, B_q)-sub- or super-stabilizable. If $\alpha < 0$, then mean $M_{\alpha,r}$ can only be (B_p, B_q)-super-stabilizable. If $\alpha = 0$, then $M_{\alpha,r}$ corresponds the logarithmic mean L, whose asymptotic expansion can be found in [**5**]:

$$L(x-t, x+t) \sim x - \tfrac{1}{3} t^2 x^{-1} - \tfrac{4}{45} t^4 x^{-3} - \tfrac{44}{945} t^6 x^{-5} + \mathcal{O}(x^{-7}).$$

Now we have

$$\begin{aligned}(L - \mathcal{R}(B_p, L, B_q))(x-t, x+t) &\sim \tfrac{1}{8}(1-p-2q) t^2 x^{-1} \\ &+ \tfrac{1}{384}\big(2p^3 - 3p^2 - 2p(5+2q) + 4q(4q(q-1)-5) + 11\big) t^4 x^{-3} + \mathcal{O}(x^{-5}).\end{aligned}$$

If we set $p = 1 - 2q$, then

$$(L - \mathcal{R}(B_{1-2q}, L, B_q))(x-t, x+t) \sim \tfrac{1}{96} q(q-1) t^4 x^{-3} + \mathcal{O}(x^{-5}),$$

and if, additionally, $q = 0$ or $q = 1$, we obtain two pairs of power means for which the resultant mean $\mathcal{R}(B_p, L, B_q)$ is equal to L:

$$\mathcal{R}(A, L, G) = L, \quad \mathcal{R}(H, L, A) = L,$$

which can be easily verified by direct computation.

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